TRUTH AND PROOF

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Abstract: Current versions of nominalism in the philosophy of mathematics face a significant problem to understand mathematical knowledge. They are unable to characterize mathematical knowledge as knowledge of the objects mathematical theories are taken to be about. Oswaldo Chateaubriand's insightful reformulation of Platonism (Chateaubriand 2005) avoids this problem by advancing a broader conception of knowledge as justified truth beyond a reasonable doubt, and by introducing a suitable characterization of logical form in which the relevant mathematical facts play an important role in the truth of the corresponding mathematical propositions. In this paper, I contrast Chateaubriand's proposal with an agnostic form of nominalism that is able to accommodate mathematical knowledge without the commitment to mathematical facts.

Keywords: Platonism. Nominalism. Truth. Proof. Mathematical knowledge.

VERDADE E PROVA

Resumo: Versões atuais do nominalismo em filosofia da matemática enfrentam uma dificuldade na compreensão do conhecimento matemático. São incapazes de caracterizar tal conhecimento como um conhecimento dos objetos descritos pelas teorias matemáticas. A esclarecedora reformulação do platonismo apresentada por Oswaldo Chateaubriand (Chateaubriand 2005) evita esse problema ao propor uma concepção mais ampla de conhecimento como verdade justificada para além de uma dúvida razoável, e ao introduzir uma caracterização adequada de forma lógica segundo a qual fatos matemáticos desempenham um papel importante na verdade das proposições matemáticas correspondentes. Nesse artigo, comparo a proposta de Chateaubriand a uma forma agnóstica de nominalismo que é capaz de acomodar o conhecimento matemático sem o comprometimento com fatos matemáticos.

Palavras chave: Platonismo. Nominalismo. Verdade. Prova. Conhecimento matemático.

1. INTRODUCTION

Mathematical proofs play a central role in mathematical practice. They are, first, the main source of *mathematical knowledge*. Mathematicians determine whether certain results hold or not mainly by devising and assessing the validity of mathematical proofs. Moreover, and equally important, mathematical proofs are the main source of *mathematical understanding*. By formulating proofs not only do mathematicians determine whether a certain result holds, but also, depending on the type of proof that is offered, they can understand *why* the result holds. This offers them understanding of the result in question. Of course, not every proof provides such understanding—some are more illuminating than others. But those proofs that describe the construction of the relevant objects tend to be particularly significant in the understanding they provide.

These points are largely uncontroversial. But they fail to settle an important issue. Do mathematical proofs offer us knowledge of the *truth* of the result they establish? The usual approach to this issue—offered typically by Platonists[1]—emphasizes that they do: proofs are the main way to establish mathematical truths. In fact, mathematicians find out the truth about the subject they investigate by devising suitable proofs. Moreover, on the usual approach, the results established by mathematical proofs are proved conclusively. After all, mathematical statements, if true, are necessarily true. Knowledge of the relevant results is taken to be infallible.

In response, one could agree that mathematicians do obtain knowledge via proofs, but object that knowledge is not of the *truth*

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[1] On its usual formulation, Platonism is the view according to which mathematical objects (i) are abstract and (ii) exist independently of us. That is, with regard to (i), mathematical objects are not located in space-time, and are causally inert; with regard to (ii), mathematical objects are not the result of our mental processes or linguistic practices—to borrow a neat phrase from Jody Azzouni (who is not a Platonist, though; see Azzouni 2004).

of the result in question. It may well be that mathematical objects do not even exist; in which case, every existential mathematical statement—such as that there are infinitely many prime numbers—is false. What mathematicians know when they know that a certain result holds is, strictly speaking, not the truth of a certain theorem, but that the result in question can be derived from other mathematical principles. Rather than strictly mathematical knowledge, what proofs offer is a sort of logical knowledge—knowledge of what follows from what (see, e.g., Field 1989).

In this case, mathematical truth doesn't enter into the picture, and the question of the connection between truth and proof doesn't arise. Moreover, the issue as to whether mathematical proofs are infallible doesn't arise either. Mathematical statements are not taken to be true—let alone necessarily true. Since on this conception, mathematical objects are taken not to exist, the resulting view can be called nominalist.

In this paper, I contrast nominalist and Platonist views about mathematics, assessing some of the benefits and costs of the views. I start with some current nominalist proposals, which, despite what is often alleged, are unable to accommodate mathematical knowledge. I then consider a recent development of Platonism, in the hands of Oswaldo Chateaubriand (2001 and 2002), which doesn't face this difficulty. The proposal introduces a notion of mathematical fact and a broader characterization of mathematical knowledge that allows it to make sense of mathematical knowledge in a very natural way. Finally, I offer an intermediary alternative—an agnostic form of nominalism—that is able to preserve the benefits of nominalism without the corresponding costs.

2. NOMINALIST CONCEPTIONS

Some nominalist views try to preserve the truth of mathematical statements without the commitment to the existence of mathematical objects. For example, some offer translation schemes from Platonist formulations of mathematical theories into modal languages, so that the existence of mathematical entities is never asserted (see Hellman 1989). On the modal-structural view, each mathematical statement S is translated into two modal statements: (i) one asserts that if there were structures of a suitable kind, S would be true in such structures (this is the *hypothetical component*); (ii) the other asserts that the structures of that kind are possible (this is the *categorical component*). In this way, the nominalist can preserve verbal agreement with the Platonist by just asserting the possibility of certain structures without ever being committed to the existence of mathematical entities.

Alternatively, the nominalist can distinguish two kinds of commitment: (i) *quantifier commitment*, which is incurred whenever we quantify over certain objects, and (ii) *ontological commitment*, which is incurred whenever we are committed to the existence of something (see Azzouni 2004). As is well known, W.V. Quine has identified these two kinds of commitments in the case of the objects that are indispensable to our best theories of the world (see Quine 1953 and 1960). Part of Quine's motivation to identify these commitments emerges from the fact that if the objects we quantify over don't exist, it's unclear how we could quantify over them. Moreover, if reference to certain objects—such as numbers and functions in the formulation of physical theories—is indeed indispensable, but these objects don't exist, it's unclear how the latter could play such an indispensable role.

Quine's identification, however, is not well supported. All of us regularly quantify over entities in whose existence we have no

reason to believe. Consider, for instance, statements such as the following (see Melia 1998, and Azzouni 2004):

(a) Average planets have 2.4 satellites.
(b) There are frictionless planes that don't exist.

In the case of (a), suppose that we are unable to determine precisely the number of planets and satellites in the universe, so we can, at best, estimate statistically the average number of satellites per planet. This means that "average planet" turns out to be indispensable to express the relation between planets and satellites we intend to express. It cannot be dispensed with in terms that are more basic. However, clearly no one should thereby be committed to the existence of average planets with 2.4 satellites. In fact, no one would believe in such objects.

In the case of (b), if we were to adopt Quine's identification of quantifier and ontological commitment, statement (b) would be a contradiction. After all, we would be asserting that existing frictionless planes don't exist. But, clearly, (b) is not a contradiction. Quine's conflation assigns two distinct functions to the quantifiers. The latter indicate existence (in the case of the existential quantifier), and they indicate the range of the objects that are talked about—all of the objects in the domain, in the case of the universal quantifier, and some of these objects, in the case of the existential quantifier (see McGinn 2000).

But there is no reason why these two functions should be collapsed. It is better to introduce an existence predicate in the language (denoted, say, by 'E'), as a way of expressing that certain objects exist, and reserve the quantifier simply to register the range of objects we are talking about (McGinn 2000 and Azzouni 2004). In this way, we can make perfect sense of cases in which we quantify over objects whose existence we are not ontologically committed to,

such as (a) above, which has an implicit universal quantifier ranging over all average planets. Moreover, we can also avoid turning statements such as (b) into contradictions. In fact, (b) can be easily formalized as: $\exists x \ (Fx \wedge \neg Ex)$, where '$F$' stands for the predicate 'frictionless plane' and 'E' is the existence predicate.

Given the introduction of the existence predicate, the question immediately arises: which objects exist? Many criteria of existence have been offered: from causal accessibility through observability to ontological independence (see Azzouni 2004). However, if we adopted the first two proposals (causal accessibility or observability), we would clearly beg the question against the Platonist. After all, for the Platonist, abstract entities exist despite the fact that they are neither causally accessible nor observable. Since mathematical objects are not located in space-time, they are not the kind of thing to which we have causal access or that we can observe. Moreover, if we adopted the third view (ontological independence) we would end up *endorsing* Platonism. After all, for the Platonist, we don't make up mathematical objects: these objects are ontologically independent of our linguistic practices and mental processes. Thus, they satisfy the ontological independence criterion.

In order to avoid these outcomes, I think nominalism should be formulated as an *agnostic* rather than a skeptic view (see Bueno 2008). It's unclear how we could establish that mathematical objects *don't* exist, as is unclear how we could establish that they *do*. The issue is better left opened. The agnostic nominalist offers only *sufficient* conditions for the existence predicate. Suppose that we have access to certain objects that is robust, can be refined, allows us to track the objects in space and time, and is counterfactually dependent on these objects (in the sense that if the objects weren't there, we wouldn't believe that they are). In this case, clearly there wouldn't be any doubt that these objects exist. In fact, criteria of this sort are regularly invoked in scientific practice and in ordinary

contexts to support the existence of various objects. The agnostic nominalist can adopt these criteria, as offering only *sufficient* conditions for the existence predicate, without begging the question against the Platonist and without ending up endorsing Platonism.

But what does it mean to quantify over objects that don't exist? It means that we are taking these objects simply as *objects of thought*. This doesn't mean that we thereby make up these objects. It simply means that we are intentionally focusing on them, considering them, and in some cases, describing them. We can do all of this easily without any commitment to the existence of these objects. Consider, for instance, our literary practices—as readers, critics, writers—and how we are all familiar with the experience of thinking about characters in a novel without ever taking these characters to exist. The brilliant detective lived in London, solved crimes in cunning ways, and one of his friends, Watson, was a medical doctor. Does Sherlock Holmes exist? Of course not! Even if we were wandering in the streets of London at the time the stories were supposed to have taken place, it would be a mistake to think that we could bump into Holmes at Baker Street. But we can certainly talk about him nonetheless—I just did.

Quantification over nonexistent objects is as easy as it gets. All it takes is simply to talk about the objects, for instance, by introducing certain principles that characterize them—nothing else is required. With some care, the objects in question may even have inconsistent properties, as it sometimes happens with poorly thought out fiction or with deliberately inconsistent mathematical theorizing.[2] And once the objects are introduced, and a logic is

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[2] One obvious care here involves the choice of the logic we use. If the logic is classical, and we are dealing with objects with inconsistent properties, we immediately obtain triviality—since everything is derivable. However, if the underlying logic is paraconsistent, we can demarcate the inconsistent from the trivial, given that with such a logic, it's no longer the

adopted, we find out what holds about them, by determining what follows from the principles given the logic. These are some of the central features of the agnostic nominalist proposal.

3. SOME TROUBLES

Most nominalist views face a difficulty that Platonist views don't. The nominalist needs to offer a nonstandard understanding of the notion of mathematical knowledge. Platonists have the significant benefit of being able to understand mathematical knowledge as knowledge of the objects and relations that mathematical theories are about, by uncovering the correct concepts that characterize these objects and relations. Given that nominalists—at least of the skeptical sort, who insist that mathematical objects *don't* exist—reject the existence of these objects, they are unable to make sense of mathematical knowledge as knowledge of the objects that are described by the mathematical theories in question. This is a significant problem for these views, since they are unable to accommodate a central aspect of the understanding of mathematics.

On the modal-structural interpretation of mathematics (Hellman 1989), mathematical knowledge is knowledge of the possibility of certain structures (recall the categorical component), and of what would hold in these structures (recall the hypothetical component). However, none of this amounts to what is traditionally taken to be mathematical knowledge, which, on this interpretation, becomes a piece of modal knowledge.

Similarly, on the mathematical fictionalist view (Field 1989), mathematical knowledge becomes logical knowledge: knowledge of what follows from what. But in order to avoid becoming a Platonist

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case that we can derive everything from a contradiction (see da Costa, Krause, and Bueno 2007).

about logical consequence, Field doesn't understand the concept of consequence in a model-theoretic way, which would commit him to the existence of models.[3] Rather, he formulates logical consequence in modal terms: in a valid argument, the conjunction of the premises and the negation of the conclusion is logically impossible. In this case, since mathematical knowledge is knowledge of what follows from what, it ultimately becomes modal knowledge. This means that mathematical knowledge is not about the objects whose properties are presented in relevant mathematical theories. Once again, mathematical knowledge is not presented as what it is supposed to be.

Even on Azzouni's deflationay nominalist view (Azzouni 2004), we are forced to have a revisionist account of mathematical knowledge. Azzouni thinks that ontological independence is the mark of the real, and given that mathematical objects are dependent on our linguistic practices and mental processes, on his view, these objects don't exist. Moreover, he believes that objects that don't exist have no properties. Thus, mathematical theories cannot be taken as describing the properties of the objects studied by such theories, given that these objects do not exist and, thus, lack any properties. As a result, on the deflationary nominalist view, mathematical knowledge cannot be what it is supposed to be.[4]

However, nominalists are not the only one in trouble with the problem of making sense of mathematical knowledge. Platonists

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[3] On the model-theoretic account, an argument is valid if, and only if, every model of the premises is a model of the conclusion. Models are, of course, abstract entities, and they are typically formulated in a set theory. Thus, they are the kind of object to which the nominalist avoids to be committed.

[4] I will return to this issue below, when I discuss whether the agnostic nominalist proposal fares better than the skeptical nominalist views *vis-à-vis* making sense of mathematical knowledge. I think it does.

also have their share of difficulties. But these difficulties are of a different sort. Platonists can explain without trouble the *nature* of mathematical knowledge: the fact that mathematical knowledge is about the concepts, objects, and relations that are described by the relevant mathematical theories. After all, Platonists posit that all of these items exist—as abstract entities—independently of us. The difficulty that Platonists allegedly have is that they are then unable to accommodate the *possibility* of mathematical knowledge. After all, the argument goes, it's unclear how we can form even reliable beliefs about mathematical objects that are causally inaccessible to us (Field 1989).

This is, however, an unfair complaint against Platonism in general. It may be a fair concern for certain formulations of the view, particularly those that invoke the standard conception of knowledge and lack a suitable characterization of the logical form of propositions. But, as we will see now, the problem vanishes given the way in which Oswaldo Chateaubriand has characterized Platonism (see his 2001 and 2005).

4. A PLATONIST ALTERNATIVE

What makes Chateaubriand's conceptualization of Platonism unique is the ingenious combination of different insights from Plato, Frege, Russell, and Gödel, bringing together the strengths of their approaches without the corresponding weaknesses. The result is an elegant, streamlined formulation of Platonism, which emphasizes: (i) the importance of facts—the way things are independently of us—(ii) the significance of characterizing properly the logical forms of propositions —and Chateaubriand assesses different strategies to do that—and (iii) the way in which propositions involve the structuring of reality (see, in particular, Chateaubriand 2001).

In the end, Chateaubriand's formulation is able to preserve central features of Platonism: the objective nature of facts (including mathematical facts) and their independence of us. The emphasis is on ontological matters, which is precisely as it should be, given that, on Chateaubriand's view, "the main question of philosophy is to understand the structure of reality" (2005, p. 442). But in order to understand that structure, we need, first, to develop a suitable conception of that structure; then we need to find ways of making sense of the latter.[5] Given the complexity of the task, Chateaubriand is very honest about the outcome: "there is no safe and final way to do it" (2005, p. 442).

But Chateaubriand also acknowledges that it is not enough to develop the metaphysics of Platonism. It's also crucial to provide a suitable epistemological account of how we can have knowledge of the abstract objects that are posited. This is a central goal of the second volume of *Logical Forms* (Chateaubriand 2005). At this point, which has traditionally been the Achilles heel of Platonism, Chateaubriand offers the most fascinating component of his account: a Platonism with a human face. Rather than pretending to offer an infallible account to explain the possibility of mathematical knowledge, Chateaubriand shows sensitivity to the intricacies and complexities of mathematical proofs. Rather than offering a quick fix to the traditional—and ultimately unworkable—account of knowledge and justification, Chateaubriand develops a novel, broader, and much more defensible proposal. He then integrates all

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[5] I think Chateaubriand would agree with the way in which Barry Stroud presented the point: "How things really are is perhaps always what is at issue in philosophy. The conception of reality that is presupposed or put to work in such philosophical inquiries is what I would like to understand. I want to explore the means by which any such conception of reality is reached." (Stroud 2000, p. 5)

of these features in an elegant, insightful way. Let me present these moves in turn.

When mathematical proofs are discussed in philosophical contexts, it's all too common to emphasize their logical components. What emerges is then an extremely idealized conception of a mathematical proof, largely dominated by considerations of a logical nature. It's pretended that an actual mathematical proof could be thought of as a finite sequence of statements each of which is an axiom, or follows from an axiom or from previous statements in the sequence by applications of suitable rules of inference. But even as an idealized account this proposal is off the mark. At best, it accommodates a couple of aspects of mathematical proofs: their alleged finite character and the fact that proofs are truth preserving. After all, these two features are clearly exemplified in this idealized, formal concept of proof from mathematical logic. But that's all.[6]

In contrast, Chateaubriand highlights four main constraints on mathematical proofs: structural, psychological, social, and ontological (2005, pp. 281-346, and 395-421). (a) The structural constraint deals with the overall structure of proofs and with the various patterns of inference that are involved in proving a certain result. This constraint is much more open-ended than the very

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[6] Interestingly, although Chateaubriand accepts the truth preservation requirement (which, as we will see shortly, is part of his ontological constraint on proofs), he challenges the requirement that mathematical proofs be finite. He argues, very convincingly, that proofs can be infinite. Despite the fact that we will *express* any proof in a finite way—that's the only way in which we can express anything—this doesn't entail that the proof itself is finite (see Chateaubriand 2005, pp. 433-436). Note the crucial work that Platonism is doing here: the infinite (abstract) proof is one thing; our finite (concrete) expression of the proof is quite another.

restricted, idealized account that emerges from mathematical logic.[7] (b) The psychological constraint highlights the fact that proofs need to be convincing: those who follow the proof are supposed to be convinced that the result holds. No final conviction, however, is required. Fallibility is the norm rather than the exception in human affairs—and this includes the activity of proving mathematical theorems. (c) The social constraint emphasizes that mathematical proofs are, in general, a social affair. Proofs depend on whole groups of mathematicians who produce, analyze, assess, revise, and eventually agree that the proofs under consideration actually prove what they were supposed to prove. The social constraint will also indicate which moves in a proof are considered acceptable given the structure of the proof. (d) Finally, the ontological constraint imposes a connection between reality and the structure of the proof. Typically, in the case of mathematical proofs, this connection is met by requiring the truth preservation of the whole structure of the proof and of each of its steps.

These four constraints apply not only to mathematical proofs, but also to justification more generally, which, on Chateaubriand's view, is also supposed to satisfy these constraints (2005, pp. 395-421). By modeling the concept of justification on this conception of proof, Chateaubriand is in a position to highlight the similarity between justification and mathematical proofs, while still allowing for differences between them. After all, the four constraints are satisfied differently in one case and in the other.

Having offered a broader conception of proof and justification, Chateaubriand also proposes a correspondingly broader account of knowledge. Knowledge is not characterized, as is usually done in contemporary epistemology, as justified true belief plus

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[7] Of course, as Chateaubriand acknowledges, if any particular mathematical proof happens to satisfy the idealized account from mathematical logic, that's perfectly fine.

some elusive additional condition. Knowledge is formulated as truth that is justified beyond a reasonable doubt (Chateaubriand 2005, p. 355), and what is counted as a reasonable doubt depends on the structure of the justification in question and on the social constraint. In particular, on this view, belief is not a relevant component of knowledge formation and assessment. In fact, the "tyranny of belief" should be avoided (Chateaubriand 2005, pp. 347-365). That is, we should reject the idea that belief is a required condition on the characterization of knowledge, as well as that we should be *forced* to believe in all sorts of things as part of the development of our philosophical and scientific theorizing. Instead of belief, a global structure of knowledge—understood as truth justified beyond a reasonable doubt—should be in place, as we assess the advantages and limitations of various theoretical proposals. Overall, the goal is to *understand*: to be able to make sense of the structure of reality. And this includes, of course, mathematical reality.

On this novel re-conceptualization of Platonism, mathematical knowledge can still be understood as knowledge of the relevant facts that exist independently of us. But our means of knowing these facts, via suitable mathematical proofs, are characterized in fallible terms. The facts in question exist independently of us, but our propositions involve the structuring of reality in a certain way. Traditional Platonists were in trouble to accommodate mathematical knowledge, given their uncritical adoption of an inadequate, unrealistically demanding concept of knowledge, in which mathematical beliefs were supposed to hold infallibly. Moreover, without a suitable analysis of the logical form of propositions, there was always a gap between the mathematical reality to be known and the means of describing it.

In contrast, Chateaubriand argues that, as with any other piece of knowledge, mathematical results will be known as long as they are true and justified beyond a reasonable doubt. In this way,

his proposal can accommodate mathematical knowledge. Two steps are central here. First, for a mathematical result to be true, suitable mathematical facts need to be in place. It's in virtue of these facts that the result is true. Thus, mathematical knowledge is about the relevant mathematical objects and relations that mathematical theories are about. This is a significant benefit that Chateaubriand's proposal has *vis-à-vis* traditional—i.e. skeptical— forms of nominalism.

Second, mathematical knowledge depends on what is counted as a reasonable doubt in the case of a mathematical proof. What would that be? The answer depends on the structure of the proof, and in particular, on the social constraint on proofs. For instance, what may be, for a specialist in functional analysis, a perfectly straightforward step in a proof, need not be obvious at all for someone not working in the field. This doesn't mean, however, that the standards in a proof—exemplified in the psychological and social constraints—are subjective or relative. All it takes to challenge the acceptability of an inferential move in a proof is a reasonable doubt: grounds to the effect that the move is not well supported.

Once these two steps are taken, it becomes clear that achieving justified truth about mathematical facts beyond a reasonable doubt is not something mysterious at all. This doesn't mean, of course, that obtaining mathematical knowledge is easy. We need to have the right concepts to begin with, and provide a suitable proof that meets the various constraints. But we have here the resources to make sense of the possibility of that knowledge. On Chateaubriand's approach, mathematical knowledge is certainly possible, given that it's actual!

5. THE RETURN OF THE AGNOSTIC NOMINALIST

Can agnostic nominalism yield the same benefits that Platonism in Chateaubriand's formulation does? I think so, and the agnostic nominalist proposal doesn't require commitment to a realm of independently existing mathematical facts.

For the agnostic nominalist, not being committed to the existence of mathematical objects doesn't entail that such objects don't exist, only that their existence plays no role in how we come to know certain mathematical facts. But what are mathematical facts in this case? The agnostic nominalist understands these facts as facts about what follows from certain assumptions regarding a given domain. The domain is specified by the introduction of suitable comprehension principles, principles that determine the meaning of the mathematical terms involved, and how to operate with the mathematical concepts in question. Mathematical facts are not understood as facts about existing mathematical objects, but as facts about what follows from certain principles and relations among concepts. (The latter, of course, are not understood as independently existing abstract entities. Recall that the agnostic nominalist allows us to quantify over entities in whose existence we have no reason to believe.)

Consider, for example, the concept of a metric. In order to operate mathematically with this concept, we need first to characterize it, that is, specify which conditions the concept satisfies. But before we can introduce sensibly the concept of a metric, we need to have already introduced some other concepts—in particular, in the usual study of metric spaces in real analysis, the concept of real numbers. Of course, to introduce *these* concepts, suitable comprehension principles would have to be introduced as well. The procedure is ubiquitous in mathematics. So, let's assume a background system of real analysis with its suitable comprehension principles. We can then specify a metric d as a two-place function defined on the

Cartesian product of a nonempty set S with values in the set of real numbers, as follows: d is always positive ($d(x, y) \geq 0$, for every x and y in S); d has value 0 precisely when its arguments are the same ($d(x, y) = 0$ if, and only if, $x = y$), and d satisfies the properties of symmetry ($d(x, y) = d(y, x)$, for every x and y in S), and triangle inequality ($d(x, z) + d(z, y) \geq d(x, y)$, for every x, y and z in S).

Once these conditions are formulated, we can then determine what follows from them. We can determine some of the facts about a metric. These facts will depend on a number of additional components. In particular, the facts will depend on the logic that is assumed in the derivations—or, given that a logic is hardly ever made explicit in mathematical practice, on some rough inference principles that are invoked. The facts will also depend on additional definitions, which introduce new concepts and refine old ones, and these concepts are then used in proofs and in the statement of theorems. Finally, the facts in some cases will depend on additional conditions specified in the assumptions of a theorem.

For example, after formulating the notion of a metric, we can introduce the concept of a metric space: the pair (S, d), where S is a non-empty set and d is a metric. We can also introduce the concept of a sequence $\{x_k\}$ in S, and of convergence of the sequence $\{x_k\}$ to x in S with respect to the metric d. The sequence converges as long as $\lim_k d(x_k, x) = 0$. We can then prove that if a sequence in a metric space converges, it converges to a unique point; that is, if $\lim x_k = x$ and $\lim x_k = y$, then $x = y$. After all, given that the metric satisfies the triangle inequality, $d(x, x_k) + d(x_k, y) \geq d(x, y)$, for all k, it follows that $d(x, y) = 0$. Thus, since $d(x, y) = 0$ if, and only if, $x = y$, we obtain the result.

This is, of course, a simple fact about sequences in a metric space. And it's very tempting to think of it as a fact about objects, such as sequences and metric spaces. There is nothing wrong with that as long as we don't reify the objects, and suddenly start thinking

that they exist. The objects are indeed introduced via the relevant comprehension principles, but their existence plays no role in the account. What matters is how the objects have been characterized and which conclusions can be drawn about them. The facts in question are facts about what follows from the comprehension principles involved.

In this way, the agnostic nominalist is in a position to make sense of mathematical knowledge as knowledge of the objects that are introduced via the relevant mathematical principles. What is taken to be true in this case is what can be obtained from such principles. Of course, given Gödel's incompleteness theorems, there will be statements that are true but cannot be derived from some mathematical principles. This is fine. There are other ways of knowing that certain principles are true, and there is some role here for a suitably formulated notion of mathematical intuition (see Bueno 2008).

Suppose then that the agnostic nominalist is able to accommodate mathematical knowledge without commitment to mathematical facts. Chateaubriand may challenge that this is an advantage, given his criticism of Ockham's razor (2005, pp. 367-394). According to this principle, entities should not be multiplied without necessity, and nominalists use this principle to deny the existence of all sorts of entities that they think are not needed to explain various phenomena.

But it's not clear that the agnostic nominalist needs to invoke Ockham's razor here. As opposed to skeptical forms of nominalism, the agnostic nominalist does not deny the existence of mathematical facts and other abstract objects. The view is neutral on the issue. Perhaps mathematical facts do exist as the Platonist argues. But if it's possible to make sense of mathematical knowledge and the objectivity of mathematics without invoking them, this is an advantage. After all, the presence of irrelevant components in an

explanation of certain phenomena indicates that the explanation is not as tightly formulated as it could be. In this sense, an explanation of mathematical knowledge that doesn't invoke facts that are not playing any role in how such knowledge is actually obtained seems a more adequate explanation. The point here, I insist, is *not* to argue that therefore mathematical facts *as understood by the Platonist* don't exist. The agnostic nominalist is *agnostic* about this issue. And clearly, it would be a poor argument against the existence of mathematical facts to claim that we can make sense of mathematical knowledge without them. Such facts, the Platonist will certainly remind us, may well exist independently of anything we think of or use in devising explanations. Thus, Ockham's razor—as a principle to reduce ontological commitment—is not invoked by the agnostic nominalist.

Does the social constraint on a mathematical proof entail that the acceptance of a proof is relative to the standards adopted by a certain community? I think it does. However, these standards are fairly stable over time, despite some changes. In general, if there is skepticism as to whether a given proof actually proves what it was supposed to prove, this means that there is a reason to doubt that the proof actually works. The issue then needs to be resolved within the mathematical community. But the agnostic nominalist will say that it is unclear that one needs Platonism to make sense of such a resolution. Ultimately, it is a matter of determining whether the steps in the proof establish the theorem. This is not a straightforward matter in the case of complex proofs, but it's not something in which the existence of mathematical objects and relations seem to play a role. Ultimately, what does the work is the adequacy of the inferences in question.

Given the ontological constraint, Chateaubriand understands the adequacy of these inferences in terms of truth preservation, which in turn depends on the relevant mathematical facts. As we

saw, for the agnostic nominalist, such facts are understood "internally", as what holds given the comprehension principles that characterize the domain under consideration. Once these principles and a given logic are adopted, it's not up to us what holds. This gives all the objectivity that we may need to make sense of mathematical knowledge. But note that there's no need here to invoke the existence of an independent domain of mathematical objects and relations—a domain of objects and relations that exist independently of the relevant comprehension principles. Whether such a domain exists or not is something the agnostic nominalist suspends the judgment about.

Is this outcome unsatisfactory? If we are ultimately interested in understanding the structure of reality, can we really live with a solution that suspends the judgment about whether mathematical objects and relations exist independently of us? I think we can. Both Platonism and skeptical nominalism, despite their many differences, give us understanding. These views indicate ways the world could be if the descriptions they provide were true. The fact that, ultimately, we are not in a position to determine which of these views is true doesn't take away the significant gains in understanding that each of them provide.

The agnostic nominalist can make sense of this. We need not believe that a certain theory is true to appreciate the conception of the world that this theory offers, and the way in which that theory explains a variety of phenomena. Newtonian theory offers a beautiful example of this, but so does Kant's conception. The fact that we can see why certain facts need not be invoked in explaining something—such as mathematical facts, understood in a Platonist way, to make sense of mathematical knowledge—also gives us understanding. And then we realize that an additional, perhaps more elusive, understanding emerges from suspending the judgment about the whole issue. We explored the terrain as much as we could. We

contrasted various alternative explanations, weighting the benefits and costs in each case, and we found out that the issue perhaps couldn't be resolved. Suspending the judgment is a natural outcome in this case. And we can see why.

6. CONCLUSION

We saw that current skeptical versions of nominalism face a significant problem to make sense of mathematical knowledge, given their inability to characterize mathematical knowledge as knowledge of the objects mathematical theories are taken to be about. Chateaubriand's insightful reformulation of Platonism avoids this problem by advancing a broader conception of knowledge as justified truth beyond a reasonable doubt, and by introducing a suitable characterization of logical form in which the relevant mathematical facts play a central role in the truth of the corresponding mathematical propositions.

In contrast, an agnostic—non-skeptical—form of nominalism is offered that is able to accommodate mathematical knowledge without the commitment to mathematical facts (as independently existing entities). For Chateaubriand's Platonism, this need not be an advantage, given the rejection of Ockham's razor. For the agnostic nominalist, however, it's unclear that Ockham's razor is at stake here. After all, the issue is not to deny the existence of mathematical facts as the Platonist understands them, but only to indicate that such facts are not playing a role in how mathematical knowledge is explained.

But suppose that the choice between these two views ends up turning on the fate of Ockham's razor. Whatever that fate is in the end, at least an alternative nominalist conception is available as a counterpart to the huge advance in the Platonist front provided by Chateaubriand.

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