CDD: 510.1

THE LEAP FROM THE EGO OF TEMPORAL CONSCIOUSNESS TO THE PHENOMENOLOGY OF MATHEMATICAL CONTINUUM

STATHIS LIVADAS

Department of Mathematics
University of Patras
Patras 26500, GREECE

livadas@math.upatras.gr

Abstract: This article attempts to link the notion of absolute ego as the ultimate subjectivity of consciousness in continental tradition with a phenomenology of Mathematical Continuum (**MC**) as this term is generally established following Cantor's pioneering ideas on the properties and cardinalities of sets. My motivation stems mainly from the inherent ambiguities underlying the nature and properties of this fundamental mathematical notion which, in my view, cannot be resolved in principle by the analytical means of any formal language not even by the addition of any new axioms to a consistent first-order axiomatical system such as the Zermelo-Fraenkel (**ZF**) Set Theory. In this phenomenologically motivated approach I deal, to some extent, with the undecidability of a fundamental statement about the cardinality of Continuum inside **ZFC** theory, namely the *Continuum Hypothesis*, and also with the underlying root of Gödel's first incompleteness result.

Keywords: Absolute ego of consciousness. Continuum Hypothesis. Gödel's (First) Incompleteness Theorem. Intuitionistic principles. Mathematical Continuum. Temporal consciousness.

O SALTO DO EGO DA CONSCIÊNCIA TEMPORAL À FENOMENOLOGIA DO CONTÍNUO MATEMÁTICO

Resumo: Este artigo procura conectar a noção de ego absoluto enquanto subjetividade última da consciência na tradição continental com a fenomenologia da matemática do contínuo (**MC**) tal como este termo foi estabelecido seguindo as idéias pioneiras de Cantor sobre as propriedades e cardinalidade de conjuntos. Minha motivação deriva basicamente das ambigüidades subjacentes à natureza e propriedades desta noção matemática fundamental as quais, no meu entender, não podem, em princípio, ser resolvidas pelos meios analíticos de qualquer linguagem formal e nem mesmo pela adição de novos axiomas a um sistema axiomático consistente de primeira ordem como a teoria de conjuntos de Zermelo Fraenkel (**ZF**). Neste tratamento fenomenológicamente motivado eu lido em certa medida com a indecidibilidade de uma tese fundamental sobre a cardinalidade do contínuo em **ZFC**, a saber, a *Hipótese do Contínuo*, e também com aquilo que está na base do primeiro resultado de incompletude de Gödel.

Palavras chave: Ego absoluto da consciência. Hipótese do Contínuo. (Primeiro) Teorema da Incompletude de Gödel. Princípios intuicionistas. Contínuo matemático. Consciência temporal.

1. INTRODUCTION

I must mote from the beginning that my approach of the question of Mathematical Continuum in this paper might indeed make sense for someone finding that mathematics fundamentally involve some particular functions of human consciousness whereas it might almost seem irrelevant to some people regarding mathematical objects as ideal objects preexisting in some kind of platonic realm or on the other extreme to some others regarding mathematics reducible to a consistent set of axioms and rules presiding a game of otherwise meaningless symbols.

My general view here is to a significant extent based on the Husserlian idea taking mathematical objects as built on perceptual objects by means of a categorial intuition (*Wesenschau*) based nevertheless on sensible intuition and thus leading to the theoretical possibility of reducing **MC** taken as a purely formal entity to the intuitive Continuum of perceptual experience. In turn, intuitive Continuum may be described after the notion of phenomenological Continuum following a phenomenological analysis of the constitutive flux of consciousness which on a yet deeper level can lead to the necessity of introducing a transcendental factor (the absolute ego of consciousness) to ensure the continuous unity of the self-constituting flux (see Husserl (1996)). This reduction to a phenomenological Continuum was largely L.E.J. Brouwer's and with some variations H. Weyl's approach to the notion of intuitive

Continuum in their scope to provide an intuitionistically oriented foundation of Mathematical Analysis (Van Atten *et a*l (2002)).

In this respect, I draw, on one hand, a parallel between L.E.J. Brouwer's 'two-ity' intuition to produce elements of any infinite choice sequence as 'durationless' points and E. Husserl's articulation of (transversal) intentionalities of a subject as noematical objects[1] as in the unity of his time flux of consciousness (see Husserl (1995)); on the other, I draw a parallel between Brouwer's assuption of existence of an underlying continuous substratum divested of any predicative character (termed the Primordial Intuition of Mathematics) on which to 'embed' objects of the 'two-ity' intuition and Husserl's description of the impredicative character of the self-constituting absolute flux of consciousness.

To ground my claims on the possibility of a phenomenology of Mathematical Continuum (**MC**) tracing its subjective origin on the transcendental (and thus impredicative) character of the temporal ego of continental tradition, I will be primarily based on Husserlian phenomenology. This seems to provide a well-articulated descriptive

---

[1] It seems purposeful here to be a bit more specific about the meaning of the phenomenological terms noetical and noematical described primarily in E. Husserl's *Ideen I* (Husserl (1995)). A noematical object manifests itself as an immanence in the flux of a subject's consciousness and is constituted by certain modes as a well-defined object immanent to the flux which can then be 'transformed' to a formal-ontological object and consequently a symbolic object of an analytical theory naturally including any formal mathematical theory. It can then be said to be given apodictically in experience inasmuch as: (1) it can be recognized by a perceiver directly as a manifested essence in any perceptual judgement (2) it can be predicated as existing according to the descriptive norms of a language and (3) it can be verified as such (a reidentifying object) in multiple acts more or less at will (Heelan (1988)). In contrast, a noetical object by hyletical-noetical perception (Wahrnehmung) can be only thought of as an aprioric orientation of an intentional consciousness by its sole virtue of being given as such 'in person' in front of a consciousness inside the open horizon of Life-World.

context on which to reduce the objects of analytical-logical formulas and consequently those of Pure and Mathematical Logic to formal-ontological objects in complete 'evacuating' abstraction from noematical objects linked in turn to corresponding intentionalities and constituted in a kind of homogenous synthesis in the unique flux of time consciousness.

I note, in passing, that it seems to be a common base in the existentialist approaches of e.g. M. Heidegger and J.P. Sartre[2] to the subjectivity of temporal consciousness with Husserlian phenomenology inasmuch as they seem to reduce to a kind of absolute subjectivity of temporal ego non-describable in terms of being in temporal objectivity without alienating itself from its mode of 'being' as a subjectivity. From this point of view, the phenomenological analysis of time consciousness and the temporality of existentialist ego can possibly lend themselves to a common theoretical ground towards a deeper understanding of the impredicative character of the continuous unity of time consciousness to the extent that they both lead to the inherent impossibility of an ontological definition of the absolute subjectivity of consciousness except by its 'auto-alienation' in objective reflection. As a matter of fact, Husserl was led by means of longitudinal intentionality (*Längstintentionalität*)[3] of

---

[2] I refer here, in particular, to the Sartrean idea of temporal ego dealt with in *L'être et le néant* (Sartre (1943)), where one is led to the impredicative character of moments of actuality and the necessity to turn to some kind of non-temporal subjectivity of the Being-for-itself (Être-pour-soi). By all accounts, one may also deduce a transcendence in Heidegger's description of the temporality of 'Being-in the World' (Dasein) precisely by the description of ecstatic temporality which cannot be characterized in terms of ontic being as it is rather an élan vital, an impetus alienating any 'being-in-itself' from its substance and transforming it into a ceaseless motion.

[3] Intentionality as a phenomenological term should not be understood as a kind of relation of a psychological content. By intentionality is meant something fundamentally deeper and aprioric. To a non-expert in Phenomenology it can be described as grounding the aprioric necessity of orientation of a subject's

the flux to a yet deeper constituting level of a transcendental type termed the absolute ego of consciousness which is a pre-reflective, non-objectifiable and thus impredicative subjectivity, the ever in-act subjectivity of the continuous unity of temporality.

My task in the next, will be to provide some clues to questions pertaining to the nature and properties of **MC** and in a broader sense to those pertaining to the notion of uncountable infinity that point to the question of constitution of a continuous unity of temporal consciousness taken as an objectivity and then to the impredicative character of the flux in-itself as an absolute subjectivity. As these phenomenologically grounded reductions of purely mathematical questions might seem at first sight as a bit far-fetched I could offer at this stage as a corroborative argument the need to generally assume some kind of pre-given infinite continuous substratum in explicit or implicit fashion in proving mathematical statements of a higher order than those involving at most countable infinity. Why this kind of assumption on the analytical-mathematical level can be taken as a reflexion of corresponding constitutional processes of a phenomenological character as those mentioned above will be made clearer in the development of my arguments in later sections. There, I review on what terms the impossibility to capture phenomenological Continuum ontologically is reflected in the impredicative character of **MC** and I also review the role of an underlying assumption of actual infinity[4] in deciding a well-known continuum statement i.e., the *Continuum Hypothesis* (**CH**) which, in fact,

consciousness to the object of its orientation. As intentional forms of the transversal intentionality of the constituting (absolute) flux of consciousness can be regarded the retention and protention attached to any original impression whereas by longitudinal intentionality we can roughly understand the constitution of the descending sequence of retentions of each original impression as a continuous whole (see Husserl (1996)).

[4] In the present context the term actual infinity is taken as a Cantorian-type infinity i.e., a pre-existing, indefinitely extending uncountable infinity.

has been proved to be independent from the other axioms of commonly acceptable Zermelo-Fraenkel & AC Set Theory (see Kunen (1982)). Taking into account this particular conjecture about the cardinality of Continuum I look into how some other actual infinity assumption such as the *Axiom of Choice* (**AC**) - proved to be also independent from **ZF** - is applied to prove the consistency of both *Continuum Hypothesis* and its negation with the other axioms of **ZFC** theory.

In Subsection 3.2, I have attempted to provide an interpretation of the undecidability of **CH** and **AC** inside **ZF** theory as a general consequence of Gödel's First Incompletess Theorem, taking into account an elusive notion of actual infinity 'creeping' in the proof of non-recursiveness of the set of theorems deduced from a consistent and recursive set of axioms extending **ZF**. In fact, the derivation of incompleteness of formal systems with at least the expressive power of formal arithmetic is partly deduced here as an effect of the non-rigorous finitistic character of the metatheoretical objects taken as formal objects inside a consistent formal theory.

My general conclusion is that there is no way to define Mathematical Continuum (**MC**) by first-order means without producing evident circularities in definition and there is no way to settle questions dealing with the cardinality or properties of **MC** without some form of *ad hoc* actual infinity assumption applied in the process owing, in my view, to the fundamentally non-analytical character of **MC** reducing on a level of phenomenological constitution to the transcendental root of each subject's continuous flux of temporal consciousness whose 'effect' on the logical-analytical level is the inherent impredicativity of the notion of continuity.

2. COULD MATHEMATICAL CONTINUUM BE REDUCED TO A PHENOMENOLOGICAL CONTINUUM?

At this point the reader might rightfully wonder what all this stuff about the temporal ego of phenomenological analysis taken as an ever in-act subjectivity of consciousness has to do with the notion of Continuum in a formal mathematical theory or anyway with any sort of mathematical activity. My answer, in the first place, to this reasonable objection is again the claim put forth in the beginning of Introduction; the acceptance of an argumentation of this kind depends, in principle, on the general philosophical stance of someone doing mathematics or rather foundational mathematics. If someone finds mathematics as basically a formal abstraction of certain functions of human mind then he might be willing to accept my discussion as meaningful and my clues as making some headway towards a deeper understanding of Mathematical Continuum.

As a matter of fact, E. Husserl provided a common ground underlying perceptual objects on one hand and mathematical objects on the other by means of categorial intuition; the latter as a special kind of intuition can lead by way of abstraction from perceptual evidences to categorial essences and in complete abstraction to mathematical entities. As it will be noted below, purely formal objects of mathematics as eidetic objects are not thought to be obtained by eidetic variation on material essences reducing thus to a possibility of common content but they are the result of complete abstraction eliminating all traits relative to a material content which is then taken to be a contingent material fulfillment. A counterargument to this Husserlian view is put up by those who insist on a fundamental difference between mathematical and perceptual intuition on the grounds that while perceptual objects are determinate and individually identifiable, that seems to be what is missing in the case of mathematical objects e. g. in the mathematical intuition of the symbol $\emptyset$ standing for the empty set (Tieszen (1984, pp. 399-400)). This counterargument can be, in the first place, easily overrun as it might be taken to refer to the empty set as a convention of formal language as much as the term absolute vacuum refers by all accounts to a conventional term within the theory of quantum physics.

But there is more to it and goes deeper into the close character of perceptual and mathematical objects on the level of intentionality. Husserl, in fact, distinguished them on the intentional level as things-substrates (*sachhaltige Substraten*) referring to perceptual objects in a sense of appropriation of some kind of corporeity and empty-substrates (*Leersubstraten*) referring to 'empty somethings' together with their corresponding syntactical objectivities. The latter class of intentional objects as 'state-of-things' (*Sachverhalte*) with all categorial objectivities grounded on them constitute the class of objects of Pure Logic as *Mathesis Universalis* e.g. syntactical elements of set-theoretical formulas, numerals, functions in their well-defined Euclidean or non-Euclidean domains, etc. (see Husserl (1995, p. 33)). For instance, taking the reading of a pointer registering the measurement of a quantum experiment as a perceptual sign, that sign regardless of its particular material content has the mode of being a 'sign-as such' and thus as an intentional object (of noetical perception) it can be called a '*state-of-things*', in other words an 'empty something' which in subsequent noematical constitution can be, as a well-defined object, assigned a unique mathematical valuation.[5] This subtle distinction between perceptual and mathematical objects can be nevertheless understood on the phenomenological-intentional level as a coherence factor in taking both of them as ultimately pure objects of intentionality irrespective of any possible material content. It is noteworthy, that K. Gödel in a supplement to his well-known paper '*What is Cantor's Continuum problem*?' figured out that there is something more than just through sensations or combinations of sensations that perceptual objects are given to us; he insisted, in particular, on 'the idea of the object itself [...] Evidently, the "given" underlying mathematics is closely

[5] To come back to the instance of null set (ø) which presumably formalizes 'ontological' nothingness, we can assert that since it presents itself in original givenness in front of the intentionality of consciousness it will be absurd to call it nothingness in phenomenological sense; for, as an original givenness it has already become a concrete fulfillment in time.

related to the abstract elements contained in our empirical ideas' (Føllesdal (1999, p. 398)).

His point was, contrary to Kant's assertions, that if these abstract elements do not follow by some kind of action of things upon our sense organs they are nevertheless not purely subjective but they must represent some other kind of relationship between ourselves and reality. D. Føllesdal, in Føllesdal (1999), takes these abstract features and primarily Gödel'semphasis on the idea of the object itself as linked to a notion of individuation of objects leading to the notions of identity and distinctness and consequently to the act of counting that makes them representable as principal mathematical elements (Føllesdal (1999, p. 399)). Given my aforementioned interpretation of the syntactical objects of Pure or Mathematical Logic as fundamentally intentional objects, this kind of individuation can be thought of as rooted to their noetical perception in the form of irreducibly individual objects of intentionality. Their individuation is then grounded on their very 'being' as solely original givenenesses of the intentionality of primary experience irreducible to anything more fundamental.

At this stage, based on this intentionality-motivated approach to perceptual and mathematical objects, we can be led to review specific mathematical objects, such as the terms of choice sequences[6] and the mathematical Continuum through a kind of synthesis of the two-ity intuition and the Primordial Intuition of Intuitionistic Mathematics. This seems to be purposeful as the intuitionistic approach to the Continuum is

[6] Choice sequences in intuitionistic context as sequences in time progression were originally taken as a means to represent points of the intuitive Continuum and are generally divided to lawlike and lawless ones.What basically distinguishes a lawlike sequence from a lawless one is that the former even when not given by a prescribed formula, it is a determinate one and thus has a fixed horizon whereas the latter is fully or partially indeterminate and one whose horizon is not fixed in advance. In the unfolding of a lawless sequence except for the obvious specification of the uniqueness of the value of each term anything can occur in its progression.

not only to a large extent modelized, at least in Brouwerian writings, after the phenomenology of temporal consciousness but also because it is exactly this approach that puts under a new point of view the fundamental difference between the two-ity intuition and the Primordial Intuition of Mathematics, the latter being roughly an intuition of Continuum.

L.E.J Brouwer in his early formation (Ph.D Thesis, 1907) and later made several comments on the intuitive Continuum that can be seen to be pretty much based on the description of temporal consciousness by Husserl. This is also the case with H. Weyl working independently and elaborating his ideas in the monograph *Das Kontinuum* (1918) (see Van Atten *et al* (2002)). I do not intend here to enter into great details in describing Brouwer's or Weyl's analysis of intuitive and subsequently mathematical Continuum put up in accordance with phenomenological principles [Van Atten *et al* offer a fine exposition on the matter in Van Atten *et al* (2002)]; I'll rather draw attention to the radical difference between the First Act of Intuitionism (two-ity intuition) and the Primordial Intuition of Mathematics (intuitive Continuum) and next to a phenomenological dimension of the graph-extensional version of the *Weak Continuity Principle* presented in Van Dalen *et al* (2002).

It is plausible to trace L.E.J Brouwer's indirect reference to the phenomenological notion of the absolute flux of consciousness or (indirectly) even deeper to its subjective ego in his discussion of the intuitive Continuum as the Primordial Intuition of mathematics: 'The substratum, divested of all quality, of any perception of change, a unity of continuity and discreteness, a possibility of thinking together several entities, connected by a "between", which is never exhausted by the insertion of new entities' (Van Atten *et al* (2002, p. 205)). The intuitive Continuum is described as a substratum in which continuity and discreteness occur as inseparable complements where it is impossible to construe one of them as a primitive entity without implicating the other in the same primitive sense.

The intuition of discreteness on the other hand, called two-ity, is the empty form of all intuitions of distinct intentional objects and can provide the basis of the discrete aspect of mathematical construction; by means of two-ity intuition we can generate natural number sequences and also any finite combinatorial objects generated from natural numbers (Van Atten *et al* (2002, p. 206)). As a matter of fact, Brouwer's two-ity can be largely interpreted by means of the transversal intentionality of the flux of consciousness (*Querintentionalität*) in the scheme original impression-retention-protention, the two latter terms meant as aprioric intentionalities respectively towards past and future (see Husserl (1996, pp. 44 & 71)). In this view, natural numbers are taken by two-ity intuition as durationless points in abstraction whereas the same cannot be claimed about real numbers considered as incomplete objects. Yet, in the flow of inner time we are not aware of any durationless now-point as there is no 'autonomous' present in original impression but a specious present composed intentionally of original impression–retention–protention. We could possibly, though, approach a durationless point in intuitive Continuum by an infinite sequence of nested rational intervals the lengths of which converge to 0. This possibility, called the 'Second act of Intuitionism', allows a modelization of intuitive Continuum on the basis of the generation of freely proceeding convergent sequences where each real number as an ideally durationless point is characterized as the species of such non-lawlike sequences. To be close to his view of real numbers as incomplete objects, L.E.J. Brouwer regarded lawless sequences, in the sense of indefinitely proceedable sequences, as better representing intuitive Continuum, e.g. a point P (representing a real number) to which a freely proceeding sequence of rational nested λ-intervals $\lambda_{n_1 \ 1}$, $\lambda_{n \ 2}$, … ,$\lambda_{n \ m}$,.. of the general form

$$[\ \frac{a}{2^{n-1}},\ \frac{a+1}{2^{n-1}}\ ]$$

converges is defined as the sequence itself and not something as a limiting point of the sequence (Van Atten *et al* (2002, p. 212)).

Nevertheless both Brouwer and Weyl handled choice sequences in the logical formulas of Continuity Principles as complete and determinate objects in infinite projection so as to provide an intuitionistic foundation for real analysis. In that sense, intuitionistic continuity principles such as the Weak Continuity Principle (**WC-N**) are, in principle, black box principles extending *ad infinitum*, at least in the unrestricted case, the horizon of finitely many intentional acts of a generating subject towards the vagueness of infinity; the latter idea presupposes the existence of an impredicative spatiotemporal substratum in the sense of the Primordial Intuition of Mathematics.

In Van Dalen *et al* (2002) the main motivation for providing a a graph-extensional version (**GWC-N**) of the Weak Continuity Principle was to extend Brouwer's Weak Continuity Principle (**WC-N**)[7] to all kinds of choice sequences (lawlike and lawless ones) considering any kind of restriction on the generation of the unfolding terms of the sequence as stemming from a noetical-noematical correlation of the intentional acts of a free generating subject with the intensional properties of the sequence in question.

Such restrictions, which in any case accept the existing initial segment of the choice sequence could be definitive "*From now on restriction* $P^i j$ *holds and will not be revised any more*" or provisional "*For an unspecified number of stages restriction* $P^i j$ *holds*", Van Dalen *et al* (2002, pp. 335-36). Formally, there is no difference except for one in the following Continuity Principle for numbers (**WC-N**) between Van Dalen *et al* and L.E.J Brouwer.

---

[7] The fundamental continuity principle (WC-N) has as a direct consequence the well-known intuitionistic theorem that all full functions are continuous and thus the Continuum is unsplittable.

$$\forall \alpha \, \exists x \, A(\alpha, x) \Rightarrow \forall \alpha \, \exists m \, \exists x \, \forall \beta \, [\overline{\beta} m = \overline{\alpha} m \rightarrow A(\beta, x)]$$

where α, β range over sequences of natural numbers, *m*, *x* over natural numbers and $\overline{\alpha} m$ stands for $< \alpha (0), ..., \alpha (m\text{-}1) >$, i.e. the initial segment of α of length *m*. The difference, in question, is that in **GWC-N** the predicate $A(\alpha, x)$ is stronger than extensional in the classical definition of the term; it is graph-extensional which means that the choice sequence α enters $A(\alpha, x)$ only by its values. In such a case what would be the role of the graph-extensionality of $A(\alpha, x)$ in **GWC-N** so as to include the widest possible range of choice sequences and what does it mean from a phenomenologically oriented view?

I propose to give the following interpretation in accordance with my general approach. Any two-ity intuition can be linked to a noetical-noematical[8] type generation of a sequence of natural numbers where these numbers are registered only as such, that is, by their distinct values as signs-in themselves. Thus **GWC-N** Principle is, by this measure, a valid continuity principle as any restrictions on the part of the sequence generating subject acting by noetical perception should be 'first-order' restrictions. Such restrictions, on a phenomenological level, might be solely considered the retention in consciousness of each original impression in actuality, that is, of each new term of the choice sequence together with the retention of a collection of the terms generated thus far as a whole, that is, of the initial segment of the sequence. Evidently, the latter retention refers to what Husserl called longitudinal intentionality (*Längstintentionnalität*) which bears already the creeping transcendence of Continuum by being the continuous unity of a whole.

[8] The phenomenological terms noetical and noematical are associated by Husserl in *Ideas I* with certain modes of intentionality, the former referring to an intentionally perceived object as solely a givenness of intentionality, the latter referring to its constitution 'thereafter' as a temporal object within the homogeneous unity of consciousness (Husserl (1995, pp. 230-231)).

Consequently, in this approach, **GWC-N** eliminates by graph-extensionality of predicate A(α ,*x*) any higher order restrictions on the generation of the terms of the sequence. Such restrictions, e.g. a provisional restriction of the type '*from now on, the choice sequence* α *is constant*' cannot be characterized as being of an intentional character. It is noteworthy that Van Dalen *et al* have proved (an alternative proof is given by A. Visser) that the original version of the Continuity Principle **WC-N** does not hold in general for extensional predicates precisely by producing a higher order restriction in the process of a strictly numerical unfolding of the terms of a choice sequence (Van Dalen *et al* (2002, pp. 340-41)).

3. TRACING THE IMPREDICATIVITY OF PHENOMENOLOGICAL CONTINUUM IN FORMAL THEORY

3.1. The notion of constitution and its role in the independence of CH

In intuitionistic theory the notion of continuity in real analysis is basically founded on continuity principles such as those already mentioned or on alternative versions of them (e.g. Brouwer's Universal Spread Law). In turn, these principles are conditioned on phenomenologically motivated assumptions such as the Primordial Intuition of Mathematics and the First act of Intuitionism.

On the other hand, in Cantorian **ZF** theory Continuum is basically introduced by the application of two axioms: the Replacement Axiom and the Power Set Axiom. There is a fundamental difference between the two; the first one defines a new set by means of a functional predicate, the second is a qualitatively different axiomatical tool generating a richer set $\wp$ (X) with a cardinality greater than that of its base set *X*; in the case of a countably infinite collection it gives rise to the extremely rich set *C* whose cardinality is defined to be the cardinality of Continuum. As P. Cohen put it, 'it is unreasonable to expect that any description of a larger cardinal which attempts to build up that cardinal from ideas deriving from the Replacement Axiom can ever reach *C*'. He was referring, of course, to the set $\aleph_1$ of all countable ordinals but also to any cardinals such as $\aleph_1$, $\aleph_\omega$, $\aleph_\alpha$, ... where $\alpha = \aleph_\omega$ produced by a piecemeal process of construction starting from $\aleph_0$ and applying at each stage the Replacement Axiom. In that case *C* would be greater than each of these cardinals and the Continuum Hypothesis would be obvious false something left to future generations to decide perhaps by seeing more clearly to the problem (Cohen (1966, p. 151)).

In my view, the 'asymmetric' character of the Power Set Axiom with respect to the other axioms of **ZF** owes much to the radical difference between two fundamental intuitions. The process of an enumeration *ad infinitum* which can be thought of as some form of intentional 'act' close to the meaning of two-ity intuition in intuitionistic theory and the process of forming subsets of infinite enumerations as objective wholes including all their elements at once. The latter seems in first view linked to L.E.J. Brouwer's intuition of Continuum as an impredicative substratum deprived of any quality (Primordial Intuition of Mathematics). On a deeper level we may reach a condition of mental constitution that should be a rather temporal constitution to ground the passing from the level of noetical perceptions of zero-level elements of sets taken as immaterial signs-in themselves to subsets forming instantly from them as objectivities in temporal unity. This could lead on a phenomenological level to first admitting certain intentionalities of the absolute flux of consciousness conditioning the immediate retention in the flux of the original presence of each zero-level element as an 'empty-something' and second the retention of any such aggregate of 'somethings' as a homogenous whole by means of longitudinal intentionality. At each stage of temporal reduction we have to turn into impredicative, that is, non-analytical forms of the flux to ground the objective unity of any aggregate of intentional objects in consciousness, in the particular case of any set (or class of sets) of elements of categorial formulas. In a next stage we are led to a pure atemporal transcendence which 'is' the absolute subjectivity behind continuous unity taken as an objective whole in temporal constitution.[9] In mathematical context it should be taken as the subjectivity behind an otherwise impredicative continuous (temporal) substratum on which may be constituted well-defined mathematical objects and further collections of such objects, classes of collections of such objects and so on *ad infinitum* in the form of objective wholes at once in original givenness.

Evidently, we cannot describe by analytical means inside any formal system what is by its nature non-analytical and ever 'in-act' for it can be never objectivated as it 'is' the absolute ego of any conceivable temporal unity. In this respect, there are lately quite a few mathematicians mostly working in mathematical foundations who have doubts about the possibility of defining Continuum as a set and who admit of at least a non-analytical character of the question of cardinality of Continuum, among them S. Feferman, who insists that **CH**[10] is an inherently vague statement that cannot be settled by any new axioms added to **ZFC** theory (Feferman (1999)). As **CH** is generally considered more than any other relevant conjecture linked to the nature of Mathematical Continuum and as its independence is proved to underlie the independence of other infinity statements within **ZFC** (e.g., Suslin's Hypothesis, the question of the product of any two **c.c.c.** spaces) I

[9] The recourse to an absolute subjectivity of a transcendental type to ground the objective unity of temporal consciousness and of its immanent objects is the common denominator of the existentialist trends mentioned in Introduction and the Husserlian view of temporal constitution. Of course the relevance of respective approaches to mathematical objects in general including the notion of Mathematical Continuum presupposes a view of mathematical objects as products of a categorial intuition based in turn on their constitution as durating well-defined objects in temporal consciousness.

[10] The Continuum Hypothesis (CH) roughly assumes that the number of subsets of a set of the power of countable infinity $\aleph_0$ is the power of continuum C, i.e. $2^{\aleph_0} = C$.

intend to show the need of assuming a notion of (temporal) constitution in the following cases: **1**) in the classical proof of the fundamental result $X \prec \wp(X)$ for any set X and **2**) in the assumption of the *Axiom of Choice* as an actual infinity axiom in the proof of the consistency of both **CH** and $\neg$**CH** within **ZFC**.

It is well-known that the Cantor theorem stating that for any non-empty set X the cardinality of $\wp(X)$ is greater then the cardinality of X ($X \prec \wp(X)$, where $\wp(X)$ is the power set of X) gives a positive answer to the question whether there are any infinite cardinalities greater than Continuum. In a phenomenologically oriented approach its proof can be reduced to a condition of temporal constitution irrespective of the order of cardinalities involved; in this point of view any higher order cardinality than that of countable infinity cardinal $\aleph_0$ is already an idealization of a second level which is not based on possible experience, taken as an idealization of the first level the definition of the set of natural numbers **N** by admitting to the possibility of an indefinitely open horizon of reiterating intentional acts of the form 'I *can do*' to produce each time a new $n \cup \{n\}$ (see Lohmar (2002)).

The classical proof consists in the construction of a 1-1 mapping from X into $\wp(X)$ which cannot be onto. Let us assume that there is such a mapping f corresponding each element x of X to the singleton $\{x\}$. Obviously it is 1-1. We then define a subset A of X in terms of $A = \{x;\ x \notin f(x)\}$. Since f is assumed to be onto $\wp(X)$ there must be $a \in X$ such that f $(a) = A$. Then a has two possibilities: to be either in A or in X $\setminus$ A. If it is in A then $a \notin$ f(a) = A, so we get $a \in A \rightarrow a \notin A$; this is impossible so a must be in X $\setminus$ A. Then $a \notin A$ so $a \in f(a)$ = A. Again we get the contradictory $a \notin A \rightarrow a \in A$, so the mapping f cannot be onto.

Although this theorem sets an upward scale of infinite cardinalities by taking each time as the new set X the class of all subsets of the previous set, the underlying mental process in the argumentation of each proof is essentially the same; the element a is successively taken in two fundamentally different levels of phenomenological perception. First, it is taken as an element of the set X this set conceived of as the general 'environment' of *a* bearing no influence on the individuality of *a*; in this sense the element *a* can be taken as an abstraction of a noetical perception directed intentionally to *a* as an'empty-something' (*Sachverhalt*). In the following stage of the proof the element *a* is taken as bearing a double nature, that of an individual-in itself and that of an object-element of an aggregate of other objects-elements of X satisfying a primitive property of inclusion expressible in formal language by the undefined predicate $\in$. At this stage we have to presuppose the constitution of *a* as a well-defined object and the simultaneous constitution of an indefinite collection of other elements x such that $x \notin f(x)$ at once in continuous temporal unity; this should correspond to a state in which *a* is noematically constituted in the temporal flux within a retentional constitution of an indefinite aggregate of other noematical objects of the same kind taken as original givennesses in the present now of the intentionality of consciousness. At the stage we have formed a subset $A = \{x;\ x \notin f(x)\}$ to which the element *a* may or may not belong we have to implicitly assume a retentional unity of an indefinite aggregate of elements x formed as a complete whole of immanent objects in the progression of the absolute flux of consciousness; there is an already 'creeping' impredicative continuity here, independently of the order of the infinite cardinality of the set X, owing to the impredicative character of the longitudinal intentionality of the flux.[11] In my view the retentional

---

[11] My argumentation here, on a constitutional level, for Cantor's result $X \prec \wp(X)$ seems to be almost in perfect consonance with D. Lohmar's view of the same question, namely of the fundamentally different character of the concept of an element as a categorial intuition and the act of constituting a collection as the complete series of unifications of its elements which is then an object of a radically different kind from the elements of this set (see Lohmar (2002, p. 238)). This goes as far as questioning the lawfullness of assuming the existence of the set of all sets since such an idealization in the sense of a complete series of unifications in (phenomenological) constitution is taken then as an element of itself.

forms of the absolute flux of consciousness (longitudinal and transversal intentionality) could possibly account on a constitutional level for probably clinging to P. Cohen's view that 'it is unreasonable to expect that any description of a larger cardinal which attempts to build up that cardinal from ideas deriving from the Replacement Axiom can ever reach C'.

I pass now to the implicit assumption of a notion of actual infinity and consequently, on a phenomenological level, of an impredicative temporal substratum in the application of the *Axiom of Choice* (**AC**) in the proof of the independence of **CH** within **ZFC**.

It is well-known that one of the common forms of **AC** states that given a non-empty class of non-empty sets $\{Xi\}$, $i \in I$, we can choose exactly one element from each set in the class to form the non-empty product $\prod_{i\in I} Xi$. It is also known that this axiom, characterized as an actual infinity axiom by certain non-standard set theorists in the sense of being conditioned on a pre-existing actual infinity of a Cantorian type, is applied to produce the independence of *Continuum Hypothesis*; as a matter of fact it is at least indirectly presupposed in Gödel's proof of the consistency of **CH** with the axioms of **ZFC** and also in P. Cohen's proof (by the forcing method) of the consistency also of $\neg$**CH** with the axioms of **ZFC**. In the rest of this subsection I'll try to sketch a phenomenologically oriented interpretation of **AC** and then show its implicit role in the 'proof' of $\neg$**CH** as it gives a strong clue to the assertion that no matter what model we are working in (e.g. a countable transitive base model M in the theory of forcing) we have to assume some actual infinity principle to prove any conjecture about uncountable infinity.

In this approach the basic underlying intuition of **AC** can be summed up as extending, in principle, over an indefinite horizon the right to 'observe' and manipulate individuals as such and at least relatively to any collection of them. On a constitutional level, we may say that as a particular individual-as such is in original givenness the object of intentionality at the lowest level of phenomenological (noetical) perception there exists a notion of ordering by the sole virtue of the intentional 'property'of the object in question to bear an outer horizon, i.e. that part of the Life-World[12] which is not the object or parts of the object, thereby defining in complementary sense the field of next potential 'observations'. Consequently a notion of well-ordering is induced by representing the intentional individual as a noematical object possibly belonging to an aggregate of other such objects in the constituting flux of consciousness. Now, what is left after discarding all other details of constitution is the possibility to 'observe' intentionally individuals at least as 'signs-in themselves' (probably with no material content) and the protentions of the intentionality towards them as aprioric potentialities defining an ever receding complementary domain of 'observations'. In addition, the *Axiom of Choice* should be conditioned on the notion of a pre-existing actual infinity which might be conceived in terms of constitution as an objective and invariably existing temporal substratum on which to 'embed' intentional individuals as noematical objects in continuous unity.

In conclusion, **AC** can be interpreted as founded first on the existence of a subject performing acts of intentional character ideally *ad infinitum* and second on the existence of a constituting flux of consciousness of the subject in question, in the continuous unity of which any noetically perceived object can be constituted as a temporal, well-defined and uniquely determined (in varying predicative situations) noematical object; any such object in appropriate predicative form can be defined both as an object-in-itself and in some kind of relation of order to any other. Grounded then as a well-defined noematical object it can be further defined as a formal-ontological object provided with proper sense

[12] The World-for us or Life-World (Lebenswelt) in Husserlian terminology can be roughly described to a non-phenomenologist as the physical world with its ever receding horizon including in intersubjective sense all knowing subjects in a special kind of presence in the World. More on this in E. Husserl's *The Crisis of European Sciences and Transcendental Phenomenology* (Husserl (1970)).

mathematical properties e.g. extensionality, a notion of order in strictly formal sense, etc.

Regarding the implicit role of **AC** in the proof of consistency of $\neg$**CH** with the other axioms of **ZFC** it might seem pointless in the present context to offer a detailed exposition of the fundamentals of P. Cohen's theory of forcing as it would be certainly lengthy, possibly difficult to comprehend while at the same time not absolutely necessary in reaching my point.[13] Instead, I'll try to be as explicative as possible in the presentation of my arguments.

Generally, in forcing techniques we rely on global properties forced to objects, such as to a function f_G of a forcing model **M**[G], by means of a *P*-generic set G over a countable transitive base model M, where *P* is a poset of forcing conditions $(P, \leq)$ in **M**. The *P*–generic set G$\in$**M**[G] can be defined to have the property of a special filter to force compatible extensions of any condition *p* over **M** and is moreover very generic in the sense of having non-empty intersection with any dense set of conditions *p* over **M**. We are going to see that the generic properties of G can lead to contradictions in case the poset $(P, \leq)$ of conditions has not **c.c.c.**[14]

[13] For a detailed exposition of forcing theory the reader may consult P. Cohen's original *Set Theory and the Continuum Hypothesis* (Cohen (1966)), K. Kunen's *Set Theory. An Introduction to Independence Proofs* (Kunen (1982)) or F. Drake's, D. Singh's *Intermediate Set Theory* (Drake & Singh (1996)).

[14] A partial order $(P, \leq)$ has the Countable Chain Condition (c.c.c.) iff every antichain (any family of pairwise incompatible elements) of the poset P is countable. Letting $P \neq \emptyset$, the elements p, q $\in$ P are defined as compatible if:

$$(\text{for } p, q \in P)\ \exists r \in P\ (r \prec p \wedge r \prec q)$$

that is, r extends both p and q in the usual intuition of extension. For example, if p, q are finite partial functions from ω to 2 and $p \prec q$ iff $q \subset p$, then p and q are compatible iff they agree on dom(p)$\cap$dom(q) in which case $p \cup q$ is a common extension of p and q.

The **c.c.c** condition is a necessary constraint to be satisfied by a set of conditions P of **M** in the process of proof of the consistency of $\neg$**CH** as it preserves cardinalities between the base model **M** and the extended model **M**[G] (Kunen (1982, p. 207)). In the proof of consistency of $\neg$**CH** with **ZFC** it is possible, based on the **Δ**-system Lemma,[15] to define a proper set of conditions, namely the set of finite partial functions Fn($\kappa \times \omega$, ***2***) from $\kappa \times \omega$ into ***2*** (κ an uncountable cardinal of **M** and ω the cardinal of countable infinity) satisfying **c.c.c**. It turns out, though, that the proof of **Δ**-system Lemma for Fn($\kappa \times \omega$, 2) needs the **AC**.

The necessity of the **c.c.c.** constraint for the poset Fn($\kappa \times \omega$, 2) may now become clear as it reduces uncountably infinite possibilities in the domain of conditions $p \in$ Fn($\kappa \times \omega$, 2) to countably many compatible extensions of these conditions. Let us keep in mind that forcing a compatible extension h for each condition p in **M** and this way *ad infinitum* can be taken as fundamentally an intentional act, let's say of a higher order than that of an irreducible intentionality to an individual-as such, and it can thus be abstracted as an act in discrete mode of a performing subject in time progression. In the mathematical context we discuss, these acts should correspond to an open-ended class of countably many compatible extensions. This is, in fact, ensured by the **c.c.c**. condition which, in the case of the proof of consistency of $\neg$**CH**, is dependent on the application of **Δ**-system Lemma mentioned above and consequently on **AC**.

I'll spare now some space just to show the key role of **c.c.c**. condition in the proof of consistency of $\neg$**CH** by citing an example [offered in Kunen (1982, p.55)] that demonstrates how the non-existence

---

[15] A family K of sets is called a Δ-system iff there exists a fixed (finite) set r, called the root of the Δ-system, such that a $\bigcap$ b = r whenever a, b are distinct members of K. The Δ-system Lemma states that if A is any uncountable family of finite sets, there is an uncountable B⊆A which forms a Δ-system (see Kunen (1982, p. 49)).

of the **c.c.c.** property of $(P, \leq)$ can lead to inconsistencies. This will also help to better clarify my approach to the structure of the proof:

Let $(P, \leq)$ be the set of finite partial functions p from ω to 2, i.e. $P = \{p\colon p \subset \omega \times 2, \quad |p| < \omega\}$ and $p \leq q$ iff $q \subset p$ as functions. If G is a filter in P then the elements of G are pairwise compatible by its property of being a filter; thus, if we define $f_G = \bigcup G$ then f_G is a function with $\mathrm{dom}(f_G) \subset \omega$. How can we conceive of f_G as being truly generic? In that case we have to apply the statement *MA(k)* which is part of Martin's Axiom **MA** and guarantees a *P*-generic set G for any non-empty **c.c.c.** partial order $(P, \leq)$.[16] Let for $n \in \omega$ $D_n = \{p \in P;\ n \in \mathrm{dom}(p)\}$. As any $p \in P$ can be extended to a compatible condition with n in its domain, D_n is a dense family and by statement *MA(k)*, $\forall n \in \omega\, (G \bigcap D_n \neq \emptyset)$. Then obviously the domain of f_G is all of ω, that is, $\mathrm{dom}(f_G) = \omega$. So, we can build a fairly generic function $f_G = \bigcup G$ by relying at first on the properties of the filter G to define the function f_G and next by forming countably many compatible extensions of finite partial functions p by adjoining to any dom(p) a finite subset of ω so that the class D_n is dense. Then, relying on the **c.c.c.** property of the particular set of forcing conditions $(P, \leq)$, we apply *MA(k)* to get a generic G intersecting all such dense sets.

The situation can become complicated, though, in case uncountable cardinals are involved and $(P, \leq)$ does not have the **c.c.c.** property, as in the case where the set of conditions is $P = \{p : p \subset \omega \times \omega_1 \wedge |p| < \omega\}$ where ω_1 is the first uncountable ordinal. As above we can easily see that there exists a generic function $f_G = \bigcup G$ with $\mathrm{dom}(f_G) \subset \omega$ and $\mathrm{ran}(f_G) \subset \omega_1$. For $\alpha < \omega_1$ let $D_\alpha = \{p \in P; \alpha \in \mathrm{ran}(p)\}$; then D_α is straightforward checked to be dense as we can always produce a compatible extension of $p \in P$ with $\alpha \in \mathrm{ran}(p)$. If there existed a generic set G intersecting each D_α for all $\alpha < \omega_1$ we

[16] MA(k) is the statement: Whenever $(P, \leq)$ is a non-empty c.c.c. partial order and $\mathfrak{D}$ is a family of $\leq_k$ dense subsets of P , then there is a filter G in P such that $\forall D \in \mathfrak{D}\ (G \bigcap D \neq \emptyset)$. **MA** is the statement $\forall k < 2^{\omega}$ (M A(k)).

would have that ran $f_G = \omega_1$ whereas dom $f_G \subset \omega$ which is obviously impossible if f_G is to be a function. But now $(P, \leq)$ lacks the **c.c.c.** property as it can have an uncountable class of pairwise incompatible conditions such as the class $\{< 0, \alpha >\}$ for $\alpha \in \omega_1$.

This is, actually, what **c.c.c.** condition is all about in this context; it eliminates an uncountable number of incompatible conditions *p* meaning that even in the presence of uncountable numbers in the range of such conditions we can nevertheless proceed with countably many compatible extensions of them and this way by applying statement *MA(k)* define a fairly generic set G. In a more intuitive nuance the **c.c.c**. property opens up the possibility to 'suppress' a uncountable infinity factor underlying the 'field' of definition of forcing conditions p_α in view of an operation of forming (in countably many steps) compatible and consistent extensions of p_α something that could, in principle, be linked to the discrete mode of acts of an intentionally performing subject.

As the **c.c.c.** condition satisfied, in general, by finite partial functions Fn*(I, J)* of a countable range *J* depends mainly on the **Δ**-system property of Fn*(I, J)* and as, in turn, the **Δ**-system property depends implicitly on the *Axiom of Choice*, I think this should be taken as a strong indication of the implicit need to turn to some form of actual infinity principle (i.e., the **AC**) to prove in an essentially countable 'operational'context a statement involving uncountable cardinalities such as the consistency of $\neg$**CH** within **ZFC**.

3.2. How to interpret the undecidability of infinity statements inside ZF theory?

A key step in proving the incompleteness of a recursive consistent extension **T** of **ZF** is to prove that the set of theorems of the extension **T** is not recursive (see Kunen (1982, §14)). In that theorem the notion of a recursive (or decidable) set plays a major role by the following Theorem **4.1** which represents recursive sets e.g. the set of natural numbers or the set of finite sequences by means of formulas of **ZF**. A recursive set might be indirectly linked to the discrete way an 'observer' applies his intentional 'observation' towards a recursively enumerable collection possibly by the intermediary of a digital device.

4.1 Theorem. *Given any recursive set* R *of natural numbers there is a formula* χ_R *(x) which represents* R *in the sense that for all n,*

$$n \in R \rightarrow (\mathbf{ZF} \vdash \chi_R (\ulcorner n \urcorner) \text{ and } n \notin R\, (\mathbf{ZF} \vdash \neg\ \chi_R (\ulcorner n \urcorner)).$$

Recursive sets of finite sequences and recursive predicates in several variables are likewise representable. (see Kunen (1982, p. 40)).

Let it be noted here that this theorem is proved in metatheory, that is, in a language referring to finitistic metatheoretical objects by means of natural numbers in the place of symbols introduced in an extension of **ZF** by definitions. Now, I prove that the set of theorems deduced from a consistent and recursive set of axioms **T** extending (the recursive set of axioms) **ZF** is not itself recursive.

4.2 Theorem. *Let* **T** *be any consistent set of axioms extending* **ZF**. *Then the set of theorems* $\{y; \mathbf{T} \vdash y\}$ *is not recursive.*

Proof: If it were recursive, then by Theorem **4.1**, there would be a formula χ (x) of **ZF** such that for any $y \in \{y; \mathbf{T} \vdash y\}$:

$$(\mathbf{T} \vdash y) \rightarrow (\mathbf{ZF} \vdash \chi\, (\ulcorner y \urcorner)).$$

And for any $y \notin \{y; \mathbf{T} \vdash y\}$:

$$(\mathbf{T} \nvdash y) \rightarrow (\mathbf{ZF} \vdash \neg \chi (\ulcorner y \urcorner)).$$[17]

Now we fix y (by Theorem **4.3**, p. 348) such that: $\mathbf{ZF} \vdash y \leftrightarrow \neg \chi (\ulcorner y \urcorner))$ (**1**). But this means that $\mathbf{T} \nvdash y$ (**2**) for obviously y would not belong to the set $\{y; \mathbf{T} \vdash y\}$ since within **ZF** it is logically equivalent to $\neg \chi (\ulcorner y \urcorner))$. Also by (**1**): $\mathbf{ZF} \vdash y$ since $\mathbf{ZF} \vdash y \leftrightarrow \neg \chi (\ulcorner y \urcorner)$. But then $\mathbf{T} \vdash y$ (**3**) as **T** is taken to be a consistent extension of **ZF**. By (**2**) and (**3**) we have that $\mathbf{T} \nvdash y$ and $\mathbf{T} \vdash y$ which means that **T** is inconsistent, a contradiction ◊

It is straightforward to see how this result fits in the proof of Gödel's First Incompleteness Theorem, namely that if **T** is a recursive consistent extension of **ZF** then it is incomplete in the sense that there is a sentence φ such that $\mathbf{T} \vdash \varphi$ and $\mathbf{T} \nvdash \varphi$. The simple proof in Kunen (1982) is as follows:

If there were no such φ, then for every φ either $\mathbf{T} \vdash \varphi$ or $\mathbf{T} \vdash \neg \varphi$ and, assuming **T** consistent, these cannot both be valid. Then we could decide whether $T \vdash \varphi$ by programming a computer to start listing all formal deductions from **T** and stop when a deduction of φ or $\neg \varphi$ has been found. But this is conditioned on the recursiveness of the set $\{\varphi; \mathbf{T} \vdash \varphi\}$ which was proved by Theorem **4.2** not to be recursive ◊

In case we take **T** to be **ZF,** its incompleteness is in fact explicitly demonstrated by the undecidability of the *Axiom of Choice* (**AC)** ($\mathbf{ZF} \nvdash$ **AC** and $\mathrm{ZF} \nvdash \neg$**AC**) whereas in case **T** is extended to **ZFC** its incompleteness is demonstrated by the undecidability of *Continuum Hypothesis* (**CH**) ($\mathrm{ZFC} \nvdash$ **CH** and $\mathrm{ZFC} \nvdash \neg$**CH**).[18]

I turn again to the steps leading to the proof of incompleteness of any recursive and consistent extension **T** of **ZF** before going on with my arguments on the possibility of a phenomenological interpretation.

---

[17] The term $\ulcorner y \urcorner$ in W.V.O. Quine's convention represents the symbol $\ulcorner y \urcorner$ by definition in formal theory corresponding to the 'object' y in metatheory.

[18] Generally, no matter how we extend **ZF** to a recursive, consistent **T** the First Incompleteness Theorem guarantees that there will always be sentences undecidable by **T**.

By Theorem **4.1**, it is proved by metatheoretical means that any recursive set, essentially any recursively enumerable process, is formally representable within **ZF** by corresponding to any metatheoretical (finitistic) object a **ZF**-formula in a free variable as a constant by definition in the place of this object. But, by Theorem **4.2**, it is proved that there can be no recursively enumerable process to check all formal deductions from **ZF** (or **T**) in formal representation within **ZF** (or within any recursive consistent extension **T** of **ZF)**, a result partly due to the non-rigorous definition of the notion of finitistic in Level 2 assertions. This is particularly important, from my standpoint, as Level 2 assertions[19] do not implicate rigorously defined finitistic objects and consequently do not fall within the range of acceptable formulas in the sense of Theorem **4.1**. Such an assertion, i.e. $\mathbf{ZF} \vdash \forall u \exists ! w\, x(u, w)$, is a key step in the proof of the following well-known theorem by which it is partly due Theorem **4.2**; this theorem also stands behind the Second Incompleteness Theorem and Tarski's Undefinability of Truth.

---

[19] An example of a Level 2 satisfaction formula within **ZF** is:

$$\mathbf{ZF} \vdash \forall x \in \omega \;\; \chi_{odd}(x) \vee \chi_{odd}(x + \ulcorner 1 \urcorner) \quad (\mathbf{1})$$

where there is no strict definition of the finitistic character of objects x whereas in a

Level 1 formula such as $\chi_{odd}(x) \equiv \exists y \in \omega\, (x = 2\ulcorner y \urcorner + 1)$ (**2**)

one should be able to check, for example, whether $\mathbf{ZF} \vdash \chi_{odd}(\ulcorner 7 \urcorner)$ or $\mathbf{ZF} \vdash \neg\, \chi_{odd}(\ulcorner 12 \urcorner)$. The incongruence of formulas (**1**) and (**2**) with respect to the finitistic character of the formal objects involved is reflected in the 'asymmetry' between the universal and the existential quantifiers in respective formulas.

We can possibly draw a parallel with the *Verifizierbarkeit* (verification) and *Falsifizierbarkeit* (falsification) of theoretical conjectures in the Popperian theory of knowledge (see Popper (1934)).

4.3 Theorem. *If $\varphi(x)$ is any formula in one free variable, x, then there is a sentence ψ such that* **ZF** $\vdash \psi \leftrightarrow \varphi(\ulcorner \psi \urcorner)$.
(see Kunen (1982, pp. 40-41)).

It seems there is some indirect 'effect' of the non-rigorously finitistic character of the metatheoretical objects represented in the satisfaction formula **ZF** $\vdash \forall u \exists!\, w\, x(u, w)$ in getting the result of the theorem above and consequently Theorem **4.2**. In turn, Theorem **4.2** is applied to prove the First Incompleteness Theorem, i.e., that there exists a sentence φ within a recursive, consistent extension **T** of **ZF** such that **T** $\vdash \varphi$ and **T** $\nvdash \varphi$.

The constraint of finitistic with regard to any formal object U is also strongly held in S.C. Kleene's approach, in Kleene (1980), so as to be able to talk about a decision procedure or, in other words, about a metamathematical effectively decidable predicate R(x, Y), where Y is a proof of the formal object A(**x**), **x** being the numeral corresponding to natural number x. This would make possible by a Gödel numbering of the metamathematical statement 'Y is a proof of A(**x**)' to correlate to the effectively decidable predicate R(x, Y) an effectively decidable number-theoretic predicate (function)

$$R(x, y) \equiv \{y \text{ is the natural number correlated to formal object } Y \text{ such that } R(x, Y)\}.$$

This way, and also by Church's thesis, mainly based on heuristic evidence, that every effectively calculable function (every effectively decidable predicate) is general recursive, we are led to an equivalence of the notions of a general recursive and an effectively decidable predicate. This leads, though, to an unsolvability of the decision problem in a formal system **S**, that is, to the non-existence of a decision procedure for determining the provability of any formula in **S** (see Kleene (1980, pp. 309 & 313)). It turns out that determining any formula of the system **S** as provable by effectively defining a corresponding formula $B(\mathbf{x})$ for any given natural number x, implicitly involves a non-rigorous notion of finitistic for the formal objects involved as was the case with the derivation of Theorem **4.2** which resulted in the non-recursiveness of the set of theorems deduced from a recursive, consistent extension of **ZF**.

The vaguely finitistic character of metatheoretical objects in establishing incompleteness results can be also noted in the original form of Gödel's First Incompleteness Theorem, presented for instance, in Kleene (1980). There relying on a Gödel numbering we can produce an undecidable formula $\mathrm{A}p(\mathbf{p})$, substituting by G. Cantor's diagonal method the numeral $\mathbf{p}$ for the free variable a in $\mathrm{A}p(a) \equiv \forall\, b\ \neg\, \mathrm{A}(a, b)$ where p is the Gödel number of the formula in free variable a, $\mathrm{A}p(a)$, and b is the the Gödel number of the proof of this formula. Therefore, the formula $\mathrm{A}p(\mathbf{p}) \equiv \forall\, b\ \neg\, \mathrm{A}(\mathbf{p}, \mathrm{b})$ asserts its own unprovability but on the (dubious) implicit assumption that Cantor's diagonal method preserves the finitistic character of metamathematical objects in an *ad infinitum* process of enumeration.[20]

So far undecidable statements of **ZF** have to do in an explicit or implicit sense with some kind of uncountable infinity e.g. **CH**, **AC**, **SH** (*Suslin's Hypothesis*), **KH** (*Kurepa's Hypothesis*), etc. and they are moreover proved to be in one or the other way interconnected; for instance, **Con(ZFC)** $\rightarrow$ **Con(ZFC+ CH)** or **Con(ZFC)** $\rightarrow$ **Con(ZFC +** $\neg$ **CH)**; **SH** follows from **MA +** $\neg$ **CH** but it also holds: $\lozenge \rightarrow \neg$ **SH** and $\lozenge$ is consistent with **GCH** (Kunen (1982, Ch. II)); **MA**, the well-

[20] There is a certain controversy around Cantor's diagonal method known also as Diagonalverfahren as it is thought to use self-referential or non-predicative concepts like 'the set of all sets'. But this is related with my view here concerning the dubious finitistic character of metamathematical objects in an ad infinitum enumeration. As to alternative versions of incompleteness proofs presented, for instance, in Enderton (1972) and Schoenfield (1967) they do not vary substantially in content and thus change nothing to my arguments.

known Martin's Axiom, is also a statement making claims about uncountable infinity.

As **CH** is a statement most directly dealing with mathematical Continuum it is worth referring to P. Cohen's conclusion in *Set Theory and the Continuum Hypothesis* that 'the problem of **CH** is not one which can be avoided by not going up in type to sets of real numbers. A similar undecidable problem can be stated using only the concept of real numbers' (Cohen (1966, pp. 151-152)). Cohen went on to state that even in postulating as a vague article of faith that any statement in arithmetic is decidable in a higher order system such as the **ZF** Set Theory by adding perhaps some extra appropriate infinity axiom there will still remain set-theoretical questions which cannot be expressed as statements about integers alone. Now, keep in mind that on pp. 347-348 I referred to the indirect effect a non-rigorous definition of finitistic in metatheoretical objects (taken as finitistic those completely captured by an exhaustive recursively enumerable process) might have on proving the non-recursiveness of the set of theorems deduced from a consistent extension of **ZF**. We may claim that a loose notion of finitistic in metatheory (as reflected in Level 2 assertions) 'plants a bug' in the structure of the proof of Theorem **4.3** whose application thereafter in Theorem **4.2** leads to the proof of non-recursiveness of the set of theorems of a consistent extension **T** of **ZF**; this latter result essentially means that the expressional depth of set-theoretical statements exceeds that of any statement involving only integers.

This result can motivate to a two-fold claim: **1**) insofar as finitistic metatheoretical objects are apprehended by some form of intuition as unique and well-defined objects or in phenomenological view as well-defined noematical objects of finitely many intentional acts in the open-ended horizon of experience they may define a recursive set such as those of Theorem **4.1** and they can be represented by means of set-theoretical formulas as 'lawful' formal objects in any consistent **ZF** extension. **2**) in the case, though, they are not rigorously finitistic and let a shade of inherent (uncountable) infinity in their apprehension in a sense contrary to the above, they let their metatheoretical non-finitary content 'slip' through the syntactical structure of relevant proofs (e.g. that of Theorem **4.3**) to produce finally the non-recursiveness of set-theoretical deductions and subsequently undecidability results on a formal-theoretical level.

Serious doubts have been expressed concerning the **CH** question, that is, whether any new axioms will settle the matter, alluding to an inherent vagueness of this hypothesis that seems to point to some kind of non-analytical character of **CH**. It is noteworthy that around 1947, In "What is Cantor's Continuum Problem" (Gödel (1947)), K. Gödel claimed that if Cantorian theory completely describes some well-defined reality then it should ultimately decide **CH** as either true or false and its eventual undecidability would mean that the difficulties of the problem 'are perhaps not purely mathematical'. It was well after his views at the time that P. Cohen (in 1962) proved the independence of **CH** from the rest of the axioms of **ZFC** (along with that of **AC** from **ZF**) thus leaving the discussion open till now as to the fundamental character of these conjectures within Cantorian theory.

My view is that there should be some inherent reason for these sentences to be proved independent from the existing axioms of **ZF**-(**C**) theory. In one or the other way these or any other sentence making claims about the nature and properties of actual infinity touch on what by its very nature is non-analytical, they touch, as we saw in earlier sections, on what ultimately grounds continuous unity as an objective whole e.g. on what is making possible to conceive the first *m* terms of an unfolding choice sequence all at once, $< a(0),\ a(1),.....,a(\text{m-1})>$, prior to the assumption of a Continuity Principle in Intuitionistic Analysis. Or, in yet another instance, in assuming the Power-Set Axiom, on the possibility to define the set $\wp(\omega)$ of all finite subsets of first countable limit ordinal ω as an actually existing collection of finitistic objects.

To come back to the temporal reduction on a phenomenological level of discussion, talking about the possibility of an objective continuous whole is a way of introducing an absolute subjectivity on which it should be rooted this objective whole and this subjectivity should not be reducible to any kind of objectivity for it would then belong to the universe of all possible objectivities. In a yet deeper leap of thought this subjectivity should be taken as a 'non-temporal' subjectivity which by essence 'constitutes' and cannot be objectively constituted and on this account it cannot be predicated even by the predicates of existence and individuality for it should then be an objectivity in constituted time. Then we are left with no analytical means to describe it and its only possible ontification is through its objective 'mirror'-reflexion as an ever-instantiated continuous whole in the progression of temporal consciousness. This transcendental ego of the most radical (or supplementary) reduction of Husserlian texts on temporal consciousness which Husserl in his later Bernauer Manuscripts (1917/18) (Husserl (2001, pp. 195-207)) identified with a rather obscure notion of primitive process (*Urprozess*) may be taken, as bold an assertion this might sound, to underlie the inherent impredicativity of objective intuitive Continuum 'reflected' on the formal-mathematical level in the impredicativity of Mathematical Continuum; for instance, in the special inclusion relation part/whole in which the part belongs to the same lowest genus as the undivided whole or in the circularities produced in such definitions where the *definiens* cannot be defined but in terms of the *definiendum* (e.g. the definition of an open interval of the real line).

This way we are led to two fundamentally distinct levels of perception: the constituted one on which to 'embed' the known predicable universe of any analytical theory naturally including any formal-mathematical discipline and the constituting one which seems to rely on a purely impredicative subjectivity. Thus, inquiring on whether the cardinality of $\wp(\omega)$ should be equal or greater to the cardinality next to countable ω (or whether *Generalized Continuum Hypothesis* (**GCH**) holds for any ordinal $a : 2^{\aleph_a} = \aleph_{a+1}$) seems to reduce on a constitutional level and irrespective of any particular cardinalities of the canonical scale involved, to the fundamentally distinct character between what is belonging to the constituted level of reality, i.e. what is predicable and analytically expressible by first-order means in a finite or ideally (countably) infinite number of steps and what is constituting this very level, objectified as a continuous unity where any analytical description necessarily engenders some kind of circularity. For example, producing a sequence of finitistic objects as subsets of ω by some digital device is a recursively enumerable process and it belongs to the constituted level. But this is done against the backdrop of the constituting level making possible e.g. to conceive the collection $\wp(\omega)$ of all subsets of ω in its totality as a constituted whole in continuous unity.

In this context of discourse we can also clearly discern the level corresponding to recursively enumerable processses and consequently to recursive sets such as the set of natural numbers involving strictly finitistic metatheoretical objects and the level corresponding to an inherent sense of actual infinity linked to non-rigorously finitistic objects in Level-2 assertions or in Cantor's *Diagonalverfahren*.

4. CONCLUSION

In this article I have tried to articulate a possible correlation between the transcendence and consequently the impredicativity on the level of constitution of temporal consciousness and the inherent impredicativity of Continuum within the first-order predicative environment of a formal theory. Doing so, I turned my attention to the subjective root of the continuous unity of temporal consciousness of Husserlian phenomenology as the possible underlying 'cause' of the impredicativity of Mathematical Continuum. Probably, this is not the kind of approach that would make a professed platonist or an unrepenting formalist eager to applause.

Yet, there is an approach nurtured some decades now towards a view of mathematical activity and of mathematical objects, in general, as intimately linked to mental processes and consequently to certain modes of functioning of the brain or even deeper of consciousness. The possibility of establishing a connection of a yet largely unexplored depth between the modes of constitution of temporal consciousness and formal questions about Mathematical Continuum seems grounded, in principle at least, on the claim put forth in the Introduction and in Section 2, that mathematical objects are abstractions, based on perceptual objects, of a certain kind of categorial intuition. Besides, the fact that there exists an impredicativity of the notion of mathematical continuity e.g. in terms of an overlapping of *definiens* with *definiendum* in relevant formal definitions is something that no present day mathematician could deny.

It is hard to tell, in view of the ontological (not kinematical) character of the descriptive context of any formal theory, the outcome that might have an in-depth review of Mathematical Continuum as fundamentally related to a temporal constitution of each subject's consciousness but it could possibly offer a whole new approach to the question of Continuum and perhaps an interconnection with other disciplines (e.g., quantum mechanics, neurobiology, etc.). The pending question in the core of this discussion would be anyway whether there is something transcendental, almost bordering to 'mystical', in the self-constitution of temporal consciousness as the ultimate source of unity in the World and its objects (including mathematical ones) or whether the impredicativity of intuitive and mathematical Continuum rather stems from the fact that we may not be entitled to a consistent and complete description of the objects of a universe - be it a material or a mathematical one - that contains the universe (and ourselves) as one of its objects.

REFERENCES

COHEN, P. *Set Theory and the Continuum Hypothesis.* Reading, Mass.: W.A. Benjamin, 1966.

DRAKE, F. SINGH, D. *Intermediate Set Theory*. Chichester: J. Wiley & Sons, 1996.

ENDERTON, H. A *Mathematical Introduction to Logic*. New York: Academic Press, 1972.

FEFERMAN, S. "Does mathematics need new axioms?". *American Mathematical Monthly*, 106, pp. 99-111, 1999.

FØLLESDAL, D. "Gödel and Husserl". In: *Naturalizing Phenomenology*. Stanford: Stanford Univ. Press, pp. 385-400, 1999.

GÖDEL, K. "What is Cantor's Continuum Problem". *The American Mathematical Monthly*, 54(9), pp. 515-25, 1947.

HEELAN, P. *Space-Perception and the Philosophy of Science.* Berkeley: University of California Press, 1988.

HUSSERL, E. *Die Krisis der europäischen Wissenschaften und die transzendentale Phänomenologie.* Transl. D. Carr. Evanston: Northwestern University Press, 1970.

HUSSERL, E. *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie.* Erstes Buch, Husserliana, Band III/I. Dordrecht: Kluwer Acad. Pub., 1995.

HUSSERL, E. *Zur Phänomenologie des inneren Zeitbewusstseins.* Transl. H. Dussort. Paris: Ed. PUF, 1996.

————————. *Die Bernauer Manusckripte über das Zeitbewußtsein (1917/18)*, herausg. R. Bernet & D. Lohmar. Dordrecht: Kluwer Acad. Pub., 2001.

KLEENE, S.C. *Introduction to Metamathematics*. New-York: North-Holland Pub.Co, 1980.

KUNEN, K. *Set Theory. An Introduction to Independence Proofs*. Amsterdam: Elsevier Sc. Pub., 1982.

LOHMAR, D. "Elements of a Phenomenological Justification of Logical Principles, including an Appendix [...] on the Transfiniteness of the Set of Real Numbers". *Philosophia Mathematica*, 10(3), pp. 227-250, 2002.

POPPER, K. *Logik der Forschung*. Wien: Springer, 1934.

SARTRE, J. P. *L'être et le néant*. Paris: Ed. Gallimard, 1943.

SHOENFIELD, J. *Mathematical Logic*. Reading, Mass.: Addison Wesley Pub., 1967.

TIESZEN R. "Mathematical Intuition and Husserl's Phenomenology. *Noûs*, 18(3), pp.395-421, 1984.

VAN ATTEN, M. VAN DALEN, D. TIESZEN, R. "Brouwer and Weyl: The Phenomenology and Mathematics of the Intuitive Continuum". *Philosophia Mathematica*, 10(3), pp. 203-226, 2002.

VAN DALEN, D. VAN ATTEN, M. "Arguments for the Continuity Principle". *The Bul letin of Symbolic Logic*, 8(3), pp. 309-347, 2002.