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REVISITING THE QUESTION ABOUT PROOF: PHILOSOPHICAL THEORY, HISTORY, AND MATHEMATICAL PRACTICE

NORMA B. GOETHE

School of Philosophy
National University of Cordoba
ARGENTINA

goethe@leibniz.uni-hannover.de

Abstract: This paper revisits some of Chateaubriand's critical considerations with regard to representing our reasoning practices in logic and mathematics by means of "idealized syntax". I focus on the persistently critical side of these considerations which aim to prepare the ground for "an interesting epistemology of logic and mathematics" that ought to make room for understanding the pragmatic dimensions of proofs as explanatory rational displays. First, I discuss the 20^{th} century "syntactic conception" of the logical and the underlying set of values it upholds. Secondly, I revisit the syntactic constraints on systematizing our formal forms of reasoning and ask about the relationship between "idealized" proofs construed as "syntactic objects" and the variety of formal forms of reasoning with its uses of the logical by the research mathematician. Finally, I consider the reasons why Chateaubriand thinks the syntactic requirements of "logical rigor" cannot be fulfilled, and why they ought not to be on the agenda. I conclude my paper by pointing to a deeper assumption which needs to be critically revisited as it stands in the way to what the author envisages as an "interesting epistemology of logic and mathematics".

Keywords: Formal forms of reasoning. Syntactic ideal of rigor. Pragmatic dimensions of proofs. Mathematical practice. Epistemology of logic and mathematics.

REVENDO A QUESTÃO SOBRE PROVA: TEORIA FILOSÓFICA, HISTÓRIA E A PRÁTICA MATEMÁTICA

Resumo: O presente artigo reconsidera algumas das considerações críticas de Chateaubriand com relação a representar nossa prática de raciocínios em lógica e matemática por meio de uma "sintaxe idealizada". Concentro-me no aspecto invariavelmente crítico dessas considerações, que têm por objetivo preparar o terreno para "uma epistemologia interessante da lógica e da matemática", a qual deve abrir caminho para a compreensão da dimensão pragmática de provas como exibição racional explicativa. Em primeiro lugar, discuto a "concepção sintática" da noção de lógica do século XX e o conjunto de valores que ela sustenta. Em segundo lugar, eu reconsidero as restrições sintáticas impostas à sistematização de nossos raciocínios formais e pergunto sobre a relação entre provas "idealizadas" construídas como "objetos sintáticos" e a variedade de modos formais de raciocínios com os seus usos do lógico pelos pesquisadores em matemática. Por fim, eu considero as razões pelas quais Chateaubriand pensa que os requisitos sintáticos do "rigor lógico" não podem ser satisfeitos, e por que eles não deveriam ser parte da agenda. Concluo meu artigo apontando uma assunção mais profunda que precisa ser reconsiderada criticamente uma vez que ela representa um obstáculo àquilo que o autor vislumbra como uma "epistemologia interessante da lógica e da matemática".

Palavras chave: Modos formais de raciocínio. Ideal sintático de rigor. Dimensão pragmática das provas. Prática matemática. Epistemologia da lógica e da matemática.

In his book *Logical Forms II: Logic, Language, and Knowledge*, Oswaldo Chateaubriand argues against the standard model-theoretic conception of logic: logic is not and cannot be merely "syntactic" or "formal" in the way this conception takes it to be.[1] Logic concerns valid reasoning and as such must focus on and deal with the notions of deductive argument and proof. Most importantly, our logical inquiries cannot abstract from the notion of truth in its relation to deductive argument and proof; for the notion of proof, Chateau-

[1] Chateaubriand (2001 and 2005). All references will be given in parenthesis indicating volume and page number, or volume and chapter.

briand reminds us, is "an epistemological notion essentially connected with the quest for knowledge, justification and truth". (I, 19) For this late 20th century philosopher of logic deeply marked by the recent history and philosophy of logic and mathematics, questions like "What is a proof?" (II, Ch. 19), and "What is the relation between proof and truth?" (II, Ch. 21) ought to be revisited against the backdrop of fundamental issues regarding the subject-matter of logic in this broad epistemological sense.

In most contemporary writing on issues in philosophy of logic, explicit elaborations of the nature of logic and the notion of proof are uncommon—but not because they can be counted on as common knowledge. As Dag Prawitz, for example, concedes in a recent article on logical consequence for an Oxford Handbook, the most basic notion of logic, the concept of logical consequence, is still poorly understood "in spite of the great advancement of logic in our time and the technical sophistification of disciplines such as model theory and proof theory", while Steward Shapiro (in a companion entry for the same Handbook) prefers to refer the issue concerning the grounds for deciding about a proposed formalization of logical consequence "to the reader's intuitions".[2] Chateaubriand would fully agree with Prawitz's observation about the concept of logical consequence, but throughout the book he is also critical of the epistemological import of the results achieved by the disciplines of model theory and proof theory. This, we recall, is the kind of broad issues that worried Wittgenstein early and late: no matter what the sophistication and theoretical interest of such formal tools may be, it would be wrong to claim that those results fully capture our actual practice in reasoning either in scientific research or everyday experience. In the meantime, we have moved ahead; but current considerations from the history of science, mathematics and logic

[2] Prawitz (2005); and Shapiro (2005).

bring those critical points back into sharper focus. I propose to revisit here some of Chateaubriand critical considerations with regard to representing our reasoning practices in logic and mathematics by means of syntactic formalization and, then, make some suggestions about the lessons we may draw from them today.

1. THE IDEAL OF A "PURELY LOGICAL" PERSPECTIVE AND THE SYNTACTIC CONCEPTION OF PROOF

1.1. The syntactic conception of the logical

Perhaps the clearest exposition of the view Chateaubriand most emphatically opposes may be found in Quine's logic textbooks.[3] Having in mind the year of publication of Frege's *Begriffsschrift*, Quine writes in the Preface to *Methods of Logic* (1950):

> Logic is an old subject, and since 1879 it has been a great one.[4]

In Quine's writings, however, we find the discipline of logic deeply transformed. The mathematization of logic imposes the design of a linear canonical notation that requires "purification" (i.e. abstraction of all "non-logical content" allowing for topic-neutral expression of generality) and makes the ideal of "logical rigor" tangible through the requirement of full explicitness of inferential steps, and the elimination of vagueness and ambiguity. Moreover, this mathematization also enforces a sharp separation between object-level and meta-level language. "Logical truth" and "logical law" appear accordingly as disjoint notions. In Quine's view, "logical truths" are particular sentences that can be "schematized" by valid schemata and are to be sharply distinguished from "logical laws" that

[3] Quine (1941 and 1950).
[4] Quine (1950, p. vii).

are about structural features of discourse.[5] Under this conception of logic, one can no longer claim that logic is a general science whose goal is to investigate "logical truths". One could perhaps say that logic studies "laws" about structural features of discourse thus carrying its study at the meta-level; however, there isn't any explanatory role left for logical laws to play as ultimate principles underlying explanatory "grounding proofs" (as Frege once conceived of them).[6] Logic deals with the "logical forms" that are common to and can be abstracted from different (particular) sentences. On this view, logical rigor demands schematization as well as interpretation. Sentences have logical properties and bear logical relations to each other thanks to the "logical forms" they share, but "logical forms" are "empty forms" or schemata that can be interpreted; and proofs are "syntactic objects" deprived of any explanatory role. Logical properties and relations such as "validity" and "logical consequence" are then "defined" by way of such "schematic" forms and their interpretation. Underlying this "syntactic conception" of logic is (formal) semantics. However, the mathematical (extensional) treatment of logical semantics thus related to logical syntax presupposes the understanding of the logical notions we use instead of really explaining them. (II, 75) In fact, formal semantics is but syntax.[7] It goes without saying that, on this syntactic conception of logic, "logical forms" do not correspond to "abstract general

[5] For a concise discussion of Quine's view, see Goldfarb (2001).

[6] See my "Frege on Understanding Mathematical Truth and the Science of Logic" (2006).

[7] The syntactic formation rules for propositional and first-order languages, Chateaubriand recalls, depend on "a rich metalanguage, which must contain a fair amount of logic and must be meaningful. (II, 75) Formal semantics is but syntax, and the Löwenheim-Skolem theorem – a fundamental result of model theory – is the "first really significant result for syntax". (II, 70)

features" of reality, as Russell once suggested[8], nor do they concern abstract general features of "our thinking about reality".[9] According to Chateaubriand, they constitute mere "meaningless syntactic structure". (II, 292)

Could this view of logic be defended by saying that it fully captures what is going on when we reason? If we focus on the rational practices underlying both mathematical and scientific knowledge, the answer would seem to be no: "syntactic formalization," while taking on a life of its own that needs to be studied in its own terms, offers less than a "*representation* of formal proof and deductive argument" (II, 292, italics added).[10] Moreover, full formalization is not, and should not be, Chateaubriand argues, the goal because what makes work in mathematics and logic possible ultimately escapes full formalization.

1.2. Technical work in logic and the mesmerizing power of "idealized extrapolations"

It is possible to read Chateaubriand's critical remarks against the syntactic view as inscribed in a familiar debate about the task of logic. The title of the book and many of the things Chateaubriand says seem to call for it. Accordingly, on the defended view the task of logic is to investigate "logical truths" that capture abstract (quasi-Platonic) logical forms, "the logically necessary features of reality",

[8] Russell (1919, p. 169).

[9] Dummett (1991, p. 2).

[10] The syntactic setting of modern logic brought to the fore novel topics of inquiry in connection with sets, recursive functions, computability, etc. pervading many areas of contemporary research with the result that "(s)yntax has become a fundamental given" (II, 74) whose strengths and limitations we need to understand.

i.e. logically necessary facts.[11] Nonetheless, my aim here is to focus instead on the persistently critical side of those remarks, which aim to prepare the ground for thinking about "an interesting epistemology of logic and mathematics" (II, 342, note 4). I ask about what we may learn from these critical points today and briefly suggest a way to deepening our understanding of the epistemological issues in logic and mathematics that Chateaubriand is challenging us to face.

As already noted, Chateaubriand's emphasis on both the power and the limitations of syntactic structures as one of the most remarkable discoveries of 20th century logic remind us of some of Wittgenstein's early and late concerns. Chateaubriand like Wittgenstein is concerned that "the modeling power of ideal syntax mesmerizes us into thinking that it is the real thing". (II, 78) The sheer breath of such discoveries leads twentieth century logicians and philosophers of mathematics to forget that syntax is but an abstraction, "an idealized extrapolation" from the different features of the phenomena that "serves the purposes of discussion and theoretical modeling" but has its own limits. And it is the limits of such tools that have not been properly evaluated. Chateaubriand sees this tendency at the root of the idea that "a proof is just an effective manipulation of strings of meaningless symbols" (II, 77) which, he claims, is but "a distortion of the phenomena". What are the phenomena in this case? "Real life" proofs – as they appear in the intellectual workshop of the research mathematician, in a court of law, and more generally, in scientific and everyday reasoning. (II, Ch. 20)

To do full justice to our rational practices in mathematics and science, Chateaubriand suggests that we need to work with an idea of "formal representation" that is much broader than linear

[11] See D. Macbeth's discussion in this volume.

"syntactic formalization", one that may be able to take into account the interplay of pragmatic-epistemic dimensions of the different forms of representations that are actually used in a variety of ways, while keeping the syntactic and semantic dimensions of language in view.[12] In particular, in order to understand the limitations of syntactic formalization, we need to focus on the pragmatic dimensions of our reasoning practice, always keeping in mind that truth indicates the goal.

2. THE AIM OF PROOF AND THE VARIETY OF FORMS OF REPRESENTATION

Chateaubriand thinks of "the logical" in a broad and rich sense that is entangled with the practices underlying both scientific and non-scientific understanding. (II, 80) As he puts it, "logic plays a large part" in the "semantics of action" which structures our natural languages; and not all know-how underlying our use of language can be cast into formal theory. This is also why formal forms of reasoning (which make use of iconic, symbolic and other form of signs *in tandem*) must be surrounded by natural language.[13] I will say more about this in section 3.1.

Deeply connected with this idea of "representation" is an understanding of "the formal" inspired by a long tradition of use in the context of the *formal* sciences (which at Leibniz's time were called the "*intellectual* sciences" in opposition to the practical disciplines). This is a broader sense of "formal" that relates to the human capacity to develop, represent, and understand and combine highly abstract forms of thought by means of signs. Paradoxical as it may sound, this notion of the formal is at the basis of the conclusion

[12] For another approach along these lines see Grosholz (2007).

[13] Breger (2000).

that not all of knowledge in the formal sciences (logic included) may be made fully explicit through "formalization".

To begin with, Chateaubriand argues that whenever we speak of "formal logic", "formal proof", and so on, to think of "formal" in terms of "meaningless syntactic structure" would be a great distortion. (II, 292) He traces the origins of systematic syntactic studies to Hilbert (and Skolem). (II, 70) But the idea that pure mathematics might be a game with signs displayed on the page was entertained earlier; and its connection with the origins of modern logic remains in fact poorly understood. Throughout history, philosophers have used the expression "formal logic", sometimes rather critically, but most of the time without worrying about the precise sense of "formal". Even today, scholars interested in so-called "informal logic" seem to assume a "vague" (open-textured) notion of what it means for logic to be "formal". In the present context, it will not help to clarify matters to say that "formal" with reference to modern "mathematical" logic means "amenable to definitive mathematical treatment". This is precisely the sense of "formal" Chateaubriand is calling into question.[14] As the debate between Frege and Hilbert shows, the issue was far from clear at the turn of the past century; and Chateaubriand's worries and extensive discussions suggest that philosophers have not settled on a clear

[14] For instance, armed with the "Tarski-Quine definition" of logical consequence we may pose the question about the soundness and completeness of a formal system. This treatment requires set-theoretic quantification over all interpretations; in the case of first-order schemata the definition requires that the set theory is as strong as basic second-order arithmetic - all that is required is the arithmetically definable set of natural numbers. These are some of the novel abstract assumptions "disguised by the usual inductive and recursive construction of syntax" which must be studied in their own terms. (II, 89)

understanding of some of the most basic notions of the logical towards the end of the century. In this regard, he is not alone.[15]

2.1. The relationship between the syntactic notion of proof and the concepts underlying our reasoning practice

With regard to the logical, two main issues underlie Chateaubriand's reflections. What, to begin with, are the constraints we face in "systematizing" the logical, for instance, the notions of "deductive argument" and "formal proof"? Secondly, and more significantly, what is the relationship between the outcome of systematization (even within a localized setting) and our understanding of these logical notions as expressed in our practice? Focusing on the notion of "formal proof" as found in late mid-twentieth century first-order logic textbooks (Church 1956, Enderton 1972) Chateaubriand addresses, in particular, the question: What is the relationship between this "theory of formal proof" and our actual practice in proving things "to ourselves and each other"? (II, 303)

To think of the logical as fully characterized by the language of "schemata" (and their interpretations) leads, Chateaubriand thinks, to an extremely restrictive view of our reasoning practice. Take, for instance, the constraints on the notion of "formal proof" found in Enderton's 1972 logic textbook. Because Enderton identifies "logical form" with "syntactic form", in the stipulated sense, he requires that: (1) formal proofs (as sequences of sentences) must be finite, and (2) formal proofs must be algorithmically checkable (and carry final conviction) without requiring "flashes of insight on the part of the checker". (II, 282) Accordingly, formal proofs are syntactic objects displayed on the page.

[15] See, for instance, van Benthem (2006).

The objection that a "formal proof" isn't merely a finite perceptible "string of signs" displayed on a surface but instead "a representation of the logical form of certain proofs, or arguments", suggests that Chateaubriand has in mind a specific understanding of "formal representation". (II, 292) This may be misleading. He is interested in the epistemology of logic and mathematics; and proofs play an important epistemological role in these disciplines. The point of proof is to reach "truth" but our "epistemic access"[16] to truth is by way of finding and proving (and thereby understanding) the results. In other words, proofs are always explanatory displays: we actually use a variety of proofs, as he puts it, "to reach truth with understanding". (II, 340)[17] Truth is the most important requirement for proof as it sets the goal for all inquiry which, in turn, requires finding ways of proving; and this is an epistemological affair. This idea requires us to recognize that "truth" in mathematics and logic cannot simply be identified with, but outruns "provability". In fact, this is an important point against Enderton's view of proof; according to the latter, truth and provability fall neatly together. Moreover, if the aim of proof is to reach (truth with) understanding, one should be skeptical about the idea that proofs are mechanical affairs able to carry final conviction.

The epistemological point of displaying a proof on the page is explanation. That proofs play explanatory roles in mathematics, not only in research and teaching but also in systematic textbook exposition, is an important point; but note that in each of these cases

[16] I am borrowing this expression from D. Macbeth (see her paper in this volume).

[17] Chateaubriand calls "truth" an "ontological constraint"; this simply corresponds to the requirement that deductive proofs must be truth-preserving. (II, 333) As truth is the goal, the said requirement makes for the normative ingredient of proof.

"understanding" (and proving) seems to play a different role.[18] Finally, Chateaubriand does not like the idea that proofs in the formal sciences exclude "flashes of insight". Interestingly enough, most of the counter-examples he uses are mathematical examples where "flashes of genius" and creativity play a central role. (II, 291) I will say more about this below.

According to Chateaubriand, then, to think of "formal" in terms of syntactic structure is a distortion of the phenomena, "real life" proofs. (II, 292) To clarify his point, he briefly distinguishes between proofs as "idealized extrapolations" - that result from a theory of proof - and "real life" proofs that consist, he says, of a great variety of ways of "proving things to ourselves and each other". (II, 303) What about the "syntactic objects", the proofs actually used by logicians that we find displayed on the page of logic textbooks? Why should they be less "real"? Aren't the practitioners of that discipline part of "real life"? The point here is that in the case of first-order logical proofs, we are dealing with proofs shaped by a notion of "formal proof" that is the outcome of twofold idealization. First, the "idealized" syntactic notion of proof as discussed in logic textbooks is inspired by the notion of modern mathematical proofs as first envisaged by Pasch (1882)[19] and Hilbert (1891-1902)[20]; secondly, the latter notion represents, in turn, an

[18] For a discussion of the different but not unrelated roles played by analytic problem-solving methods, on the one hand, and deductive methods of proving results, on the other, see Cellucci (2002) and see also Grosholz (2007, pp. 40-46).

[19] Pasch (1882/1926).

[20] Hilbert's lecture notes on the foundations of geometry (1891-1902) were recently published by Springer. See Hallet & Majer (2004).

idealization of "real life" proofs as paradigmatically used by German research geometricians towards the end of the nineteenth century.[21]

Science, in other words, requires idealizations but "idealizations", once introduced, also become a part of reality however abstract they may be. We need, then, to understand the newly conceived abstractions, and to do that we need to move one level higher up. In so doing, however, we are moving further away from the concerns we were supposed to address by systematization to begin with. Chateaubriand wants to return to where the modern notion of "syntactic proof" started: the workshop of the mathematician. The "real life" phenomena Chateaubriand is talking about here are those produced in the intellectual workshop of the research mathematician at a specific moment.

The practices of the research mathematician, his proofs as well as his work with surrounding methodologies, need to be studied against the backdrop of styles of thinking and research traditions that are typically entrenched within the context of a particular moment of inquiry. So far this has been mainly the domain of the history of science, but Chateaubriand insists that to look closer at such localized settings should not spell the end of the epistemology of mathematics and logic. (II, 324) His own way of addressing some of the issues – e.g. "What is a proof?" – suggests, however, that even he is struggling to break out of the general framework of "formal" abstraction characteristic of much modern philosophy of mathematics.

In fact, the idea of "finite and effectively checkable" proofs is not all there is to proof; that idea corresponds to a certain

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[21] See Mancosu (2005, p. 14). As Mancosu points out, Pasch was one of the pioneers of the development of geometry characterized by the rejection of diagrammatic tools as relevant to geometrical foundation. However, Pasch did not agree with Frege with regard to a purely logical foundation of arithmetic, as the Frege-Pasch correspondence shows. See Frege (1986).

"idealization" of proofs in the context of mid-twentieth century first-order logic (plus set theory). These considerations can seem to suggest in turn a "paradigmatic" (free-standing) philosophical question: What is a proof? Chateaubriand says that the syntactic view does not capture the "essence of proof", but what is the "essence of proof"? Does it even make sense to pose this question in isolation of a specific research program, let us say, late nineteenth century German rigorization of mathematical analysis?

2.2. The "syntactic" constraints on systematizing our reasoning practice

We have seen that one of Chateaubriand's central concerns is the constraints on systematizing the logical. This motivates in turn concerns about the requirement of "finiteness": Why should those "representations of the logical form" of our proofs and arguments be limited to the case of finite structures? Proofs by mathematical induction offer a good illustration of this point. But, Chateaubriand is concerned also about the requirement of "algorithmic checkability" because it may easily mislead us into thinking that the main explanatory goal of "real" proofs, which is to reach (truth with) understanding, may be replaced by mindless (mechanical) verifiability.

Concerning the requirement of "finiteness", Chateaubriand notes that it is the identification of proof with the actual, material presentation of it (marks on paper) that leads to strict "finitism" about proofs; "strict finitism" about proofs follows from the (fallacious) identification of a notation with its material, perceptible aspects.[22] But granting that the actual sequence of sentences that

[22] To deal with this issue, philosophers of logic introduced the type/token distinction, but as Chateaubriand argues, for the distinction to hold we need to assume (again) a strong idealization of what characterizes a "type" in opposition to its "token". (II, Ch.14)

appears printed on the page in any presentation of a theorem is finite, the mathematician is trained to work with and convey infinite structures by using such finite "paper tools".[23] The modern mathematician has learned to bring the finite into rational relation with the infinite by using a variety of notations and other forms of representations (diagrams, tables, etc.). Such tools are essential i.e. irreducible instruments; and they are more than perceptible marks displayed on a surface. However, the syntactic view holds that the "figure" is dispensable and conceives of proofs as syntactic objects that are written up as strings of signs arranged in a finite and inspectable structure. As Neil Tennant put it, this view "is now a commonplace".[24] But a formal proof ought to be "a representation of the logical form of certain proofs, or arguments". (II, 292) Hence there is no reason for these representations to be limited to finite sentential structures. Nor does the requirement of algorithmic checkability apply, unless, that is, we think exclusively in terms of first-order logic (plus set theory). But why privilege first-order logic (with its canonical linear notation)? It would be foolish to deny the interest and depth of first-order logic results; almost half a century ago, those novel results were enthusiastically acknowledged. But once the results were in place logicians were ready to move on. From the perspective of today's logicians, to assume that first-order logic proofs is all there is to formal proof would be foolish, so why should the philosopher of logic and epistemologist insist upon such restrictive view today?

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[23] The expression "paper tools" goes back to Klein (2003).

[24] See Tennant (1986, p. 304): "It is now commonplace to observe that the diagram (...) is dispensable (...) for the proof is a syntactic object consisting only of sentences arranged in a finite and inspectable array".

2.3. Styles of thinking, research traditions, and the design of proofs

Fully aware of the different attitudes and styles of thinking underlying mathematical practice, Chateaubriand further reminds us to keep in mind an important distinction between proofs in "the idealized sense in which they are usually characterized in theories of proof" ("idealized extrapolations") and "actual proofs" that is to say, "proofs that we use in proving things to ourselves and to each other" (II, 303), "real life" proofs. But upon closer inspection, this distinction is hard to pin down; according to historians of mathematics, it is a philosophical idea ultimately related to the ideal of context-independence of mathematical results and the "eternity of mathematical truth".[25]

In the context of his criticism of the syntactic conception of proof, Chateaubriand is in fact eager to emphasize the distinction between these two senses. It isn't by accident that the most compelling counter-examples Chateaubriand comes up with are cases of "flashes of genius", and people like S. Ramanujan who never received any kind of mathematical instruction. Otherwise he mostly ignores the distinction conceding that in the case of mathematics (and logic), the distinction between "idealized" proofs and "real life" proofs is never sharp enough, because the "ideal" vision of what a proof should look like – as conveyed by teaching or reading textbook expositions– helps to shape our actual uses of proofs.

Actual proofs as we know them are multi-dimensional visual objects displayed on the page, and from the point of view of their materiality and design they are like any other scientific object, "artifacts". This brings out another important point: like any other

[25] See Chemla (2005); see also my discussion of this work "Modes of Representation, Working Tools, and the History of Mathematics" (Goethe 2008).

scientific object, actual proofs are historically placed in a particular context of intellectual debates, theoretical design and production; and as such they are influenced not only by the writing technology but most importantly, by a highly stylized "conception" of what a proof should look like.[26]

3. REVISITING THE IDEAL OF LOGICAL RIGOR

3.1. Why syntactic formalization cannot fully capture the concepts underlying our reasoning practice

The concern of logic, as Chateaubriand understands it, is not just any consistent formalization of *inference* but instead the epistemological study of forms of representation that take into account our reasoning practice with its uses of the logical. Given that the logical notions are intertwined with our uses of proof and the goal of science, that is, truth, the notion of proof, in turn, cannot be merely syntactic, as the standard conception claims, but ought to be seen as "an epistemological notion essentially connected to the quest for knowledge, justification and truth" (I, 19). In particular, logical deduction is "not a purely syntactic notion by the simple argument that preservation of truth is a necessary condition for logical deduction and for deductive proof in general".[27] (II, 295) Our ordinary practice in reasoning and proving makes the same point: that the syntactic requirements for proof do not apply.

But there is a deeper reason why Chateaubriand thinks the syntactic conception of the logical cannot be right. He writes:

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[26] As already noted, the modern conception of "formal proof" that relies on a syntactic characterization of proofs as sequence of sentences is associated with Hilbert but the basic idea is explicit in Pasch (1882). See note 21.

[27] By the same token, the notion of "truth" is not a linguistic notion. A related objection was advanced by Etchemendy (1990).

> We reason and prove things about formal logical consequence without these reasonings and proofs being formulated in an algorithmically verifiable way. In these proofs *we use our understanding and insights* about the logical notions *in exactly the same way that mathematicians use their understanding and insights* about the mathematical notions that concern them. (II, 292-3)

We thus arrived at the most crucial point in Chateaubriand's argument: in our reasoning practice (however abstract and sophisticated) we move back and forth most fluently guided by forms of know-how or implicit knowledge that have been internalized (by different forms of learning experience); but this easiness in reasoning – Leibniz called it "blind thinking"[28] - does not mean that we proceed "mechanically" in the sense of an omniscient algorithmic agent. These forms of acquired "know-how" are of course never explicitly formulated in an algorithmically verifiable mode. One may wonder whether formalization of it should be a goal, whether this capacity can be made fully explicit in one way or another, even in principle. It seems to me that the main point here is the one recently made by Grosholz following Breger: that "formal forms of reasoning must be surrounded by natural language that explains their significance"; and the acquired know-how that permits mathematicians to engage in problem-solving by using and combining signs resists full formalization.[29]

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[28] Leibniz argues that our practice in reasoning - this easiness and fluidity –substantially relies on the acquired ability to work with all kind of signs which, in turn, requires pen and paper, the essential materials to develop modern mathematical writing. (This includes writing in natural language, not just symbolic writing, uses of tables, diagrams, etc.) "Blind thinking" is often read as "algorithmic" or "mechanical"; but this represents a misconception of Leibniz's understanding of this matter.

[29] The mathematician often works with many modes of representations - diagrams, tables, drawing, different forms of notations (symbolic and iconic), and often combines these various tools without reducing them to a

In Chateaubriand's view "our understanding and insights about logical notions" mesh with the use of different modes of formal representation which may not be fully reduced to a linear universally applicable notational system as the syntactic view requires. We thus reached the crucial point that sets limits to syntactic formalization and the requirement of logical rigor which underlies the modern conception of formal proof. For the notion of modern deductive rigor requires making explicit "everything essential to inference". This notion, we recall, includes the requirement that "in any complete proof" of a theorem "figures" (diagrams) and other non-linguistic forms of representation are "dispensable". [30]

3.2 Stepping back inside the intellectual workshop of the research mathematician: learning experience, problem-solving and the importance of seeing for understanding

Having stated the main reasons why he is not convinced by the "syntactic view" of proof espoused by Enderton (and Church), Chateaubriand then, in an unexpected Gestalt switch, goes on to confront this view with some remarks made by the mathematician Hardy (1929) about understanding and the process of proving in mathematics.

Hardy compares the mathematician with an observer who gazes at a landscape; in his highly metaphorical depiction, "seeing"

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unique form of expression. Grosholz argues that "the way in which we combine the formal languages employed in problem-solving cannot be completely formalized, and no mathematician would be interested in doing so." See Grosholz (2007, p. 51).

[30] Pasch claimed that this stringent requirement for proofs is "fulfilable". See Pasch (1882/1926, p. 90); see also Mancosu (2005, p. 15).

by himself and "pointing to an object", so that someone else also sees it, appear as fundamental activities in finding and proving results:

> I have myself always thought of a mathematician as in the first instance an *observer*, a man who gazes at a distant range of mountains and notes down his observations. His object is simply to distinguish clearly and notify to others as many different peaks as he can. There are some peaks he can distinguish easily, while others are less clear. He sees A sharply, while of B he can obtain only transitory glimpses. At last he makes out a ridge which leads from A, and following it to its end he discovers that it culminates in B. B is now fixed in his vision, and from this point he can proceed to further discoveries. In other cases perhaps he can distinguish a ridge which vanishes in the distance, and conjectures that it leads to a peak in the clouds or below the horizon. But when he sees a ridge he believes that it is there simply because he sees it. If he wishes someone else to see it, *he points to it*, either directly or through the chain of summits which led him to recognize it himself. When his pupil also sees it, the research, the argument, the *proof*, is finished.[31]

Chateaubriand confronts this vivid depiction with the logical ideal of rigor and "algorithmically checkable proofs" which "must be the sort of thing that can be carried out without brilliant flashes of insights" thus transforming proofs into "mechanical affairs that replace understanding by verifiability". (II, 291)

The point of Hardy's depiction of an observer who placed in beautiful scenery is able to see "into this world" directly and most pleasurably is twofold. First, it emphasizes the importance of "seeing" for understanding in the process of finding results, proving and showing them to others. Secondly, it puts into images the ways of proceeding of the working mathematician who uses, as Chateaubriand puts it, his "understanding and insights about the mathematical notions that concern him". (II, 293) According to Hardy, even if this is "not the whole truth", there is good deal in it,

[31] Hardy (1929, p. 18; emphasis added).

for the comparison gives us a good approximation to the way we learn and make discoveries in mathematics, ways that may include "flashes of insight".[32] Finally, the imagery also visualizes the point that "the main interest a mathematician has in proofs is aesthetic". (II, 304)

As Chateaubriand reminds us, people often "see" their results in various ways, as in the case of Ramanujan (also greatly admired by Hardy) who got many deep results in unorthodox ways and without any systematic mathematical instruction. Then we have the cases of celebrated conjectures most difficult to prove, such as Fermat's theorem: "Did Fermat see his last theorem? Did he have a proof of it?" (II, 324) Chateaubriand thinks questions like these and the phenomena related in Hadamard's work ought to be taken more seriously by philosophers and not simply be left to psychology, sociology, or history.[33]

Where the syntactic view focuses exclusively on the epistemic value of "justificatory uses" of proof, Chateaubriand is also interested in ways of learning, problem-solving methods, the fruitfulness of new ideas, "flashes of insight", and the aesthetic value of mathematical experience. In particular, he emphasizes the variety of our practices of proofs and refutations, and deductive, inductive, and abductive forms of reasoning; he makes in fact some challenging points that are of current interest, for instance, why should we think of inductive inference in analogy with the classical axiomatic deductive scheme? Does it make sense to formalize inference? What are the limits of formalization? Can all forms of reasoning be reduced to and *explicitly* stated in linear one-dimensional discursive text? What about "visualizations", and other forms of flow and storage of information?

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[32] Ibid., p. 23.

[33] Hadamard (1945/1996).

3.3. By way of conclusion: what stands in the way of an "interesting epistemology of logic and mathematics"

We are currently witnessing a "renaissance of interest in visualization in logic and mathematics"; but according to some scholars, the new emphasis on visualization has not brought about a major shift in the role of visual presentations for the general framework that assumes the sharp divide between ways of learning and finding results, on the one hand, and the epistemic virtues of justification by rigorous proof, on the other.[34] It is just this sharp distinction, which (in its purely *normative* dimension) goes back to Kant and is widely accepted among contemporary epistemologists as plain matter of fact, that underlies the modern concept of logical rigor and formal proof.[35]

Take for instance, Needham (1997). In the Introduction to his book, he writes that many of the arguments in the book "are not rigorous", at least as they stand, but "an initial lack of rigor is a small price to pay if it allows the reader to see into this world more directly and pleasurably than would otherwise be possible".[36] Moreover, there seems to be agreement among scholars that "many modern fields of mathematics admit visual presentations which do not, of course, claim to be logically rigorous, but (...) offer a prompt introduction into the subject matter".[37] The ideal of the logical

[34] See Mancosu (2005, p. 13).

[35] In the history of mathematics, the issue relates to the (entangled) distinction between problem-solving methods of analysis and synthetic methods of grounding by deductive proof; however, it was Kant's normative understanding of logic and epistemology that relegated the issues of learning and finding results to empirical psychology. This is of course a reading of Kant's writings, nonetheless, one that had great impact upon modern epistemology.

[36] Needham (1997, p. xi).

[37] Fomenko (1994, p. vi), as quoted by Mancosu (2005, p. 20).

rigor of "formal proof" comes into the picture when philosophers of mathematics (with an interest in traditional foundational issues) sit down to discuss the epistemic virtues of justification by proof – the so-called purity and rigor of proof -, but before doing that they need to draw the distinction as part of their own methodology and relegate learning and discovery to psychology (and history/or sociology of mathematics). It is against the backdrop of a strong emphasis on the epistemic virtues of justificatory practices that the modern notion of formal proof was designed. The said distinction underlies the syntactic approach and is so deeply entrenched in contemporary normative epistemology that it is easily overlooked. It really should be one of the critical targets of any one seriously concerned with working towards an "interesting epistemology of logic and mathematics" (II, 342, note 4). I doubt that this is an issue that can be addressed by normative epistemology rather than by revisiting detailed case studies in the history of logic and mathematics; and it is precisely here where philosophers are learning to make use of the history of science.

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