Logic, Language, and Knowledge. Essays on Chateaubriand's ***Logical Forms***
Walter A. Carnielli and Jairo J. da Silva (eds.)

CDD: 511.3

WHICH CAME FIRST: THE LOGIC OR THE MATH?

MARK WILSON*

University of Pittsburgh
1001 Cathedral of Learning
PITTSBURGH, PA 15260
USA

mawilson+@pitt.edu

Abstract: Many authors, including Oswaldo Chateaubriand, maintain that "properties" should be structured in logical grades, where the least abstract quantities comprise the lowest ranks of a hierarchy that embraces more abstract and mathematized qualities only at higher levels. But applied mathematicians warns that no quantities can be expected to possess crisp, real world extensions unless they have already been processed with a fair amount of set theoretic machinery beforehand.

Keywords: Properties. Sets. Hierarchy. Applied mathematics.

QUEM VEIO PRIMEIRO: A LÓGICA OU A MATEMÁTICA?

Resumo: Muitos autores, incluindo Oswaldo Chateaubriand, sustentam que "propriedades" deveriam ser estruturadas em uma gradação lógica, onde as quantidades menos abstratas ficariam num nível mais baixo de uma hierarquia que abarca qualidades mais matematizadas e abstratas somente em níveis mais elevados. Mas matemáticos aplicados advertem que não se pode esperar de nenhuma quantidade que elas possuam extensões precisas, como do mundo real, a menos que já tenham sido tratados de antemão com uma boa dose de maquinaria conjuntista.

Palavras chave: Propriedades. Conjuntos. Hierarquia. Matemática aplicada.

---

* I am grateful for conversations with Pen Maddy and Bob Batterman on the considerations canvassed in this essay.

(i)

Oswaldo Chateaubriand was one of my first philosophy teachers and I learned as much from his kindly guidance as from anyone else that I have subsequently encountered in--well, more years than either of us would care to recount. So it is a great honor to be invited to contribute to this celebration of his exquisite *Logical Forms*. Back in those early days, a basic methodological dilemma presented itself (Oswaldo was then working on the problem of counterfactual conditionals): to what degree should philosophical problems about sound reasoning be addressed largely within the orbit of logical notions bequeathed to us by Frege and Russell? As *Logical Forms* beautifully demonstrates, Oswaldo has chosen to develop the logic-focused tradition in a sophisticated modern manner. I followed the other fork in the road: I have spent much of my career investigating the semantic contribution of *non-logical forms of reasoning* to our thinking (as provided in physics' differential equations or the "first this happens, then that" thought patterns of ordinary life). When I sat down to read *Logical Forms*, this orientation caused me some confusion, for I found myself confronted with two *hierarchies of properties*, rather as the British astrophysicist Arthur Eddington once sat down to multiple tables:

> *I have settled down to the task of writing these lectures and drawn up my chairs to two tables. Two tables! Yes; there are duplicates of every object about me--two tables, two chairs, two pens... One of the* [se tables] *has familiar to me from earliest years. It is a commonplace object in the environment I call the world... Table No. 2 is my scientific table. It is a more recent acquaintance and I do not feel so familiar with it....* (Eddington 1935, p. xi)

In fact, my puzzlement is intimately related to Eddington's.

Chateaubriand describes a tower of metaphysical entities constructed in familiar logicist-style fashion: non-abstract physical objects at the bottom; their first-order properties immediately above them; the latter's second-order traits above these, and so on. I have no trouble with any of this; I am simply wondering *which* properties actually inhabit those lowest floor flats. Following Frege and Russell, Oswaldo presumes that the "non-temporal abstract constructions" of mathematics only appear in levels higher than this. But this assertion makes me uneasy: "Gee, aren't the ministrations of mathematics required *before* any putative trait can be permitted first floor occupancy within logic's great skyscraper? And this point of view gives rise, *à la* Eddington, to divergent conceptions of the "first-order properties" wanted in the logical hierarchy, as follows:

The logicist picture: From daily experience, we gain a basic knowledge of physical objects and their everyday traits. Upon this foundational basis, we build (or "posit) a richly extended hierarchy of further sets and/or higher order properties. The typical assertions of working mathematicians are to be treated as disguised claims about the behavior of the denizens of these upper layers of architecture. This portrayal is allied to Eddington's "familiar world."

The applied mathematician's picture: From daily experience, we obtain only a rough and partial knowledge of ill-defined "physical objects" and an unsatisfactory range of under-specified "traits." To improve upon this situation, we must call upon the clarifying virtues of basic mathematical constructions. Amongst these "mildly transcendental" concoctions, the ability to appeal freely to large collections of conditions in the manner of the iterative conception of set has emerged as crucial to our descriptive prospects. This picture represents the property analog of Eddington's "scientific world."

According to this second point of view (which I embrace), the logicist presumption that "the typical assertions of working mathematicians can be treated as disguised claims about the behavior of the upper layers of hierarchy" casts the true epistemological situation in a misleading light, for "the typical assertions of working mathematicians" are *needed simply to arrange the lowest floors of our contemplated logical hierarchy in adequate order*, in advance of any attempt to erect further superstructure on top of those founding layers. And the tensions between our two "pictures" become evident when we read Chateaubriand's description of how we come to grips with "first-order properties" in ordinary life. He provides a charming narrative of how his young son Victor gradually manages to obtain an improving conceptual grip upon predicative specimens such as "being a tiger" or "weighing five pounds."[1] However, an "applied mathematics" critic will complain that Oswaldo has terminated his story prematurely, for the mild improvement processes he discusses cannot possibly bring Victor's loosely specified traits to the acme of perfection required in the bottom floor of an acceptable logical hierarchy. Orthodox classical logic demands first-order traits with crisp and well-defined extensions and such exactness cannot be reached simply through the humble improving processes that Oswaldo catalogues. To properly satisfy logic's demands, Victor probably needs to go to the university, where he can acquire the requisite improvement tools, including a good deal of contemporary set theory. And so we critics complain to Chateaubriand: "You've stopped detailing the story of epistemological improvement with respect to physical concepts at exactly the point when important mathematical issues become philosophically salient."

Virtually nothing to be questioned in this paper hinges upon any opinion eccentric to Oswaldo; allied assumptions concerning

[1] *LF*, Vol. 2, Chapter 13. Here I presume that Victor has benefitted from all of the additional assistance that immersion in society can provide, whether of a Kripkean or Wittgensteinian cast. In the book, Victor is only two years old!

mathematics' placement with respect to logic's hierarchy of increasingly higher-order traits are widely shared across analytic philosophy. However, apprehensions of an "applied mathematics" stripe have enjoyed a long philosophical heritage of their own, although doubts in this vein seem rarely acknowledged within mainstream analytic thinking. It would be nice to see these tensions more clearly resolved. Insofar as I can determine, the author of *Logical Forms* might cheerfully acquiesce in our alternative picture of what proper "first-order properties" must be like, although the concession should force a retraction of stray assertions here and there (*e.g.*, the remarks about upper floor "non-temporal abstract constructions"). But a fair number of philosophical projects afloat within our profession cannot be so easily sustained, I think, once the applied mathematician's objections are acknowledged. In particular, I have in mind ambitions that seek to eliminate or curtail substantial stretches of set theoretic architecture as physically unnecessary or extravagant.[2] I believe that such undertakings are predicated upon a fundamental misunderstanding of the role that sets play within modern physical thinking. Unfortunately, we won't be able to canvass any of these further methodological ramifications within the present essay.

(ii)

The easiest route to appreciating the applied mathematician's worries begins within some basic methodological considerations that appeared robustly within Descartes' reflections upon the potential scope of applied mathematics. Learning how later mathematicians

---

[2] I have in mind the many frigates that have set sail under the colors of "Quine's indispensability argument" or "Benaceraff's problem of mathematical knowledge."

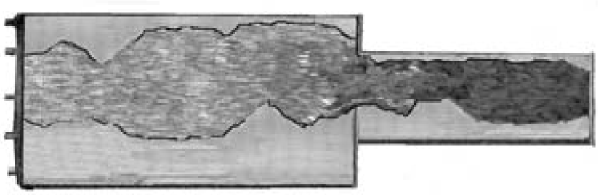
have addressed these concerns will help us appreciate set theory's special contributions to the logical clarity of first-order properties. Let us first observe that poor Descartes has become a popular whipping boy amongst contemporary epistemologists, scorned as the careless Pandora who unleashed a plague of skeptical worries upon us all. It is not adequately acknowledged that intimately entangled within those familiar Cartesian musings are a rich set of legitimate considerations with significant bearing upon applied mathematics' descriptive prospects. In true applied mathematician spirit, Descartes claimed that unreflective reliance upon our everyday arsenal of "familiar world" predicates would pose great difficulties with respect to sound reasoning about the physical world. He held that any data couched in everyday conceptual terms needed to be run through a rigorous filter of mathematical correction before such information could be accepted as descriptively reliable. As is well known, he distinguished "clear and distinct ideas" such as ***exact equilateral triangle*** from the "confused" notions of common sense like ***red*** or a child's ***looks roughly triangular***. Descartes observed that the latter notions are not, in their own right, very productive deductively. Suppose that we have a pile of cuckoo clock parts before us. If we are merely told how all of its parts are colored, we will know very little about how our assembly of pieces might behave: the colored conglomeration indicated might do almost anything. But once we know the exact geometrical shape of each piece and how they interlace, we can calculate exactly how the assembly will behave over time: that the unwinding of the mainspring will cause lever A to pivot 30^0 and so on. As a philosophical rationalist, Descartes happened to believe that we can know all (or, as we shall see, *almost all*) of the reasoning principles relevant to the "clear and distinct" notions on an *a priori* basis, for he believed that the full mechanics pertinent to interlaced rigid bodies was largely a matter of Euclidean geometry and conservation assumptions tied to the capacity to perform work. But we are interesting in the bearing of "clear and distinct ideas" upon inferential productivity, not their apparent aprioricity.

Why then, Descartes asked, should a benevolent God have equipped us with sensory systems that feed us information primarily encoded within "confused idea" formats? His reply hinges upon our stupidity: God isn't a deceiver but He also didn't make us capable of solving hard geometrical problems within a short period of time. Suppose we take a peach in hand and ask, "Do I dare to eat this?" If we were truly smart like God, we could swiftly map out blueprints for the fruit's internal geometry and our own digestive system and compute how the two systems will mechanically interact (food poisoning constituting an undesirable matching). But we dull-witted humans will clearly starve before we complete the required inferential assay. Accordingly, our nervous systems supply us with crudely averaged estimates of the fruit's surface condition that allow us to make rude, but practicable, decisions about its gustatory merits (***pinkness*** signaling "acceptable"; ***brownness***, unacceptable). To be sure, genuine information about the fruit's microscopic physical condition is supplied within our visual "looks brown" reports, but that data has become complexly encoded (= "confused" in Cartesian lexicon) in a manner that optimizes swift decision over accurate representation. As any system designer immediately recognizes, effective "quick decision" schemes usually work ably over a circumscribed range of circumstances and behave quite erratically beyond those bounds. We should therefore expect that a concept such as **brownness** is informationally adjusted only to the mild terrestrial environments in which we normally find ourselves. It may become quite unclear what the predicate "brown" should properly signify if considered as a classifier of material objects located across a range of more unusual conditions.

Observe that these Cartesian concerns are fully analogous to our applied mathematician's worries about the "concepts" native to a "familiar world" logical hierarchy: insufficient evidence has been offered that such traits are properly well-defined everywhere (*i.e.*, possess the clear, universe-wide extensions needed in classical logic). Modern scientific study of the informational encoding embodied in our everyday judgments of color suggest that these Cartesian worries are well founded. Typically, our nervous systems blend together an assessment of a patch's own reflectance characteristics with an allied evaluation of how its immediate surroundings are illuminated. In the special case of **brownness**, the brain categorizes a patch as "brown" only if the prevailing illumination is assessed as generally brighter; otherwise, the brain issues a verdict of "orangish" (evaluations of ***blueness***, in contrast, are less surround-sensitive). Consider those self-illuminated angler fish that appear brightly orange within their deep sea environments, yet appear dull brown when hauled to the deck of a fishing trawler. Are they "really brown" within their natural habitat (that response seems wrong) or do they instead "turn brown" when brought to the surface (that seems incorrect as well). In fact, some measure of *descriptive fiat* seems plainly obligatory in such cases (and one can easily concoct allied scenarios of the same ilk). If so, we can hardly pretend that familiar ***brownness*** can meet the demands of sharpness that standard logic places upon its "properties." The core problem does not stem from "brown"'s borderline vagueness as normally understood, but from the fact that ***brownness*** doesn't seem to be clearly defined over large stretches of our universe. And the problem traces exactly to the considerations that Descartes diagnoses: our nervous systems have chosen to *blend together* physical information in a manner suited only to the parochial needs of quick decision within our narrow niche of earthly crust.

In contrast, if a sound physics can inductively assure us that its "unconfused" traits make sense everywhere, then we needn't entertain allied reservations with respect to their suitability for sound reasoning. A trait like ***being perfectly circular*** may be rarely instantiated within irregular Nature, but we can reason about the trait with no logical qualms whatsoever.

Although such Cartesian considerations suggest that we must scrutinize the ground level properties we allow into our logical hierarchy with scientific care, no need for set theoretic thinking is evident within these concerns. However, we can discover such reasons if we turn to another aspect of Descartes' remarkable thinking upon these issues. The thesis that the Book of Nature is covertly written in a mathematical pen was shared by many of his contemporaries, but Descartes noticed a deeper problem affecting this claim which his compatriots usually ignored or minimized. Descartes' worries trace to the observation that many common physical processes cannot be plausibly described as the evolution of any group of objects displaying a *finite geometry*. Consider a fluid moving along a tube with a constriction in the middle. Plainly the fluid must adjust its configuration to accommodate the altered pipe geometry. However, Descartes tolerated no spaces between his finite-sized fluid particles. If so, how could the flowing matter fit into the narrowed aperture without first fracturing into some kind of infinitesimal dust and then reassembling?



Descartes claimed that adjustments requiring this level of complexity simply lie beyond our comprehension. True, such infinitely detailed processes occur rather commonly in nature and God Himself can foresee how they will turn out, but we humans, comprising Creatures of a Very Small Brain, must perforce lose their details in a computational fog that we can never adequately penetrate. In Descartes' own terminology, the unfolding physical processes become "indefinite" during such infinitesimally complicated moments. As his disciple Rohault explained:

> [Aristotle's followers] *did not consider that equality and inequality are properties of finite things, which can be comprehended and compared by human understanding, but they cannot be applied to indefinite quantities which human understanding cannot comprehend or compare together, anymore than it can a body with a superflies, or a superflies with a line.* (Rohault 1988, p. 33)

We can remain assured that, whatever transpires during these "indefinite" intervals, the whole operation will still unfold according to the same geometrical principles that God permits us to understand in finite circumstances (otherwise He would prove a deceiver), but we should not expect to inferentially track those same complex steps ourselves. This viewpoint leads Descartes to a position that I have elsewhere dubbed *mathematical opportunism*: Nature offers only restricted occasions under which we can inferentially track her unfolding processes with mathematical tools. (Wilson 2000)

(iii)

As intimated, appeals to *set theoretic constructions* play a vital role in dispelling these Cartesian limitations upon our descriptive capacities. Historically, this alternative view is intimately entangled with the rise of a form of contra-Cartesian *mathematical optimism* that becomes viable once *evolutionary partial differential equations* (p.d.e.s) and their kin get developed after 1750. With these new mathematical tools, unknown to Descartes, mathematicians began to anticipate that *every* real life physical process might be isomorphically copied (at least in theory) by a mathematical trajectory solving a suitable set of governing equations. The rise of this p.d.e.-based "optimism" is commonly regarded as a critical shift in mathematicians' expectations with respect to the capacities of their subject matter. As such, the change in attitude is generally attributed to Euler and his contemporaries, although it actually takes a much longer time before other ingredients essential to an adequate optimism fall in place.[3] These further adjustments in our mathematical artillery enforce further degrees of "mild transcendentalism" with respect to our human descriptive capacities. The main burden of this essay is explain what this "mild transcendentalism" entails.

For our purposes, we needn't understand how evolutionary p.d.e.'s operate in any detail, except to observe that they characterize how a system behaves on an *infinitesimal level* with respect to a given moment in time and along all three spatial dimensions.[4] It is precisely the *infinitesimally focused features* of such equations that

[3] Here I have in mind the fancier constructions that I later describe in connection with "generalized p.d.e.s" and "the factoring of satellite systems" (Kant's problems with physical infinitesimals fall into this category as well). Unfortunately, I cannot develop this side of the story adequately in this essay.

[4] *Ordinary* differential equations, on the other hand, work with only one independent variable, rather than four. As we'll see, even these simpler equations stand to us in a "mildly transcendental" remove (indeed, the same can be argued even for various aspects pertaining to algebraic equations). I stress the p.d.e.s because they display the modeling flexibility required in Eulerian optimism.

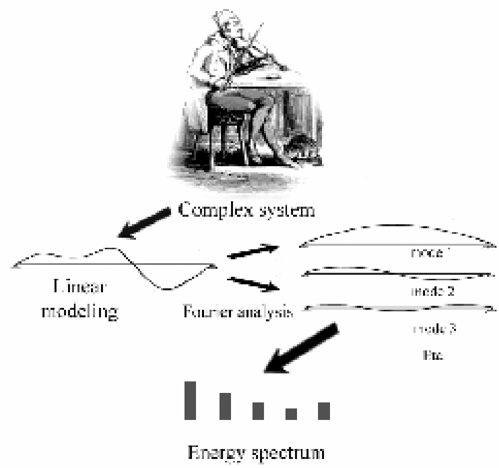
permit an escape from Descartes' dilemma about fluid flow, because p.d.e.s can move the fluid forward at a level that Descartes would have viewed as describing an "indefinite" and infinitely divided "dust." Indeed, there is a well-known set of simple p.d.e.s-- the Navier-Stokes equations--that, with suitable initial and boundary conditions, capably capture (to the best of our current knowledge) *every* flow pattern that macroscopic water is likely to manifest (which is quite an accomplishment, given that such behaviors range from slow, organized creeping to complete chaotic turbulence). These formulas can perform these remarkable descriptive feats precisely because, in fealty to the old maxim that "physics is simpler in the small," they only attempt to delineate the causal factors presently impinging upon each infinitesimal region of fluid. They do *not* attempt to *describe directly* how these localized conditions assemble into larger, macroscopic patterns.

More generally, the Navier-Stokes equations are simply one member of a large family of evolutionary equations capable of capturing the local influences acting upon virtually any form of macroscopic material that one is likely to encounter in everyday life. Do you want your fluid to act springy like a polymer? Or to flow easily when brushed, yet not drip under gravity, like a good paint? Or, to recover slowly from a pipe constriction like a toothpaste? Any and all of these desiderata can be easily accommodated by suitable choices of p.d.e.s akin to the Navier-Stokes equations. Such realistically detailed p.d.e. modelings are actively studied by applied mathematicians within industry today.

It is the broadened capacities for physical coverage offered by p.d.e.s that inspires Eulerian optimism with respect to mathematics' descriptive capacities: we can now hope that there is a mathematical modeling isomorphically adequate to every physical process, no matter how complex or recondite. However, it must be remembered--and this observation is critical to what follows--that extracting useful physical information from such equations by humanly tractable procedures represents another matter entirely. From a generic point of view, p.d.e.s are absolutely hell to solve or to otherwise utilize as reliable founts of information. In practice, quite idiosyncratic techniques must usually be developed before an array of simple p.d.e.s will disgorge even the smallest nuggets of useful information. Quite commonly, we must first project our core governing equations into various simpler sets of formulas, by dropping "insignificant terms," linearizing, averaging, appealing to symmetries and so forth. Even within these simpler, reduced variable satellite modelings, we must often manufacture new mathematical gizmos so that their solutions sets can be factored into manageable components. And it is only at these reduced modeling stages that many vital physical properties emerge clearly and in full definitional force.[5] Consider a violin, for example, which represents a very complex vibrational system. Theoretically, one can write down a set of continuum physics p.d.e.s with complexly varying coefficients that will mock the violin's real life behaviors with great similitude. However, such equations are completely intractable as they stand. It is only after considerable reductive processes have been applied to these starting formulas that we reach a mathematical level where we can capably extract concrete facts about the instrument. For example, it is only at this reduced stage of analysis that our violin's vital qualities traits will emerge clearly from the woodwork through Fourier analysis: the hidden traits that supply a nicely bowed note with its characteristic musical "color." However, we must carefully scrutinize (when we can!) how these computationally compliant satellite modelings relate back to the fuller but intractable

[5] Batterman (2001), stresses allied observations in a very persuasive manner.

models from which they spring, for we sometimes need to know when the properties featured in our reduced structures lose their physical salience, as occurs whenever Junior bows his fiddle in that awful manner that sets our teeth on edge. In other words, the entanglements between an original but directly intractable p.d.e. modeling and its reduced modeling satellites must be set in place if the boundaries of the "physical quantities" we discuss in our physical investigations are to maintain firmly marked boundaries (even if we humans have trouble computing precisely where they lie).

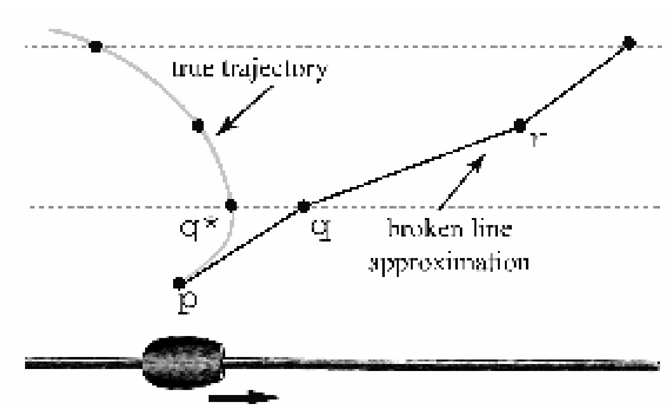


Let us now see why the invocation of p.d.e.s arranges mathematical claims upon a different descriptive plane than Descartes had assumed. The key lies in the fact that such equations directly describe what occurs within its target systems upon an *infinitesimal level* alone, rather than in a format directly verifiable by ordinary computational means. In fact, the early practitioners of the calculus were troubled by the infinitesimal focus of differential equations so much that they often presumed that such relationships could be meaningful only if they represented limiting formulas extractable from preexistent relationships manifested in otherwise familiar, finite mathematical figures (in other words, they saw the infinitesimal relationships as simply a convenient shorthand for registering a host of finitary facts about figures such as circles or ellipses). But Euler and his followers began to approach these issues in an opposed manner: p.d.e.s became regarded as autonomous sorts of mathematical critter that can grow their solution sets on their own recognizance, regardless of whether mathematicians have been previously familiar with the resulting curves and surfaces. According to this new point of view, p.d.e.s. simply march ahead through mathematical time and space and carve up the landscape as they see fit, quite independently of our feeble capacities to anticipate what patterns they will mark out. Indeed, based upon his substantive experience with such formulas, Euler suspicioned that differential equations rarely carve out solution sets that coincide with familiar curves or surfaces and, in the 1840's , Liouville was able to rigorously establish a number of theorems to this general effect, using "rate of growth" techniques familiar to logicians within recursion theory.

Granting the p.d.e.s this descriptive autonomy subtly shifts our conception of "mathematics" away from concrete human inferential capacities. For Descartes, the term "mathematics" approximately meant "physical situation amendable to the rigorous tools of precise human reasoning." And this point of view forced him into his mathematical opportunism: applied mathematicians must search for the rare arrangements when Nature accidently allows her activities to be adequately mirrored within our paltry toolkit of precise descriptive predicates. By allowing the infinitesimally focused p.d.e.s "to do their own thing," Eulerian mathematicians erect what is, in effect, an intermediate kingdom of potential behavior (call it "Pure Mathematicsland") interposed neatly betwixt our limited human reasoning capacities and Nature's actual range of behaviors. That is, we humans can easily grasp the p.d.e.

formulas themselves, but we must then allow them to grow their solutions "under their own recognizance" out in Pure Mathematicsland, the results of which can potentially mirror all of Nature's varied behaviors with perfect isomorphism. Sitting back in Human Computationland, however, we must struggle to figure out what our p.d.e.s have been up to in their autonomous wanderings.

As such, this "trust in p.d.e.s" picture considerably tempers Descartes' gloomy assessment of our descriptive prospects. But this widened descriptive scope comes at a price, for p.d.e.s achieve their match-ups with Nature in a *mildly transcendental way*. To appreciate what I have in mind here, consider a simple ordinary differential equation such as dx/dt = f(x), as applicable to a bead sliding along a wire. Our formula states that the velocity of the bead is linked at every temporal moment to its current position by an f-rule, stating that when the bead is located at point p, its current velocity must be f(p). Observe that this formula fixes how the bead behaves *at each infinitesimal moment* in space and time and tells us nothing directly about any longer interval. However, what we generally want to know precisely how these localized constraints *fit together into finite patterns* that we can actually observe. If the f in "dx/dt = f(x)" is readily computable, we might attempt to deduce our bead's movements along the wire over a short span of time t by starting at time t_0 and utilizing f(p) as a *plausible estimate* for the system's velocity over the entire t interval. That is, we draw a graph where the bead shifts by the distance it would have traveled over the t time period *if* it had maintained a constant f(p) feet/sec velocity the entire time. Suppose that this calculation predicts that the bead will shift to point q. Starting over again, we now graph where the bead will relocate if it now maintains a constant f(q) velocity, leading it to position r. And so on (this graphing technique is called a *finite difference approximation* to the differential equation; real life computer routines commonly utilize more sophisticated variants upon this basic strategy). By these inferential methods, we can hope to extract useful information about our bead's observable, finite movements from our infinitesimally focused differential equation. And allied finite difference techniques can be applied to true p.d.e.s as well.



But observe that this manner of reasoning is quite fallible, for suppose that our equation's f-rule alters the bead's velocity considerably by supplying it with some sharp kick inside the t interval. Rather than landing at q, the bead will actually travel to some completely different location q*. Indeed, if one is not careful in real life computing, the blind application of even the best finite difference routines will cheerfully plot supposed "solutions" that bear absolutely no resemblance to the true curves carved out by the target differential equations themselves. These difficulties trace to the fact that we finite humans can only inspect how our curve behaves at staggered t intervals, allowing sufficient wiggle room for our infinitesimal differential equation to decide to do something utterly different to our bead in the moments when we're not "looking" at it (i.e., calculating).

However, we can still plausibly presume that, if we could only draw our little straight lines estimates over increasingly shorter time steps t, the *complete collection* of broken line estimates we would frame will eventually close in on the particle's true trajectory as a cocooning envelope. But plainly only a god could actually draw the complete infinite mesh of estimates required, forcing us, as mere mortal geometers, to terminate our labors at some finite step size t. And that work stoppage allows our bead ample opportunity to receive an unexpected kick that we haven't yet investigated. In fact, we can (and *must*[6]) appeal to the *complete collection* of broken line estimates as a means of *defining* what the mathematical claim "curve **C** *solves* the equation dx/dt = f(x, t)" actually means (that is, a suitably regular curve **C** solves the equation if the full collection of telescoping estimates eventually converges to **C** under some suitable standard of "convergence"). That is, rather than accepting the differential equation itself as the core autonomous Eulerian agency, we directly cite the good behavior of an infinite set of finite difference approximations as the labors that must occur out in Pure Mathematicsland if a finite curve is properly traced by a differential equation. Indeed, this account is essentially what the standard Cauchy-Weierstrass δ/ε treatment of "dx/dt" demands. Due to its infinitary character, our big jumble of improving broken line estimates stands in a *transcendental* relationship to us, because we can't directly deal with so many conditions ourselves.

However, the relationship is only "*mildly* transcendental" because, if we're lucky, with sufficient cleverness we might find adequate control over an otherwise unmanageable collection. For

[6] P.d.e.s cannot be trusted unto themselves to always draw curves reliably as some of them admit no solutions whatsoever and even the tamest sorts frequently stop converging after a finite time. Our sets of broken line estimates allow us to resolve these "existence" worries on a firm basis.

example, close inspection of the "f(x)" clause often reveals a limited capacity for growth over the t interval, indicating that our broken line estimate can never lie too far from the correct curve carved out by our equation (such bounding estimates on error are called "*a priori* inequalities"). If we can cite such mitigating factors, we can often establish that our finite difference estimates provide a very reliable picture of the equation's true solutions. Unfortunately, the greater complexity of p.d.e.s often renders obtainable reassurances of this "a priori" type rather vapid (the known bounds on reliable numerical estimates for a generic Navier-Stokes situation are extremely short) and we must then look for other methods to extract trustworthy information from our p.d.e.s. Indeed, we are often interested in long-term "trend" questions that numerical approximation schemes cannot answer, no matter how well behaved they prove in the short run. Such requirements usually force us to attack our p.d.e.s with the "reduce to simpler systems" techniques described earlier. Applied mathematicians have displayed amazing suppleness in devising these roundabout tricks.

But note the big improvement in our descriptive prospects over that offered within gloomy Cartesian opportunism: no longer must we presume that fluids in pipes pass through misty "indefinite" states that we can never penetrate by any descriptive means whatever. Instead, we now allow our p.d.e.s to capture the fluid's behaviors directly within its "mildly transcendental" meshes of approximating conditions, while we struggle to catch up inferentially with what our p.d.e.s hath wrought (which is not entirely impossible, for the reasons we have sketched). Trusting to our p.d.e. meshes, we can insure that the physical quantities associated with such modelings will possess the crisp and fully defined character required in the traits we treat in standard classical logic. Unlike Descartes, we no longer have to rely solely upon familiar geometrical thinking as the only provider of "clear and distinct" ideas; we can extend our "well-defined" confidence to the much richer set of quantities carved out by p.d.e. systems during their mathematical meanderings.

Although I don't have the space to go into such matters in detail, further appeals to large sets of conditions become required within many other parts of applied mathematics. Indeed, simple p.d.e.s cannot always serve the modeling needs we have outlined without further supplementation. Place a fluid that resists flexing (such as a molten polymer) inside Descartes' constricting pipe. Sometimes such liquids cannot flow around the sharp corner in the tube without developing a kink in the affected region. Inspecting how our finite difference estimates behave in this locale, we discover that they do not converge to well-defined velocities along the kink--our polymer melt acquires a finite shift in direction as it passes through the crimped region. This loss of a well-defined velocity indicates that our allegedly "governing p.d.e.s" cannot properly govern what physically occurs in this region. To handle such problems (and they arise fairly commonly in practice), modern approaches often repair such lapses by complicated functional constructions that sit on top of the usual p.d.e. conditions (giving rise to what are often called "generalized p.d.e.s"). In fact, the properties required in these gizmos are commonly specified by appeal to a new infinite set of conditions that relate the true polymer flow to a large collection of hypothetically smoother flows; the result is, essentially, one calculus-type construction piled upon another in a rather elaborate pattern.

The necessity of considering further layers of infinitary construction becomes greatly magnified when we consider the satellite modelings that arise as intractable original modelings become mapped into simpler structures. As we observed in the violin case above, vital qualities such as the instrument's overtone characteristics commonly emerge only after those simplified models have been factored into a range of "basic behaviors." But these "basic behaviors" often need to constructed (to avoid the risk of trafficking in undefined quantities) by further layers of equivalence class construction allied to those required for our polymer flow (modern harmonic analysis is full of this). In pursuing these satellite constructions, we can be potentially wafted to virtually any region within wider Pure Mathematicsland (the key to unlocking some of our violin's essential secrets, for example, emerge only after its regular functional traits have been extended over the complex numbers). I stress this fact because one commonly encounters misleading claims in the philosophical literature to the effect that the only sets "physics needs" lie at some comparatively low level within logic's analytical hierarchy. Such contentions usually trace to some observation to the effect that the basic physical laws of, e.g., classical electrodynamics can be formulated employing quantifiers that fall within the indicated bounds. Yes, but we've noted that articulating a collection of p.d.e.s and extracting salient information from them represent projects of divergent degrees of difficulty. In dealing with p.d.e.s, physics needs all the help it can get and that help might potentially stem from any arcane corner of mathematics that might offer assistance in "factoring" and allied chores. In that sense, physics "needs" all of classical mathematics.

(iv)

Summing up, our current faith that the attributes discussed in science possess the "clear and distinct" qualities required in physical traits obedient to classical logic tacitly relies upon a confidence that p.d.e.s and their accompanying set theoretic underpinnings correctly allocate the needed "crispness" to all of the quantities that fall within their dominions. Just as Descartes opined, we must strain the loose "traits" of everyday thinking through mathematical filters before they can be considered suited to classical logic's strong demands. Kant, of course, framed allied conclusions long ago after meditating upon related physical considerations.[7] In particular, he believed that the only way we can understand the notion of (scientific) "object" is to view the notion as the hypothetical end-product of an ever-improving set of regulative conditions founded upon Newtonian physics. In Kant's hands, these observations became an "idealism" because he presumed that the only source of these rectifying rules could be the human mind. But Eulerian optimists view these same concerns in a different way: it strikes us as a brute empirical fact that Nature's behaviors generally run somewhat obliquely to our computational capacities. The "mildly transcendental" story we have told of how p.d.e.s manage to mirror physical behaviors captures the precise degree of our descriptive estrangement nicely. Such observations do not force us into idealist despair, for our predicament remains rather mild. After all, we know that there are lots of sets of natural numbers that we cannot effectively enumerate even though simple fluid flows governed by p.d.e.s can readily pick out those same classes in an analog fashion. Doesn't Eulerian optimism simply qualify as an empirical observation of this same basic character, albeit sketched at a different scale? Our computational position within Nature may not be everything we might wish for, but it ain't that bad either. Indeed, we should thank our lucky stars that our degree of direct descriptive disengagement stops at the mildly transcendental, in contrast to Descartes' totally unbridgeable "indefiniteness."

---

[7] Kant wasn't troubled by p.d.e.s *per se*, but the peculiar sorts of physical infinitesimal required in continuum mechanics. See my "Back to 'Back to Kant'" (forthcoming).

Of course, we possess no *guarantee* that applied mathematicians will continue to be successful in their endeavors, for at some point we may simply hit a Cartesian brick wall in our descriptive ministrations (it is rather amazing that by simply allowing Nature a somewhat looser mathematical leash, we seem able to capture her unruly behaviors using largely p.d.e.s and their modern generalizations). And we might also suspect that set theory *per se* is not quite the right glue to hold the entire edifice together (the grounds for this speculation are currently stronger). However, at present, such uneasy doubts should be simply regarded as the Night Thoughts of an Eulerian optimist. For the time being, we should presume that it is within the skein of set theoretic articulation that the "clear and distinct" properties of modern physics will find their proper articulation and delineation.

It is this portrait of our current scientific proceedings that engenders the "two hierarchies of quantities" dilemma with which we began--it supplies a firm sense in which the "math" must come before the "logic." As indicated at the outset, our alternative portrait of the first-floor inhabitants within Oswaldo's hierarchy does not cast doubt upon the soundness of the architecture itself, although our considerations may diminish the lust of those philosophical reformers eager to lop off its higher extremities.

REFERENCES

BATTERMAN, R.W. *The Devil in the Details.* Oxford: Oxford University Press, 2001.

CHATEAUBRIAND, O. *Logical Forms. Part II: Logic, Language and Knowledge*. Campinas: UNICAMP, 2005. (Coleção CLE, v. 42)

DICKSON, M., DOMSKI, M. (eds.). *Synthesis and the Growth of Knowledge.* Chicago: Open Court Press, forthcoming.

EDDINGTON, A. *The Nature of the Physical World.* New York: MacMillan, 1935.

ROHAULT, J. *Rohault's System of Natural Philosophy*. London: Taylor and Francis, 1988.

WILSON, M. "Back to 'Back to Kant'". In: M. Dickson and M. Domski (eds.) (Forthcoming).

————. "The Unreasonable Uncooperativeness of Mathematics in the Natural Sciences." *The Monist*, 83(2), pp. 296-314, 2000.