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HOLISM AND CONTEXTUALITY: A QUANTUM-LIKE SEMANTICS FOR MUSIC

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Abstract: Quantum theory and quantum computation give rise to some characteristic holistic semantic situations, where the meaning of a *whole* determines the meanings of its *parts* (and not the other way around, as happens in the semantics of most traditional logics). *Quantum computational logics* are new forms of quantum logic that have been suggested by the theory of quantum logical gates in quantum computation. In the standard semantics of these logics, sentences denote quantum information quantities (systems of qubits, or, more generally, mixtures of systems of qubits), while logical connectives are interpreted as special quantum logical gates (which have a characteristic *reversible* and *dynamic* behavior). An abstract version of this semantics can be naturally used to analyze different kinds of semantic phenomena where *holistic*, *contextual* and *ambiguous* patterns play an essential role. In this framework we analyze some characteristic features of musical languages.

Keywords: Quantum computation. Music.

HOLISMO E CONTEXTUALIDADE: UMA SEMÂNTICA AO ESTILO QUÂNTICO PARA A MÚSICA

Resumo: A teoria quântica e a computação quântica dão origem a algumas situações semânticas de característica holística, onde o significado de um *todo* determina o significado de suas *partes* (e não reciprocamente, como acontece na semântica da maioria das lógicas tradicionais). *Lógicas quânticas computacionais* são novas formas de lógica que têm sido sugeridas pela teoria das portas lógicas quânticas na computação quântica. Na semântica padrão dessas lógicas, as sentenças denotam quantidades de informação quântica (sistemas de qubits ou, mais geralmente, misturas de sistemas de qubits), onde os conectivos lógicos são interpretados como portas lógicas quânticas especiais (as quais têm uma característica *reversível* e um comportamento *dinâmico*). Um versão abstrata dessa semântica pode ser usada naturalmente para analisar diferentes tipos de fenômenos semânticos onde padrões *holísticos*, *contextuais* e *ambíguos* desempenham papel essencial. Neste escopo, analisamos algumas características das linguagens musicais.

Palavras chave: Computação quântica. Música.

1 Introduction

After Frege's *Sinn und Bedeutung* the *compositionality-principle* has been often considered one of the basic assumptions of logical semantics: *the meaning of any compound expression should be described as determined by the meanings of its parts*. In fact, a compositional (analytical) behavior represents a characteristic feature of classical logic and of a number of non-classical logics as well. At first sight, Kripke-semantics seems to provide some counterexamples to compositionality. We need only think of the truth-conditions for negated sentences in many weak logics: generally, the truth-value of a negation $\neg\alpha$ with respect to a possible world w cannot be described as a function of the truth-value of the positive sentence α with respect to w. However, in such cases, compositionality can be recovered at a deeper level, be referring to *all* the worlds that are *accessible* to w. A strong compositionality-principle seems to be hardly compatible with the semantics of natural and of artistic languages, where *holistic* and *contextual* features often play a relevant role: generally, the meaning of a compound expression determines the meanings of its parts, that are strongly context-dependent. Furthermore, meanings are, to a certain extent, always ambiguous.

Is it possible to investigate holistic, contextual and ambiguous semantic phenomena in the framework of a scientific theory? Strangely enough the mathematical formalism of quantum theory (QT) and of quantum computation provides some abstract structures that can be successfully used to this aim. We will try and apply this approach to a formal analysis of musical compositions.

2 A holistic semantics for quantum computational logics

As is well known, the basic concept of quantum computation is the notion of *qubit*, which can be regarded as the quantum variant of the classical notion of *bit*. In classical information theory, one bit measures the information that is transmitted (or received), whenever one chooses one element from a set consisting of two elements (for instance, from the set consisting of the answer *YES* and of the answer *NO*, or from the set consisting of the number 1 and of the number 0). In quantum information, one cannot generally refer to precise answers (like *YES* or *NO*). The typical ambiguous answer is represented by a *quantum perhaps*, that can be described as a quantum *superposition* of the answer *YES* and of the answer *NO*.

From the physical point of view, a qubit can be regarded as the *pure state* of a single particle, while a system of n qubits (also called *quregister*) corresponds to the state of a compound system consisting of n particles. The idea is that a single particle (like an electron) can physically carry the information-quantity represented by one qubit. In order to carry the information stored by n qubits we need, of course, a compound system consisting of n particles.

From the mathematical point of view, qubits are particular vectors, whose length is 1, that "live" in the two-dimensional Hilbert space $\mathcal{H}^{(1)} = \mathbb{C}^2$ (based on the set of all ordered pairs of complex numbers). Hence, the mathematical form of a qubit is usually written as follows:

$$|\psi\rangle = a|0\rangle + b|1\rangle,$$

where $|0\rangle = (1, 0)$ and $|1\rangle = (0, 1)$ are the two elements of the canonical orthonormal basis of the space. From an intuitive point of view, the vector $|0\rangle$ represents the classical bit 0 (the answer *NO* or the truth-value *Falsity*), while the vector $|1\rangle$ represents the classical bit 1 (the answer *YES* or the truth-value *Truth*). According to the standard interpretation of quantum superpositions, the *quantum perhaps* $|\psi\rangle$ represents an information that might give rise to the answer *NO* (corresponding to the state $|0\rangle$) with probability $|a|^2$ and might give rise to the answer *YES* (corresponding to the state $|1\rangle$) with probability $|b|^2$.

What about systems of n qubits (n-quregisters)? As we have seen, an n-quregister is a possible state of a compound system, consisting of n particles. The mathematical environment for such a system can be represented as a special product (called *tensor product*) of n two-dimensional Hilbert spaces. On this basis, the mathematical representative of an n-quregister is identified with a unit-vector of the product space

$$\mathcal{H}^{(n)} = \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{n-times}.$$

Hence, the general form of an n-quregister is usually written as follows:

$$|\psi\rangle = \sum_i c_i |x_{i_1}, \ldots x_{i_n}\rangle,$$

where x_{i_j} is a classical bit, while $|x_{i_1}, \ldots x_{i_n}\rangle$ is an element of the canonical basis of the space $\mathcal{H}^{(n)}$, representing in this framework a classical register (a sequence of n bits).

Quregisters are pure states, hence *maximal pieces of information*, that cannot be consistently extended to a richer knowledge: even a hypothetical omniscient mind could not know more. In quantum computation, one cannot help referring also to non-maximal pieces of information that correspond to *mixtures of quregisters* (also called *qumixes*). These are mathematically represented by *density operators* ρ of a space $\mathcal{H}^{(n)}$. Of course, quregisters can be described as special cases of qumixes.

Quantum computation has recently suggested some new forms of quantum logic that have been called *quantum computational logics* (**QCL′s**). The basic semantic idea of can be sketched as follows[1]:

- the sentences of **QCL′s** are supposed to represent quantum information quantities (qumixes).

[1] Technical details can be found in [1], [2].

- The logical connectives are interpreted as *quantum logical gates* (briefly, *gates*).

What is exactly a gate? As is well known, in QT the dynamic evolution of quantum objects is governed by Schrödinger-equation. Accordingly, for any times t_0 and t_1, a pure state $|\psi(t_0)\rangle$ of an object at time t_0 is transformed into another pure state of the same object at time t_1 by means of a unitary operator U (which represents a reversible transformation):

$$|\psi(t_1)\rangle = U(|\psi(t_0)\rangle.$$

Gates are special examples of unitary operators that transform quregisters into quregisters in a reversible way. Hence, from an intuitive point of view the application of a sequence of gates to an input-quregister can be regarded as the dynamic evolution of a quantum object that is processing a given amount of quantum information. By definition gates are unitary operators whose domains consist of vectors of convenient Hilbert spaces. However they can be naturally generalized also to qumixes.

The classical logical operations (*not*, *and*, *or*, ...), which are normally described as irreversible, can be easily transformed into reversible gates. At the same time, we will have uncountably many new logical operations that cannot have any counterpart in classical logic. Just these strongly non-classical logical operations are responsible for the deep parallel structures that represent the main cause of the efficiency and of the speed of quantum computers. Of course, any particular example of quantum computational logic will select a system of gates that are considered significant from a logical point of view.

In the holistic semantics of **QCL′s** (see [4]) a *model* (or *interpretation*) of the language is a map `Hol` that assigns to any sentence α a qumix that represents the informational meaning of α:

$$\alpha \;\mapsto\; \mathtt{Hol}(\alpha).$$

As expected, any model `Hol` shall preserve the logical form of the sentences, by interpreting any connective $\circ$ of the language as a corresponding gate G°. Furthermore, the qumix $\mathtt{Hol}(\alpha)$ should live in a Hilbert space whose dimension depends on the logical form of α. The simplest examples of sentences are *atomic* sentences (which cannot be decomposed into more elementary sentences). Accordingly, the meaning of such sentences shall live in the simplest Hilbert space: the two-dimensional space $\mathcal{H}^{(1)} = \mathbb{C}^2$ (where all qubits are located). A molecular sentence with n occurrences of atomic sentences can be regarded as a linguistic description of a compound physical system consisting of n particles. In fact, we need n particles in order to carry the information that is expressed by our molecular sentence. On this basis, it is natural to assume that the meaning of such a sentence lives in the product-space $\mathcal{H}^{(n)}$. Let us call *atomic complexity* of a sentence α (abbreviated as $At(\alpha)$) the number of occurrences of atomic sentences in α. The space $\mathcal{H}^{\alpha} = \mathcal{H}^{(At(\alpha))}$, where all possible meanings of α shall live, represents the *semantic space* of α.

The holistic features of our semantics depend on the fact that any model `Hol` assigns to any sentence α a *global* meaning, that cannot be generally inferred from the meanings assigned by `Hol` to the atomic parts of α. What happens here is just the opposite with respect to the standard behavior of compositional semantics: $\mathtt{Hol}(\alpha)$ determines the meanings of all its parts, which turn out to be essentially *context-dependent*. As a consequence, any sentence may receive different meanings in different contexts. Going from the *whole* to the *parts* is here possible because all logical operations are reversible: one can go back and forth without any dissipation of information!

A fundamental role in this semantic game is played by the notion of *entanglement*, one of the most mysterious aspects of QT, which is mathematically based on the characteristic properties of tensor products. What is exactly entanglement? From an intuitive point of view the basic features of an *entangled state* $|\psi\rangle$ can be sketched as follows:

- $|\psi\rangle$ is a maximal information (a pure state) that describes a compound physical system S (say, a two-electron system);

- the information determined by $|\psi\rangle$ about the parts of S is non-maximal. Hence, the states of the whole system is a pure state, while the states of the parts (which are determined by the state of the whole and are usually called *reduced states*) are proper mixtures. Once broken into its parts, the puzzle cannot be composed again! It may also happen that the state of the compound system (although representing a maximum of information) describes the parts as essentially indiscernible objects, that cannot satisfy any characteristic individual property. One obtains, in this way, an apparent violation of Leibniz' indiscernibility principle.

A typically entangled state is, for instance, the following quregister (which lives in the space $\mathbb{C}^2 \otimes \mathbb{C}^2$):

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle) + \frac{1}{\sqrt{2}}(|1\rangle \otimes |0\rangle).$$

Here $|\psi\rangle$ describes a two particle-system, where both parts might be either $|0\rangle$ or $|1\rangle$ with probability $\frac{1}{2}$.

Entanglement-phenomena can be naturally used to model some typical holistic semantic situations in the framework of our quantum computational semantics. We can consider entangled quregisters that are meanings of molecular sentences. As an example, consider a conjunctive sentence having the form

$$\gamma = \alpha \wedge \beta.$$ [2]

The following situation is possible:

- the *meaning* $\texttt{Hol}(\gamma)$ of the conjunction γ is a quregister, which represents a maximal information (a pure state);
- the meanings of the parts (α,β) are quantum-entangled and cannot be represented by two pure states (two quregisters).

We can say that the *sharp meaning* of the conjunction determines two *ambiguous meanings* for the parts (α, β), which are represented by two mixed states. In other words, *the meaning of the whole determines the meanings of the parts, but not the other way around*. In fact, one cannot go back from the two ambiguous meanings of the parts to the quregister representing the meaning of the whole. The mixed state representing the ambiguous meaning of α (of β) can be regarded as a kind of *contextual meaning* of α (of β), determined by the *global context*, which corresponds to the quregister $\texttt{Hol}(\alpha \wedge \beta)$ (the meaning of the conjunction $\alpha \wedge \beta$).

In spite of its appealing properties, the standard version of the quantum computational semantics is strongly "Hilbert-space dependent". This certainly represents a shortcoming for all applications, where real and complex numbers do not generally play any significant role (as happens, for instance, in the case of natural and of artistic languages).

Is it sensible to look for an *abstract quantum computational semantics*? This question admits a positive answer (see [8]). One can define a notion of *abstract qumix structure*, where abstract qumixes, quregisters and registers are identified with some special objects (not necessarily living in a Hilbert space), while gates are reversible functions that transform qumixes into qumixes. From an intuitive point of view, abstract qumixes and quregisters represent pieces of information that are generally *uncertain*, while

[2] In the framework of **QCL's** a conjunction $\alpha \wedge \beta$ is dealt with as a metalinguistic abbreviation for a more complex ternary conjunction that represents a reversible logical operation (see [1]).

(abstract) registers are special examples of quregisters that store a *certain* information. Any (abstract) qumix is associated to a given length n, and lives in a subdomain $\mathfrak{D}^{(n)}$ of the domain $\mathfrak{D}$ of all possible qumixes.

As expected, one can show that the concrete (Hilbert-space) qumix-structures are special examples of abstract qumix structures.

3 The language of scores: a bidimensional syntax

To what extent can some basic ideas of the quantum holistic semantics be successfully applied to a formal analysis of music? Let us first discuss some characteristic syntactical features of musical languages. As is well known, musical scores are very complicated examples of symbolic languages. It is interesting to analyze how information is coded by musical scores in comparison with the standard formal languages that are used for scientific theories. The most important differences seem to be the following:

- Formal scientific languages are basically *linear*: *words* and *well-formed expressions* are represented as *strings* consisting of symbols that belong to a well determined alphabet. The relevant syntactical relations can be adequately represented by a *Turing Machine*, which refers to a one-dimensional tape.
- Scores, instead, are two-dimensional syntactical objects, which have at the same time a horizontal and a vertical component. Any attempt to linearize a score would lead to totally counter-intuitive results. From a semantic point of view, the characteristic two-dimensionality of musical notation seems to be significantly connected with the deep parallel structures that have an essential role in our perception and intellectual elaboration of musical experiences.

Is it possible (and interesting) to represent a musical score as a peculiar example of a formal language? In a sense, are scores formalizable? One can positively answer to this question, by introducing the notion of formal representation of a musical score. From an intuitive point of view, we can imagine the formal structure of a score as a configuration of symbols that are written on a piece of graph-paper: each row of the paper corresponds to what shall be performed by a particular instrument (say, the first violin); while columns correspond to what shall be played at the same time. Each cell of our paper can be regarded as a container for an atomic information. From the mathematical point of view, such two-dimensional syntactical configurations can be naturally represented by means of some special matrix-like structures. Accordingly, any score-measure can be represented as follows:

$$\begin{pmatrix} Ins_1 : A^{[r]}_{11} \ldots A^{[r]}_{1n} \\ Ins_2 : A^{[r]}_{21} \ldots A^{[r]}_{2n} \\ \ldots \\ Ins_m : A^{[r]}_{m1} \ldots A^{[r]}_{mn} \end{pmatrix}$$

where each row corresponds to a particular instrument (or voice), while columns correspond to what shall be played simultaneously. Any $A^{[r]}_{ij}$ represents a *score-atom*: a piece of information for the i-th instrument at the j-th position of the measure in question. For the sake of simplicity, one can assume that the number of rows remains constant through the score. Whenever a given instrument *tacet*, we suppose that some appropriate pause-symbols are written for the instrument in question.

The traditional musical notation is highly complicated: in fact, reading a score turns out to be quite difficult for non-professional musicians. The language of music is, without any doubt, much more complex and rich than the standard formal languages used by scientific theories. Roughly, we can identify at least the following classes of symbolic expressions that play a fundamental role:

a) names for the different notes that represent different sound-pitches, (for instance, the diapason-a, corresponding to the approximate frequency of 440 Hertz);

b) names for the different pauses;

c) meter-indications (like $\frac{4}{4}$, $\frac{6}{8}$, and so on);

d) metronomic indications;

e) tempo-indications (like *Allegro*, *Adagio*, *Vivace*, and so on);

f) dynamic-indications (like *piano*, *forte*, *crescendo*, *diminuendo*, and so on);

g) prescriptions about the sound-emission (for instance, *legato*, *staccato*, *pizzicato*, and so on);

h) names for the different instruments and for the different kinds of voice (violin, viola, soprano and so on).

Traditionally, note-names are indicated by referring to the staff-notation and to some conventions that make essential use of the clefs, the accidentals and so on. In this framework, each occurrence of a note-name is associated with a certain rhythmical value (say, $\frac{1}{4}$). A number of indications are given either at the beginning of a movement (for instance, *Allegro*) or at the beginning of a measure or at the beginning of a line (which consists of a sequence of measures). Of course, all initial indications are supposed to be distributed over what is following, if they are not changed in the course of the score. Any formal version of a score shall reflect all these pieces of information, possibly adopting some different conventions. For instance, one could use some arithmetization-techniques, by assuming that note-names are conventionally represented by particular natural numbers.

An important role is played by *musical phrases*, which seem to behave like *well-formed expressions* of a formal language. From a syntactical point of view, musical phrases can be described as special *linguistic objects* that represent fragments of a formal score. Any formal definition shall take into account the following characteristic properties of concrete musical phrases (briefly *phrases*):

- A phrase generally consists of a (small) number of measures.
- A phrase does not necessarily begin at the beginning of a measure and does not necessarily end at the end of a measure.
- Phrases are generally *transversal* with respect to the instruments of the score: an instrument may begin a given phrase, while other instruments will continue it.
- A phrase does not generally concern *all* the score-atoms contained in the measures where parts of the phrase in question appear. Hence, phrases can be formally represented as pieces of score with "holes", which will be also called *empty score-atoms* (and indicated by the symbol $\clubsuit$). Of course, empty score-atoms (which represent "holes") should not be confused with pause-symbols.

Accordingly, a phrase can be identified with a special sequence of score-columns, that may contain empty score-atoms. We will require the following natural condition: all the columns that constitute a phrase shall contain at least one non-empty score atom.

As an example, we might refer to the celebrated *incipit* of Beethoven's Ninth Symphony, where the measures 2-5 played by the first violins, the violas and the contrabasses give rise to a phrase in this sense. Here, we suppose that the corresponding measures of all other instruments consist of score-holes ♣.

But what might be the interest of formalizing musical languages? As is well known, in the case of scientific theories, formalization is not aimed at providing some perfect languages that should substitute the "old rough" languages used by the scientific community. Formal languages are heavy and unreadable (if not accompanied by some translation-rules into the natural language). In a similar way, any attempt to substitute a traditional score with its formal version would be absolutely unreasonable! In both cases (science and music), the basic aim of formalizing languages is bringing into light some deep linguistic structures that represent significant invariants in a variety of different kinds of expressions and notation-systems. Identifying the elements that have a fundamental role in our information encoding process represents the first step for any successful theoretic analysis.

4 Musical interpretations in an abstract holistic semantics

Let us now turn to semantic problems. How can we describe the relationship between a score and the class of all its (possible or real) interpretations? What do we exactly mean by *interpretation* of a musical score? We are now dealing with a quite critical concept, often discussed by musicians and musicologists, who have proposed different perspectives and solutions.

As is well known, the sound-world is a typically *relational* world: the meaning of a single note, of a chord, of a musical phrase is always context-determined. There is no doubt that music requires a holistic and contextual semantics. Is it possible (and interesting) to look for a formal characterization of the intuitive concept of *musical interpretation* by using some abstract tools suggested by our quantum holistic semantics?

Consider a score **S** (say, the score of Beethoven's Ninth Symphony), and let **FS** be a formal version of **S**. By abstracting from what happens in the concrete cases, let us first try and identify the elements that play a fundamental role in any possible interpretation of our score.

a) **The choice of the musical phrases**
Formal scores (as well as real scores) do not contain any explicit division into phrases. All the signs written in a score correspond to atoms that can be combined giving rise to phrases according to a number of different modalities. The choice of the significant phrases is not determined at the syntactical level but rather at the semantic one. For any relevant score-fragment, any interpretation generally collects the signs occurring in that fragment by forming a few phrases that create a kind of dialogue in the framework of a characteristic "musical design". Consider a complex instrumental composition (say, a symphony by Brahms or by Mahler): the conductor (and the listener, as well) does not generally perceive distinct sounds that correspond to the different rows in the score. On the contrary, he (she) seems to "collect" them into some holistic patterns that behave like "individual voices". Of course, this does not forbid the (good) conductor to recognize possible mistakes of single performers in an analytical way.

Accordingly, from a formal point of view, the following assumption seems to be sensible: the first element of a given interpretation is a partition of the score into a number of musical phrases. We will call *score-covering* any set Phr of phrases that satisfies the following condition: each score-atom occurs exactly in one element of Phr (hence, Phr turns out to *cover* the whole score).[3] Needless to say, any score admits, in principle, a number of different coverings. As a consequence, one can reasonably assume that the choice of a covering Phr represents the first characteristic element of a given interpretation, giving rise to a significant bridge between musical syntax and semantics. Just in this sense the common musical jargon uses to refer to the *phrasing* of a particular interpreter.

b) **The tempo-choice**
Another element that represents a characteristic feature of any particular interpretation is the tempo-choice. As is well known, different performances

[3]Notice that each score-atom is labelled by an index. Hence, different occurrences of one and the same note-name give rise to different atoms.

of one and the same score may have quite different durations. All metronomic indications (contained in the score) generally represent only approximate prescriptions: should they be faithfully respected, the result would be a quite boring interpretation!

How can we describe, from an abstract point of view, the tempo-choice in the framework of a particular interpretation? We can assume that the second characteristic element of a given interpretation is a *temporal function* $Temp$ that assigns a time interval to each column occurring in the score. Let

$$\mathbf{c}_r^j = \begin{pmatrix} A_{1j}^{[r]} \\ A_{2j}^{[r]} \\ \dots \\ A_{mj}^{[r]} \end{pmatrix}$$

be the j-th column of the r-th score-measure. We will write:

$$Temp : \mathbf{c}_r^j \ \mapsto \Delta t.$$

Of course, as happens in all experimental sciences, the lengths of such time-intervals are always determined up to a certain *error*: as a consequence, any $Temp(\mathbf{c}_r^j)$ is, in fact, a fuzzy interval. On this basis, the function $Temp$ automatically determines the duration of the performance of the whole score. From the musical point of view, $Temp$ realizes all the dynamic choices of the performers: any *accelerando*, *ritardando*, *rubato*, which often do not have any clear syntactical counterpart in the written score.

c) **The choice of the musical meanings**

The choice of the musical meanings certainly represents the most critical element for an abstract concept of musical interpretation. What do musical compositions mean? Does music have any content? As is well known, this question has been deeply discussed by musicians, musicologists and philosophers, who have proposed different answers. A basic problem seems to be the following: suppose we accept that music is associated to some *meanings*. Are these meanings always *internal* to music, or do they rather essentially refer to some *external worlds*? Whatever is our answer to this preliminary question, we are faced with the following problem: is it possible to define those particular objects that we call the *meanings* of a musical composition?

In fact, looking for simple definitions seems to be a somewhat naive attitude both in the case of music and of scientific theories. A more fruitful position, that has represented a winning trend for a number of contemporary semantic theories, refuses any research for general *explicit definitions*.

Instead of trying to define the entities we are talking about, one attempts to describe their behavior by creating some convenient structures, where our entities are supposed to live. On this basis *meanings* are identified with some abstract objects that belong to special worlds, often called *models* of the theories under investigation. Following this perspective, we will not try to answer the question "what are musical meanings?". We will only attempt to formally describe some characteristic properties thereof.

Our starting point is the following hypothesis: we assume the existence of a *universe* (indicated by $\mathbf{M}$) of ideal objects called *musical meanings*. The elements of $\mathbf{M}$ may possibly refer to some extra-musical worlds that are supposed to be distinguished from them. From an intuitive point of view, musical meanings can be thought of as a kind of *intensional* objects, that are quite similar to the intensional meanings associated to the expressions of our natural languages. Generally, a musical meaning cannot be directly identified with a physical sound-event. One is rather dealing with an *idea* that permits us to create a particular sound-event. Just these ideas represent an important link between the score (which is a syntactical object) and a performance (which corresponds to a physical event). In a sense, they play the role of a kind of *invariants* (or *quasi-invariants*) with respect to the concrete performances that are realized in different times. It is not a chance that we use to speak, for instance, of "Herbert von Karajan's interpretation" of Beethoven's Ninth Symphony, without necessarily referring to a particular historical performance.

How is the universe of all musical meanings structured? It is natural to assume the following hypothesis: $\mathbf{M}$ includes a subuniverse $\mathbf{M}^{At}$ consisting of all *atomic meanings* that may represent the *intension* of a single note. As an example, in the case of a string quartet, the atomic meanings will be represented by all possible ideas of a single sound that can be performed by a violin or by a viola or by a cello. Apparently, when a quartet-performer plays a single note, he (she) selects a sound-idea from this atomic universe. We will suppose that the set $\mathbf{M}^{At}$ contains tho privileged elements: the *silence-meaning*, which corresponds to a pause-symbol, and an *empty meaning*, which represents the meaning of a score-hole $\clubsuit$.

As we know, single notes are combined giving rise to score-columns and to phrases (which are column-sequences possibly containing score-holes $\clubsuit$). In order to associate meanings to all molecular syntactical objects, we shall refer both to vertical and to horizontal combinations of atomic meanings. Let v be the *vertical complexity* (i.e. the number of rows) of our score: any sequence $(\mathbf{m}_1, \ldots, \mathbf{m}_v)$ of v atomic meanings will represent a possible meaning of a score column. Such molecular meanings (expressing ideal chords) will be called *vertical meanings*. We will assume that $\mathbf{M}$ contains all possible sequences $(\mathbf{m}_1, \ldots, \mathbf{m}_v)$; in other words, $\mathbf{M}$ is supposed to be closed under the v-th cartesian power of $\mathbf{M}^{At}$.

Any *analytical* semantic description would stop here, identifying the set of all vertical meanings (of complexity v) with $(\mathbf{M}^{At})^v$ (the v-th cartesian power of $\mathbf{M}^{At}$)). In this sense, molecular meanings are represented as determined by their parts. As we have seen, quantum holistic semantics gives rise to a different situation: generally, the meaning of a global expression determines the meanings of its parts, and not the other way around! Furthermore, meanings may be entangled: hence our global understanding of a given expression might be more precise and sharp than the contextual pieces of information we have about its parts.

Let us try and apply these ideas to the semantics of music. Our basic hypothesis is the following: the universe of all vertical meanings (the ideal chords) is not realized by the v-th cartesian power of $\mathbf{M}^{At}$. Besides the *analytical meanings* (represented by all sequences $(\mathbf{m}_1, \ldots, \mathbf{m}_v)$), we assume the existence of *holistic meanings*, which are not determined by the combination of their parts. Following the example of quantum semantics, holistic meanings can be thought of as *superpositions* of analytical meanings, that ambiguously describe a variety of coexistent semantic situations. We will indicate by $\mathbf{M}^{\downarrow}$ the set of all possible vertical meanings, which may be either analytical or holistic; and we will use the symbol $\mathbf{m}^{\downarrow}$ for a generic element of $\mathbf{M}^{\downarrow}$.

As happens to compound quantum systems, any global vertical meaning $\mathbf{m}^{\downarrow}$ determines its parts, but generally not the other way around. We have seen that, in the case of quantum objects, one can define the *reduced state function red*: if ρ is the state of a system consisting of n parts, then $red_i(\rho)$ represents the state of the i-th component of our system (for any i s.t. $1 \leq i \leq n$). It is reasonable to assume that a similar function also exists for musical meanings. Accordingly, we will call *vertical contextualization function* the following map:

$$cont_i : \mathbf{m}^{\downarrow} \;\mapsto \mathbf{m} \in \mathbf{M}^{At} \;\text{ (for any } 1 \leq i \leq v).$$

Hence, $cont_i(\mathbf{m}^{\downarrow})$ is the atomic meaning that represents the contextual meaning of the i-th part of $\mathbf{m}^{\downarrow}$. In the particular case where $\mathbf{m}^{\downarrow}$ is an analytical meaning having the form $(\mathbf{m}_1, \ldots, \mathbf{m}_v)$, we shall require the following natural condition:

$$cont_i(\mathbf{m}^{\downarrow}) = \mathbf{m}_i.$$

In other words, the contextual meaning of the i-th part of $\mathbf{m}^{\downarrow}$ is just its i-th element. In fact, analytical meanings turn out to behave like *factorized states* in QT (states that can be represented as tensor products $\rho_1 \otimes \ldots \otimes \rho_n$ of the states ρ_i associated to their parts).

As an example, let $\mathbf{m}^{\downarrow}$ be a vertical meaning corresponding to a c-major chord (which consists of the three notes c, e, g):

Suppose we are reasoning in the framework of a harmony-treatise, where we are not concerned with any interpretation-problem. In such a case we will naturally represent $\mathbf{m}^{\downarrow}$ as an analytical meaning identified by the sequence of three atomic meanings. Hence will have three vertical contextualization-functions such that:

$$cont_1(\mathbf{m}^{\downarrow}) \;\mapsto\; \mathbf{m}_1;\; cont_2(\mathbf{m}^{\downarrow}) \;\mapsto\; \mathbf{m}_2;\; cont_3(\mathbf{m}^{\downarrow}) \;\mapsto\; \mathbf{m}_3,$$

where $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{m}_3$ represent three atomic meanings corresponding to the three notes c, e, g, respectively.

A similar procedure can be also applied to horizontal combinations. In this way we obtain some abstract configurations that represent harmonic and melodic structures at the same time. Let o be the *horizontal complexity* of our score (which is determined by the number of the score-columns). Any sequence

$$(\mathbf{m}_1^{\downarrow}, \ldots, \mathbf{m}_r^{\downarrow}) \text{ (where } 1 \leq r \leq o),$$

represents a possible horizontal meaning, that may be the interpretation of a phrase whose (horizontal) complexity is r. Sequences of this kind will be called *horizontal analytical meanings* (of complexity r).

Of course, analytical meanings will not exhaust the universe of all possible horizontal meanings. As happens in the case of vertical meanings, we will have horizontal meanings that cannot be identified with sequences of vertical meanings. Such meanings, that are not elements of the r-th cartesian power of $(\mathbf{M}^{At})^v$, will be called *horizontal holistic meanings*. We will indicate by $\mathbf{M}^{\rightarrow}[r]$ the subset of $\mathbf{M}$ that consists of all (analytical and holistic) meanings of complexity r. The symbol $\mathbf{m}^{\rightarrow}[r]$ will represent a generic element of $\mathbf{M}^{\rightarrow}[r]$, while $\mathbf{m}$ will denote any musical meaning living in the universe

$\mathbf{M}$. Apparently, all vertical meanings (which consist of a single column) are special examples of horizontal meanings whose complexity is 1.

As we have done in the vertical case, we will also assume *horizontal contextualization functions* that permit us "to go from the whole to the parts". Let $\mathbf{m}^{\rightarrow}[r]$ be a horizontal meaning: for any i (s.t. $1 \leq i \leq r$), $cont_i(\mathbf{m}^{\rightarrow}[r])$ will represent the i-th *contextual meaning* of $\mathbf{m}^{\rightarrow}[r]$. As expected, the contextualization-functions can be naturally generalized to all *molecular parts* (consisting of sequences of columns) of a given meaning $\mathbf{m}$.

As an example, we might refer again to the *incipit* of Beethoven's Ninth Symphony (Fig.1). Now we will consider the first five measures of the whole score.

Here the dominating phrase (*e*-*a*, *a*-*e*, *e*-*a*) is performed by the first violins, the violas and the contrabasses. One is dealing with a musical idea that has been compared with a sudden, striking event. In this connection, one has sometimes evoked the image of a meteorite-fall, but we could also think of the origin of the Universe. This dominating phrase emerges from a kind of indistinct background: the empty fifth chords realized by the horns and by the *tremoli* of the second violins and of the cellos. Needless to observe, any analytical analysis would be fully inadequate to describe such a semantic situation. One cannot avoid referring to a global musical meaning, while the different contextualization-functions permit us to obtain different partial meanings for the component parts.

b) **The relationship phrases-meanings**

The set $\mathbf{M}$ of all musical meanings represents a kind of *virtual* universe: the score-interpretation shall select from this universe a particular meaning for each phrase that belongs to the score-covering Phr. Accordingly, any syntactical phrase α will be associated to a *semantic phrase* represented by a meaning $\mathbf{m}$ ($\in \mathbf{M}$) that shall respect the linguistic form of α.

We know that in the semantics of scientific theories, any model of a theory T associates to any expression of the language of T a meaning that lives in the system of objects created by the model in question. In a similar way, in our musical semantics it is natural to require the existence of a map $Real$ that assigns to any phrase α of the score-covering Phr a convenient meaning:

$$Real : \alpha \ \mapsto \ \mathbf{m} \in \mathbf{M},$$

where the phrase α and $Real(\alpha)$ are supposed to have the same horizontal complexity r.

On this basis, the contextualization-functions, applied to $Real(\alpha)$, will determine a contextual meaning for all subphrases of α. We require that

Figure 1: The *incipit* of Beethoven's Ninth Symphony

the realization-function *Real* satisfies the following (natural) condition: the contextual meaning of any occurrence in α of the score-hole $\clubsuit$ is always the empty meaning.

To what extent shall the map *Real* preserve all pieces of information that are written in the score? In principle, any "faithful interpretation" should respect *all* score-prescriptions. We know however that this is not generally the case in real performances, both for technical mistakes and for interpretation-choices of the performers. How to describe formally the "faithfulness-degree" of a given interpretation? As happens in the case of experimental sciences, we can require that *Real* preserves the score-prescriptions *up to a certain accuracy*, that can be conventionally represented by a numerical parameter ε_{Real} (which can be easily defined). Of course, ε_{Real} will represent one of the characteristic features of the interpretation in question.

The conditions we have required guarantee the contextual behavior of *Real*: phrases that occur in different parts of the score generally receive different contextual meanings. This reflects what happens in the case of real performances, where repeated musical phrases are generally interpreted according to different modalities.

e) **The choice of extra-musical meanings**

So far we have considered musical meanings as a kind of autonomous ideal objects that may possibly refer to some extra-musical worlds (which are sharply distinguished from them). We know however that evoking some extra-musical events (like emotions, feelings, descriptions, ...) represents, in many cases, an important characteristic property of a given interpretation. And referring to extra-musical situations turns out to be an essential component of interpretations, whenever the score includes a text (as happens in the case of lyric operas, *Lieder* and symphonic poems).

How can we find a formal counterpart for such correlations between musical and non-musical events? We can assume that the last characteristic element of a given interpretation is represented by a map Wor that assigns to any phrase α of the score-covering Phr a world w consisting of non-musical meanings:

$$Wor:\ \alpha\ \mapsto\ w.$$

It seems reasonable to admit that $Wor(\alpha)$ may be the empty set. In fact, there are musical compositions (or parts thereof) whose meanings are completely internal to music. As an example, we might think of many Bach's compositions (like the *Goldberg Variations*).

How to describe the values of the map Wor? We should try and represent the vague, ambiguous and subjective features that seem to characterize all feelings and all concrete situations that are evoked by music. Classical semantics could not certainly be used to this aim. However, we can have recourse to a special kind of *fuzzy possible world semantics*, where generally all objects and all relations are supposed to have an ambiguous behavior. In this framework, $Wor(\alpha)$ can be identified with a particular *fuzzy possible world*. Needless to say, the creation of the fuzzy worlds to be associated to the different score-phrases, represents a characteristic choice of any interpreter.

Summing up, any interpretation $\mathcal{I}$ of a formal score **FS** can be now represented as an abstract structure whose form is the following:

$$\mathcal{I} = (Phr, Temp, \mathbf{M}, Real, Wor),$$

where:

- Phr is a score-covering of **FS**;
- $Temp$ is the function that assigns to any score-column of **FS** a time-interval;
- **M** represents the universe of all musical meanings, which includes the sets of all atomic, of all vertical and of all horizontal meanings. For any meaning **m** (of **M**), the (vertical and horizontal) contextualization-functions assign contextual meanings to all parts of **m**.
- $Real$ is the realization map that assigns a semantic phrase (an element of **M**) to any syntactical phrase α of Phr, by preserving the score-prescriptions (contained in α) up to a certain accuracy ε_{Real}.
- Wor is the map that assigns to any phrase α of Phr a (possibly empty) fuzzy world w, which represents the extra-musical meaning of α.

We require that $\mathcal{I}$ satisfies all the conditions we have illustrated above.

Following the example of (formalized) scientific theories, we can now identify any musical composition with a pair $(\mathbf{FS}, \mathbf{K})$ consisting of a formal score **FS** and of the class **K** of all possible interpretations $\mathcal{I}$ of **FS**. On this basis, the *history* of a given musical composition can be reconstructed as a kind of "journey" through the class **K**. Since the set of all historical journeys is *de facto* finite, while **K** is in principle infinite, one can easily understand the abstract reason why musical compositions are essentially *unfinished and open works*.

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