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PROOF AND PRACTICE: RESPONSE TO NORMA GOETHE

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Abstract: Norma Goethe addresses my criticisms of the notion of formal proof as a representation of the practice of proving, and in the process revisits large portions of my discussion of proof. I agree with many of her comments, and direct my response to two specific issues. The first concerns the essential features of proof, and the second the distinction between actual proofs and idealized proofs.

Keywords: Proof. Mathematical practice. Knowledge. Justification.

PROVA E PRÁTICA: RÉPLICA À NORMA GOETHE

Resumo: Norma Goethe tece seus comentários entorno de minhas críticas à noção de prova formal como representação da prática de provar e retoma diversos aspectos da discussão de prova em meu livro. Concordo com muitas de suas considerações e dirijo minha réplica a duas questões específicas. A primeira é sobre as características essenciais das provas, e a segunda sobre a distinção entre provas atuais e provas idealizadas.

Palavras chave: Prova. Prática matemática. Conhecimento. Justificação.

Norma addresses my criticisms of the notion of formal proof as a representation of the practice of proving, and in the process revisits large portions of my discussion of proof. I agree with many of her comments, and will discuss just a few specific issues.

1. THE ESSENCE OF PROOF

As part of my discussion of the syntactic view of proof espoused by Enderton and Church, I say on p. 293 that whereas the analysis of logical consequence in terms of finite sequences of effectively checkable steps is an important contribution to logic and philosophy, it is wrong to conceive of this analysis as expressing the very essence of proof. Although these authors do not claim to be giving a full analysis of the essential characteristics of proof, it is clear that the conditions of finiteness and effectiveness are considered by them to be essential. In fact, at the end of the passage I quote on pp. 286-287, Church explicitly argues that effectiveness guarantees final conviction, which he considers an essential feature of proofs (Church 1956, p. 53):

> Indeed it is essential to the idea of a proof that, to any one who admits the presuppositions on which it is based, a proof carries final conviction. And the requirements of effectiveness ... may be thought of as intended just to preserve this essential characteristic of proof.

But, Norma asks, referring to my statement (p. 374):

> [W]hat is the "essence of proof"? Does it even make sense to pose this question in isolation of a specific research program, let us say, late nineteenth century German rigorization of mathematical analysis?

I agree there is an important historical dimension to the notion of proof in mathematics, and what is considered to be a proof varies with time, place, and programmatic aims. I also agree it is a tricky business to try to characterize the essence of anything, but I do think it makes sense to ask about essential features of proof from a general conceptual outlook. This is precisely what I attempt to do with my analysis in terms of the four basic constraints I introduce in Chapter 20. I will briefly recapitulate the considerations that motivate the constraints, and then comment on their relation to some of the issues Norma raises in her paper.

As I begin to argue already in the Introduction (p. 19), a fundamental constraint on proofs is truth-preservation. If we start from truths and arrive at a falsehood, then we do not have a proof. An essential aspect of any conception of proof is to guarantee truth-preservation. Evidently, we may start from some established truths and arrive at a false conclusion, but if this happens we either conclude there is something wrong with our "established truths", or there is something wrong with our "proof". This is exactly analogous to what happens with claims to knowledge, which are disqualified when the proposition allegedly known turns out to be false.

The way proofs guarantee truth-preservation depends basically on their structure, and everyone seems to agree with the general idea that proofs are composed of steps, or stages, which may vary considerably but whose combination is supposed to ensure the truth-preservation of the whole. The stages in the structuring of a proof are relative to the knowledge and interests of specific groups, and must be convincing and agreed upon to guarantee the trustworthiness of the proof.

This relativization to specific groups not only takes into account different levels of knowledge and ability—as between professional mathematicians and undergraduate students, for instance—but also historical periods and special research programs, such as mentioned by Norma. In fact, with a few exceptions I discuss below, Norma's ensuing discussion of my criticisms of the syntactic view, and of my "unexpected Gestalt switch" to Hardy's phenomenological description of the process of proving, is clearly in tune with the general conceptualization I propose. At the end of the section she remarks (p. 381):

> Where the syntactic view focuses exclusively on the epistemic value of "justificatory uses" of proofs, Chateaubriand is also interested in ways of learning, problem-solving methods, the fruitfulness of new ideas, "flashes of insight", and the aesthetic value of mathematical experience.

Although I refer mostly to Hardy's views, I think they are representative of the views of many mathematicians, even with respect to the aesthetic value of mathematical experience. Dyson's recent lecture "Birds and Frogs" gives an eloquent description of the ways of mathematicians, whom he divides into two categories (2009, p. 212):

> Some mathematicians are birds, others are frogs. Birds fly high in the air and survey broad vistas of mathematics out to the far horizon. They delight in concepts that unify our thinking and bring together diverse problems from different parts of the landscape. Frogs live in the mud below and see only the flowers that grow nearby. They delight in the details of particular objects, and they solve problems one at a time.

In the course of the lecture he also emphasizes the aesthetic component of mathematical work, as did Hardy. I will quote two passages. One about Besicovitch (p. 216):

> The Besicovitch style is architectural. He builds out of simple elements a delicate and complicated architectural structure, usually with a hierarchical plan, and then, when the building is finished, the completed structure leads by simple arguments to an unexpected conclusion. Every Besicovitch proof is a work of art, as carefully constructed as a Bach fugue.

The other about Hermann Weyl (p. 217):

> Characteristic of Weyl was an aesthetic sense which dominated his thinking on all subjects. He once said to me, half joking, 'My work always tried to unite the true with the beautiful; but when I had to choose one or the other, I usually chose the beautiful'.

2. PROOFS *VS* THEORIES OF PROOF

Norma suggests that the distinction I emphasize between idealized proofs, as characterized in theories of proof, and the actual proofs used in the day to day activities of mathematicians "is hard to pin down", and argues (p. 376):

> In the context of his criticism of the syntactic conception of proof, Chateaubriand is in fact eager to emphasize the distinction between these two senses. It isn't by accident that the most compelling counter-examples Chateaubriand comes up with are cases of "flashes of genius", and people like S. Ramanujan who never received any kind of mathematical instruction. Otherwise he mostly ignores the distinction conceding that in the case of mathematics (and logic), the distinction between "idealized" proofs and "real life" proofs is never sharp enough, because the "ideal" vision of what a proof should look like – as conveyed by teaching or reading textbook expositions– helps to shape our actual uses of proofs.

This is a misinterpretation, however, because the contrast I make is not between an intuitive genius like Ramanujan—who according to both Hardy and Littlewood had no conception of proof—and formal theories of proof; but, rather, between the latter and the actual proofs that appear in books, papers, courses, talks, conversations, etc. It is only in the context of logic books that we encounter examples of proofs as characterized in the syntactic conception; and even in logic books, the proofs that are not merely exemplifications of the definition of formal proof, but are used to prove meta-theoretical results such as completeness, undecidability, etc., have the same character as ordinary mathematical proofs—which, in fact, they are.[1]

In my remarks about insight, it was never my intention to contrast the view that the verification of a proof must be purely mechanical—"without brilliant flashes of insight", as Enderton (1972, p. 101) puts it—with the brilliant insights that may be present in the work of gifted mathematicians. I was thinking, rather, of the everyday kind of insight we all have when studying or carrying on a proof—i.e., the sort of thing we often express by locutions like "Oh, I see!"

Also, contrary to what Norma says, it seems to me that the distinction between effectively checkable formal proofs and actual proofs is quite sharp, and even when mathematics textbooks spell out proofs in great detail they are never instances of the formal idealization. It is true, however, as she remarks, that a formal ideal of proof influences "our actual uses of proof", though never to that extent.

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[1] As can be seen from a recent issue of the *Notices of the AMS*, however, there are some remarkable developments on the mechanization of formal proofs. The lead article is Hales (2008).