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PROOF AND INFINITY: RESPONSE TO ANDRÉ PORTO

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Abstract: The main issue André Porto raises in his paper concerns the use of dot notation to indicate an infinite set of hypotheses. Whereas I agree that one cannot extract a unique infinite expansion from a finite initial segment, in my response I argue that this holds for finite expansions as well. I further explain how my remarks on infinite proof structures are neither motivated by the impact of Gödel's incompleteness theorems on Hilbert's program, nor by a negative view of strict finitism.

Keywords: Infinite expansions. Formalization. Strict finitism.

PROVA E INFINITUDE: RÉPLICA À ANDRÉ PORTO

Resumo: O problema central que André Porto discute em seu artigo diz respeito ao uso da notação de pontos para indicar um conjunto infinito de hipóteses. Mesmo estando de acordo não ser possível extrair uma expansão infinita a partir de um segmento inicial finito, em minha réplica argumento que isto vale igualmente para expansões finitas. Explico também que minhas observações sobre estruturas de prova infinitas não são motivadas pelo impacto dos teoremas de incompletude de Gödel no programa de Hilbert, e tampouco por uma visão negativa do finitismo estrito.

Palavras chave: Expansões infinitas. Formalização. Finitismo estrito.

The main issue André raises at the end of his paper concerns the use of dot notation to indicate an infinite set of hypotheses. He argues that one cannot extract a unique infinite expansion out of a finite initial segment. I begin with some remarks about this and work my way backwards.

1. INFINITE EXPANSIONS

I agree with André that one cannot extract an infinite expansion from an initial finite segment. In fact, I forcefully argue this point myself in connection with the discussion of rules, both on pp. 76-77 and on pp. 90-92 (and in note 34). I also argue that the same issue arises for finite expansions; for just as I cannot extract a unique infinite expansion from a finite initial segment, I cannot extract a unique expansion of length $(10^{10})^{10}$ from an initial segment *actually given to me* (pp. 287-288).[1]

The issue about infinite expansions does not affect my discussion, however, because my point is that I can give an infinite set of hypotheses, and an infinite proof, in exactly the same way I can give *any* infinite set; namely, by describing it in an appropriate way. And, as I said, I would have to do the same for very large finite sets, and even for not so large accessible finite sets for which I do not want to bother to list all the members (say a set of 1000 elements). In fact, unless one has a very specific reason for listing the elements of a set—say, the primes up to a million in a table of primes—one would simply describe the set by means of a condition that identifies its elements; as I just did.

My idea of allowing formalizations that include deductions with an infinite structure, and deductions involving steps that are not algorithmically checkable, is certainly not as "wild" as André

[1] Thus, André's examples at the end of the paper apply just as much to a set of $(10^{10})^{10}$ hypotheses as to the infinite set of hypotheses.

suggests. A formal deduction of length $(10^{10})^{10}$—of which there are infinitely many according to the standard view—is just as much an abstract entity as is a formal deduction of length ω. And even if one does not want to consider these things to be abstract entities, at the very least they are idealizations, rather than actual physical phenomena. The actual physical phenomena are the provings, and it makes no difference for these whether we are dealing with infinite sets or with (very large) finite sets.

2. STRICT FINITISM

Another issue raised by André is strict finitism. He suggests that as a counterpart to my infinitary position I raise the "threat of strict finitism" as something "dreadful" and "most unwelcome". On the contrary, I am quite sympathetic to strict finitism, and, as can be gathered from the discussion on pp. 90-92 mentioned above, only question those positions adamant in rejecting infinitary methods, on account that, e.g., one *cannot actually have an infinitely long proof*, and at the same time allow for proofs of any finite length; as if one *could actually have a proof of length* $(10^{10})^{10}$. In fact, I find these two opposing positions to be quite coherent; either one takes a theoretical attitude toward the representation of actual phenomena, and allows them to be represented by notions and processes of a high degree of abstractness, as is the case with infinitary methods, or else one wants to restrict oneself to what can in fact be done, and represent it in some way. In the exchange of letters about the Axiom of Choice by Borel, Baire, Lebesgue, and Hadamard, we find the two positions very clearly stated. I agree with Hadamard, and Zermelo, but I think that the strict finitist position is a perfectly legitimate alternative.[2]

[2] See note IV in Borel (1914). The five letters are translated in Appendix I of Moore (1982).

If, as the strict finitist maintains, to exist in mathematics is to be *defined*, then it makes no sense to allow for the existence of entities that cannot be *actually defined*. In this connection one should keep in mind the important distinctions made by strict finitists. If a finite number is a number that can actually be reached from 1 iterating the successor operation, then $(10^{10})^{10}$ is *not* a finite number. This does not mean that $(10^{10})^{10}$ does not exist, for it can be defined using the exponentiation operation. However, it is just as "transfinite" vis-à-vis the successor operation as is ω, which can be defined as the unique order type that satisfies certain conditions.

3. FORMALIZATION

André begins his paper quoting the following passage from p. 292:

> A formal proof, or deduction, is a representation of the logical form of certain proofs, or arguments, and there is no reason for these representations of logical form to be limited to finite structures.

Although I agree with many of his ensuing comments, I should mention that my motivation for allowing infinite representations for proofs was not related to Hilbert's program and the incompleteness theorems.

My idea was, rather, to allow infinite representations of certain aspects of the actual practice of proving. Thus, for the example on p. 285, I think the proof I give is an actual proof whose structure should be represented by an infinite sequence of steps. And, evidently, just as I can describe the infinite set of hypotheses in a finite way—without using elision dots—I can also describe the infinite sequence of steps in a finite way without using elision dots. In both cases I would have to use quantification over all positive integers.

REFERENCES

BOREL, E. *Leçons sur la Théorie des Fonctions*. Paris: Gauthier-Villars, 1914.

MOORE, G. *Zermelo's Axiom of Choice: Its Origins, Development, and Influence.* New York: Springer, 1982.