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PROOF IN MATHEMATICS: RESPONSE TO JAIRO JOSÉ DA SILVA

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Abstract: The paper by Jairo José da Silva is mainly concerned with the character of mathematical proof and with the nature of mathematics and its ontology. Although there is a fair amount of agreement in our views, I focus my response on three issues on which we disagree. The first is his view of mathematical proof as generally unconstrained by language and by a previous proof apparatus. The second is his discussion of Brouwer's views on proof and formalization. The third is his nominalistic account of structuralism.

Keywords: Mathematical proof. Formal proof. Structuralism. Brouwer.

PROVA NA MATEMÁTICA: RÉPLICA À JAIRO JOSÉ DA SILVA

Resumo: O artigo de Jairo José da Silva explora principalmente o caráter das provas matemáticas e a natureza ontológica da matemática. Apesar de haver bastante concordância em nossos pontos de vista, o foco de minha réplica são três questões em que discordamos. A primeira é sua visão da prova matemática como completamente livre de restrições impostas pela linguagem e por um aparato prévio de prova. A segunda é sua discussão de Brouwer em relação à prova e à formalização. A terceira é sua formulação nominalista do estruturalismo.

Palavras chave: Prova matemática. Prova formal. Estruturalismo. Brouwer.

Jairo's paper is mainly concerned with the character of mathematical proof and with the nature of mathematics and its ontology, and I will center my comments on these two issues.

1. PROOF

Jairo says that my initial criticisms (in Chapter 19) of the conditions of finiteness and effectiveness are the main question I raise about formal proofs, suggests I don't go far enough, and proposes a more radical criticism. In my opinion, a more radical criticism of the notion of formal proof as a representation of real mathematical proofs, as well as the emphasis on the explanatory character of proof, are clearly expressed in the various chapters of my book. In fact, elaborating this broader criticism is largely what I focus on in chapters 19-25 by developing my proposal for an analysis of proof and justification in terms of structural, psychological, social, and ontological features.

In particular, Jairo's emphasis on understanding as a fundamental characteristic of proofs expressed on p. 190:

> Besides showing *that* something is true, a proof in mathematics must ideally show *why* it is true. Aristotle had already, long ago, called our attention to the fact that proofs must be *explicative* (whenever possible),

is in complete agreement with my remark at the end of Chapter 20 (p. 314):

> Helping us understand is an essential feature of proofs, for we not only want to know *that* a theorem is true, but *why* it is true.

Although I find many aspects of Jairo's discussion very congenial, I also disagree with some of his claims. Thus, on p. 187 he states:

> … whereas formal proofs *presuppose* a formal context, a formal system—language, rules, and axioms—that must *already* be in place *before* proofs within the system are devised (for the context frames and imposes constraints on proofs), mathematical proofs often create their own context, and are not *a priori* constrained by a language or by a previously designed proof apparatus.

I think the first part of this remark is true only in the strict sense of 'formal system' characterized in logic books. In a broader sense of 'formal', as this term is normally meant in formulating ordinary mathematical theories, mathematical proofs do presuppose a fairly well established formal context of linguistic conventions, notations, techniques, previous results, etc., and it is only in *exceptional circumstances* that they create "their own context".

Similarly, I do not see why in the next paragraph Jairo restricts the "logical and epistemological role of proofs" to "guaranteeing truth and producing knowledge", as if clarifying concepts, building connections, inducing new discoveries, etc., were not part of the logical and epistemological role of proofs. Moreover, to say that "mathematical proving activity is a *free* enterprise" *in this sense*, does not seem to me to distinguish sufficiently mathematics from physics, for instance, or from various other scientific enterprises.

Jairo's main point is that "*mathematical proving cannot be confined to a proof apparatus fixed beforehand*," which relates to the point he makes a page or so later that "[as] a rule mathematicians just do not work within the limits of pre-designed systems, domains or structures." The first claim is quite true, although even the strictest adept of formal systems of proof would agree that formal systems are extended and modified to accommodate mathematical innovations. The second point, on the other hand, is rather questionable, as can be seen from the textbooks (including advanced textbooks and monographs) used to impart to students the "pre-designed systems, domains, and structures" of the various mathematical disciplines—

analysis, topology, group theory, and so on. It seems to me, therefore, that mathematics is a lot more formal than Jairo gives it credit for.

2. BROUWER

Jairo considers Brouwer to be a paradigm proponent of his view on formalization (p. 187):

> Brouwer was right in believing that formalization plays no role in mathematical *practice* and that we cannot predetermine mathematical proof techniques. Brouwer's views on proofs are the most faithful account of the real character of mathematical proofs we can find in the traditional philosophical literature.

In the remaining part of the paragraph, however, he takes away all that is specific to Brouwer's view of proofs as being mistakenly derived from his "foundational goals" and "mystical prejudices":

> His foundational goals (not mentioning mystical prejudices), however, impose unreasonable restrictions on some well-established mathematical methods. I think that Brouwer's mistake is to conjoin a peculiar interpretation of mathematical existence ... with the belief that mathematical theories are *contentual*—that is, theories of *determinate* mathematical domains of *objects*.

But the most central aspect of Brouwer's conception of mathematics, and of proof, is that mathematical activity, as well as proof, is purely mental and non-linguistic,[1] and one would be hard put to sell this idea to the community of mathematicians. In fact, when (part of) the mathematical community got around to developing Brouwer's intuitionistic mathematics in a more systematic way, they did so by a process of formalization—either in the strict

[1] See, e.g., the quotations from "Historical Background, Principles and Methods of Intuitionism" in Chapter 25 (p. 445, note 3).

sense one finds in the works of Kleene, for instance, or by the more informal process one finds in the works of Heyting and others.

Thus, I think that when Jairo rejects Brouwer's "peculiar interpretation of mathematical existence", suggesting instead a structuralist view based on "empty" forms not existing independently "in themselves", all that is left of Brouwer's insights is that he does not believe in formalization.

2. STRUCTURALISM

After the discussion of proof, Jairo turns, in the last part of his paper, to metaphysical issues about mathematics. Here, again, there is significant agreement on some issues. He begins by rejecting the view of mathematics as dealing with particular abstract objects such as numbers, which is a view I also hold, and defends a structural view, which I do as well. One difference between us is that whereas I consider structures to be abstract relations, Jairo takes a "nominalistic" view of structures. He says (p. 194-195):

> It is, as always, a matter of dispute among metaphysicians the ontological character of structures. Are they Platonic entities (*ante rem* realism) or simply Aristotelian ones (*in re* realism)? I.e., do they exist independently of the domains they in-form, or are only aspects of them? I think the natural approach is to consider the term "structure" only as a way of speaking and give reality only to structural descriptions, which are nothing but assertions of a language. Of course, since we assume that different descriptions can describe the *same thing*, there is a way in which there is *something* they describe. There are ways, however, of giving this entity a sort of existence, like that of cultural artifacts (Mahler's eighth symphony, for instance), that escapes both Plato's and Aristotle's models.

But *what is* this *something* described? If *all* that is real are the structural descriptions, then there is no *entity* described, and it seems to make no sense to talk about "giving this entity a sort of existence". I find this kind of formulation very obscure, and I think it is essentially an attempt to eat one's cake and have it too. If what is real are the structural descriptions, then one should state one's position exclusively in terms of those, and not pretend there is a *demimondaine* "something" described.

It is this basic difficulty that led me to conceptualize structures as relations in intension. The structure of the natural numbers, for instance, is the intensional successor relation, characterized (intensionally) by the second-order Peano axioms. Even if there were no particulars at all, of any kind—mental, physical, logical, etc.—there would still be the structure of the natural numbers.

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