CDD: 149.94

ON THE SEMANTICS OF MATHEMATICAL STATEMENTS*

GUILLERMO E. ROSADO HADDOCK

Department of Philosophy
University of Puerto Rico at Río Piedras
P.O. Box 21572
SAN JUAN, PUERTO RICO 00931-1572

grosado@uprrp.edu

Abstract: Husserl developed – independently of Frege – a semantics of sense and reference. There are, however, some important differences, specially with respect to the references of statements. According to Husserl, an assertive sentence refers to a state of affairs, which was its basis what he called a situation of affairs. Situations of affairs could also be considered as an alternative referent for statements on their own right, although for Husserl they were simply a sort of referential basis. Both Husserlian states of affairs and situations of affairs are extensional. Tarskian semantics can be rendered as a sort of state of affairs semantics. However, to assess adequately the existence of dual theorems in mathematics and, more generally, seemingly unrelated interderivable statements like the Axiom of Choice and its many equivalents, states of affairs (and truth-values) are not enough. We need a sort of refinement of the notion of a situation of affairs, namely what we have called elsewhere an abstract situation of affairs. We are going to introduce abstract situations of affairs as equivalence classes of states of affairs denoted by closed sentences of a given language which are true in the same models. We first sketch the procedure for a first-order many-sorted language and then for a second-order many-sorted language.

Key-words: Semantics. Sense and reference. Husserl.

SOBRE A SEMÂNTICA DOS ENUNCIADOS MATEMÁTICOS

Resumo: Husserl desenvolveu – independentemente de Frege – uma semântica do sentido e da referência. Contudo, há algumas diferenças importantes, especialmente com respeito às referências de enunciados. De acordo com Husserl, uma frase assertiva refere-se a um estado de coisas, que era sua base, o que ele chamou uma situação de coisas. As situações de coisas também poderiam ser consideradas como um referente alternativo para

* Originally published in *Manuscrito*, v. 19, n. 1, p. 149-175, 1996.

enunciados embora, para Husserl, elas fossem simplesmente um tipo de base referencial. Tanto os estados de coisas como as situações de coisas husserlianas são extensionais. A semântica tarskiana pode ser traduzida como um tipo de semântica de estado de coisas. Entretanto, para avaliar adequadamente a existência de teoremas duais em matemática e, de maneira mais geral, enunciados interderiváveis e aparentemente sem conexão, como o Axioma da Escolha e seus muitos equivalentes, os estados de coisas (e valores de verdade) não são suficientes. Nós precisamos de um tipo de refinamento da noção de uma situação de coisas, a saber, o que nós chamamos de uma situação abstrata de coisas. Nós vamos introduzir as situações abstratas de coisas como classes de equivalência de estados de coisas denotadas por sentenças fechadas de uma determinada linguagem, as quais são verdadeiras nos mesmos modelos. Nós esboçamos o procedimento primeiro para uma linguagem multi-sortida de primeira ordem e então para uma linguagem multi-sortida de segunda ordem.

Palavras-chave: Semântica. Sentido e referência. Husserl.

§1. INTRODUCTION

In our paper 'Remarks on Sense and Reference in Frege and Husserl'[1] we considered two different Husserlian alternatives to Frege's specific choice of truth values as the referents of statements in a semantics of sense and reference, namely, a semantics in which the referents of statements are Husserlian states of affairs, and a semantics in which the referents of statements are what Husserl called *Sachlagen* which we have translated as 'situations of affairs'. The first of these two alternatives is the one adopted by Husserl in his *Logische Untersuchungen*[2] whereas the second alternative can be extracted from *Erfahrung und Urteil*[3] and is also present in some of his recently published lectures.[4] It should

[1] Rosado Haddock (1982).

[2] Husserl (1975) and (1984a). See, e.g., the Fourth Investigation §11. In the First Investigation, however, there seems to exist some confusion between the two notions.

[3] Husserl (1939). The distinction between 'state of affairs' and 'situation of affairs' is already present in *Logische Untersuchungen,* specifically in the Sixth Investigation. See, e.g., § 48.

[4] See, e.g., Husserl (1987).

be emphasized, however, that Husserl does not oppose a state of affairs semantics to a situation of affairs semantics. For Husserl both states of affairs and situations of affairs are components of his semantics and intimately related to each other. States of affairs are the referents of statements, whereas situations of affairs are a kind of 'abstract' referential basis. Moreover, although, semantically, states of affairs seem to be prior to situations of affairs, since we arrive at the latter through the former, ontologically situations of affairs are prior, being a kind of ontological foundation for states of affairs. Since we believe that this view is essentially correct, so we tacitly adopt it throughout.

Both Husserlian alternatives[5] to Fregean semantics are extensional, and each of the three choices for the referents of statements originates a group of transformations of sentences determined by the invariance of their respective referential choice. The Fregean group, which is the largest, contains the other two as subgroups, whereas the group of transformations determined by the invariance of the situation of affairs contains as a subgroup the group of transformations determined by the invariance of the state of affairs, and all those three groups contain as a subgroup the group of transformations determined by the invariance of the thought expressed by the statement. All these inclusions are proper.

Hence, according to Husserl, the statements 'The morning star is a planet' and 'The evening star is a planet' do not refer to a truth value, but to the state of affairs that Venus is a planet. Moreover, although John might not know that the evening star is the morning star, the state of affairs that Venus is a planet is not influenced by John's beliefs. (Of course, John can have contradictory beliefs without knowing it, e.g., if he believes that the morning star is a planet but that the evening star is a planet.) In the same vein, the inequalities $5+3 > 6+1$ and $9-1 > 6+1$,

[5] It should be clear from what was said above that what we are calling 'Husserlian alternatives' do not strictly represent Husserl's position, since he did not oppose states of affairs to situations of affairs.

although having different senses, refer to the same Husserlian state of affairs, namely, that the number that we usually refer to by the numeral '8' is larger than the number that we usually refer to by the numeral '7', or, briefly, that 8>7. But, as Husserl has observed, those states of affairs involve some sort of 'categorization', and a distinction can be made that goes deeper into the objectual realm. The relation 8>7 and its inverse 7<8 have something in common (apart from and more immediate than their truth-value), namely, the proto-relation that 8 is a larger number than 7. To the two states of affairs 8>7 and 7<8, which could be called 'signed relations', there corresponds a sort of unsigned proto-relation, which is what Husserl calls 'situation of affairs'.

§2. SITUATIONS OF AFFAIRS

In the realm of non-mathematical objectual examples that could serve as illustrations for the Husserlian distinction between what state of affairs and situation of affairs are. Probably, not as clear-cut as mathematical examples. An example somewhat similar to the arithmetical one given above would be the one consisting of the pair of statements 'Joe is taller than Charles' and 'Charles is shorter than Joe', or any other pair of statements obtained from them by substituting any comparative for 'taller' and its inverse comparative for 'shorter', e.g., by the pair 'older/younger'. (It should be clear that any relation and its inverse could serve as well to illustrate this distinction.) Thus, 'Joe is taller than Charles' and 'Charles is shorter than Joe' refer to different states of affairs, but these states of affairs have the same situation of affairs as underlying proto-relation. Another possible, maybe less clear, example occured to us some thirty years ago when the first heard of the Frege-Husserl distinction between sense and reference (in the Husserlian version) and we *intuited* – as was later confirmed by a study of Husserl's Sixth Logical Investigation – that by going deeper into the objectual realm a somewhat similar but more ontological distinction could be made. If Joe buys a car from Charles, we can say that 'Joe bought a car from Charles' (or, briefly, that 'Joe bought a car') or that 'Charles sold a car to Joe' (or, briefly, that 'Charles sold a car'). The states of affairs referred to by the two statements are different, but the underlying situation of affairs is the same. As we already noticed then, the abridged versions, although somewhat informationally incomplete and, thus, allowing the possibility of some sort of referential ambiguity (e.g., respectively, in the case that Joe has bought more than one car in his life) serve to illustrate another point, namely, that contrary to the other examples given above, it does not always have to be evident that two different states of affairs have the same underlying situation of affairs. Peter can be a neighbour of Joe and a colleague of Charles, and know both that Joe bought a car and that Charles sold a car and that Charles sold a car, without knowing that Joe and Charles participated in a transaction in which Joe bought exactly one car and Charles sold exactly one car, and the car bought by Joe is the same car sold by Charles, say, on the 20th of September 1965.

In our paper 'On Frege's Two Notions of Sense'[6], after arguing that Frege really had two notions of sense, namely, the official one of 'Uber Sinn und Bedeutung' and one essentially not distinguishable from his old notion of conceptual content[7] – a notion that is a forerunner of Husserl's notion of a situation of affairs –, we tried to introduce a sort of 'explanans' of Husserl's notion, which could be fruitful in investigations on the semantics of mathematics, namely, the notion of an abstract situation of affairs. In our paper 'On Husserl's Distinction between State of Affair *(Sachverhalt)* and Situation of Affairs *(Sachlage)*,[8] we developed these ideas somewhat further. We introduced that notion of an abstract

---

[6] Rosado Haddock (1986, pp. 31-41).

[7] Compare *Begriffsschrift* with Frege's letter to Husserl of the 30th of October to the 1st of November 1906, in Frege (1976, pp. 101-104). Frege's letter to Husserl of the 9th of December of the same year, see Frege (1976, pp. 105-106), is alfo relevant to this issue.

[8] Rosado Haddock (1991, pp. 31-48).

situation of affairs to grasp that two dual sentences – e.g., the Ultrafilter Theorem and the Prime Ideal Theorem – have in common, and then generalized this rendering to what any pair of interderivable statements have in common, even if, as in the case of the Axiom of Choice and many of its mathematical equivalents, they seem to talk about very different things, or more precisely, they refer not only to different, but to seemingly unrelated states of affairs. Of course, they also have their truth-value in common, but this they have in common with statements like 'Berlin is the German capital on the 1st of January 1996' or with its negation, without being interderivable with either of them. Hence, if we are going to account for interderivability phenomena in mathematics, it should be clear that neither a semantics that considers Husserlian states of affair as the referents of statements, but ignores abstract situations of affairs, not a Fregean semantics that ignores both states of affairs and abstract situations of affairs, and considers truth-values as the referents of statements, can do justice to the semantics of mathematical statements.

In the above mentioned papers we spoke somewhat loosely of a semantics for mathematical statements, and took for granted an axiom system, e.g., ZF-set theory, without any specification of the language in which such an axiom system should be expressed. In what follows such a concern will be addressed, at least to show the way for future research.

However, since some people might think that the problems of the semantics of mathematics have been answered by Tarki's seminal monograph 'The Concept of Truth in Formalized Languages'[9], and by the ensuing development of model theory, a few words should be said on this issue.

[9] In Tarski (1983). This is an English translation of the expanded German version (1935) of the original Polish monograph of 1933, although the ideas date from 1929. (Cf. The "Bibliographical Note", Tarski (1983, p. 152n.))

§3. BRIEF DIGRESSION ON TARSKI

Fregean semantics is clearly much simpler than its Husserlian counterparts, and that was probably one of the main reasons why Frege adopted it. But even in his attempt to apply his semantics in the logical system *Grundgesetze der Arithmetik,*[10] he was not very successful. Besides many other points that could be and have been levelled[11] against Frege on the adequacy of his semantics in that work, in 'On Frege's Two Notions of Sense',[12] we pointed ou that his infamous Axiom V cannot be adequately rendered either in terms of the Fregean referents of statements, i.e., their truth-values, not in terms of his official notion of the sense of statements in 'Uber Sinn und Bedeutung'[13] and *Grundgesetze der Arithmetik,* but only in terms of his second notion of the sense of statements, namely, what he had called 'conceptual content' in *Begriffsschrift,*[14] and which, as we already mentioned, antecipates Husserl's notion of a situation of affairs. (A similar rendering applies to Frege's second and third attempts at defining the concept of number in *Die Grundlagen der Arithmetik*[15].) The only logical languages in which Fregean semantics seems adequate are the languages of propositional logic and very similar systems (e.g., many-valued propositional logics, which require minor adjustments to the Fregean semantical framework, essentially, the allowance of more than two truth-values).

If we somehow abstract from the notion of sense, as we in fact have been doing for most of our discussion (since it does not play any role at all in most of our considerations), we can subsume Tarskian semantics under a state of affairs semantics as described above. At first sight, some people might think that Tarskian semantics should be

---

[10] Frege (1962).

[11] See, e.g., Kutschera (1989).

[12] See footnote 6 above.

[13] Frege (1982), reprinted in Frege (1991, pp. 143-162).

[14] Frege (1964).

[15] Frege (1986).

rendered as a sort of Fregean semantics. But such a belief would be based on the unwarranted idenfication of Fregean semantics with extensional semantics. As the first pages of both Tarski's famous monograph cited above[16] and his less known paper 'Truth and Proof' [17] make clear, Tarski wanted his work on the notions of truth to be assessed as a contribution to a tradition going back to Aristotle, according to which a statement is true if and only if the state of affairs described (or referred to) by the statement is a fact, and is false if that is not the case. (This has been correctly rendered as a version of the so-called correspondence theory of truth – a conception against which Frege argued[18] most emphatically –, but, of course, the intersection between states of affairs semantics and correspondence theories of truth is clearly not empty[19].)

It is in this spirit that the famous Convention T and Tarski's fundamental notion of satisfaction are conceived. Thus, e.g., a so-called open sentence of the calculus of classes $x_k \subseteq x_l$ is satisfied by an infinite sequence s of classes if and only if it is the case that the k-th member of that sequence is a subclass of the l-th number, i.e., if the state of affairs assigned by the sequence s (more correctly, by a function r_s determined by s) to $x_k \subseteq x_l$ is the case. Similarly, in the predicate calculus a so-called open sentence $P(x_i)$ is satisfied by an infinite sequence s^* of individuals if and only if it is the case that the individual assigned by s^* (more correctly, by r_{s^*}) to x_i has the property referred to by P in the interpretation, i.e., if the state of affairs that the object assigned to x_i by r_{s^*} has the property

[16] See footnote 9.

[17] Tarski (1969), reprinted in Tarski (1986, pp. 399-423).

[18] Freg (1918), reprinted in Frege (1991, pp. 362).

[19] They, however, do not necessarily coincide, since situations of affair semantics can also be seen in a loose way as correspondence theories, and since some somewhat narrow conceptions of correspondence theories (e.g., one that would requires some sort of isomorphism between expressions and their referents) could admit only some sorts of states of affair semantics.

assigned in the interpretation to *P* is the case. As is well known, the notion of truth for sentences is then obtained as a special case of the notion of satisfaction.

Hence, to fix our concepts a little, we can say that in a first-order model *M* different sequences of members of the domain of *M*, together with the fixed interpretation of relational and function constants, offer different evaluations of the open sentences of the (first-order) language. We identify states of affairs with (the results of) such evaluations. In the case of a second-order model *M**, different pairs (or, in general, *n*-tuples) of sequences (together, possibly, with the fixed interpretation of relational and function constants) offer different evaluations of the open sentences, and we identify states of affairs with (the results of) such evaluations. On the other hand, closed sentences (i.e., sentences with no free variables) have essentially one evaluation (or interpretation) in each model. Thus, e.g., a closed first-order sentence like 'For all x_i, if x_i is an *F*, then x_i is a *G*' is true in a model *M* if and only if it is the case that for all sequences *s* of objects of the domain of *M,* if their *i*-th object has the property which is the intepretation of *F* in *M*, then it has the property which is the intepretation of *G* in *M*. The state of affairs that is referred to in *M* by such a statement is precisely the (complex) state of affairs that all objects in the domain of *M* that possess property *f* also possess property *G*. It should be clear that even if that is the case in a model *M* for the language under discussion, it does not have to be the case in another model M^-, in which the referent of the closed sentence is a different state of affairs, since the members of the domain of M^- are, in general, different from the members of the domain of *M*, as also are the interpretations of *F* and *G* in *M* and M^-. Similarly, a closed second-order sentence 'For any property X_i, there exists a property X_j such that for all individuals x_k, x_k has the property X_j if and only if it does not have the property X_i is true in a second-order model *M** if and only if for any property in the model there is a property in the model which is possessed by exactly those objects that do not have the first property. The state of affairs that is referred to in M^* by such a statement is the (complex) state of affairs that in M^* each property has a complementary property. As before, even if that is the case in a model M^* for the language under discussion. It does not have to be the case in a different model M^+ (e.g., in a Henkin model).

§4. ABSTRACT SITUATIONS OF AFFAIRS FOR MANY-SORTED FIRST-ORDER LANGUAGES

Although Tarski's seminal monograph on the semantics of formalized languages is worthily considered the basis of the semantics of those languages, this does not mean that it is adequate for all problems in that area. Firstly, although it was not Tarski's fault, most developments in the semantics of formalized languages occurred within the realm of classical first-order model theory, even though it should be evident that the variety of mathematical entities makes any account of a substantial part of mathematics in a usual first-order language looks very artificial. The belief in set theoretic reductions of mathematics and, more importantly, the enthronement of the Skolemian prejudice against higher-order logics are probably responsible for that development. Ironically, a famous theorem of Lindström that could be used by Skolemicists to canonize first-order logic opened the way for the systematic study of the model theory of the most varied extensios of first-order logic[20]. However, so far as we know, the problem posed by the semantic rendering of the interderivability of seemingly unrelated mathematical statements has not been addressed by any of those developments. It is precisely for this task that we intend to introduce abstract situations of affairs in the semantics of mathematics.

Since the uniqueness and the legitimacy of reductions (set-theoretic, categorical or whatsoever) in mathematics can be called into

---

[20] See, e.g., J. Barwise and S. Feferman (1985). See also C. C. Chang and H. J. Keisler (1990) especially pp. 127-135.

question, and, moreover, since although useful for many purposes, such reductions can throw more obscurity than light on issues like the one under discussion, we should admit from the very begining that there are different sorts of mathematical entities (and possibly properties and relations that are defined for some sorts of mathematical entities but not for the others). This clearly excludes from the outset usual first-order languages from becoming the language in which to formalize a substantial part of mathematics. (It is a consequence of such a conviction that in our discussion of interderivable statements we should avoid considering statements like the Löwenheim-Skolem-Tarski theorems, whose validity can be affected by such an exclusion.) Probably the most adequate language for such an investigation would be a higher-order many-sorted language. This would do justice both to the intuition that there is natural hierarchical order between some entities and properties in mathematics, and the intuition that there are diverse sorts of mathematical entities, e.g., a group and a topologial space, that could not be put in a hierarchical order without doing some violence.

For our illustrative purposes, however, we will by begin considering a first-order many-sorted language and then extend our discussion to a second-order many-sorted language. Hence, let us suppose that $L_1{}^S$ is a denumerable first-order many-sorted language with individual constants $c^i{}_1$, $c^i{}_2$,..., $1 \leq i \leq m$, the index i indicating that the constant is a name of sort i; individual variables $x^i{}_1$, $x^i{}_2$,..., $1 \leq i \leq m$, the index i indicating the sort of entities which the variable represents; relational constants $P^k{}_1$, $P^k{}_2$,..., for each $k \in \mathbb{N}$, where the index k indicates the arity of the relation; function letters $f^k{}_1$, $f^k{}_2$,..., for each $k \in \mathbb{N}$, where the index k indicates the arity of the function; and, finally, the well known logical connectives, quantifiers ranging only over individuals, and the usual auxiliary signs. Let us now consider an intepretation $M = \langle D, \langle R_i{}^n\rangle_{i\in\mathbb{N}}, \langle F_i{}^n\rangle_{i\in\mathbb{N}}\rangle$ where $D = D_1 \cup D_2 \cup \ldots \cup D_m$ is the domain of objects of the m sorts of entities corresponding to the m sorts of individual constants available in $L_1{}^S$; $\langle R_i{}^n\rangle_{i\in\mathbb{N}}$ is a set of n-ary relations for each $n \in \mathbb{N}$; and $<F_i^n>_{i\in\mathbb{N}}$, is a set of n-ary operations for each $n \in \mathbb{N}$. Let us consider an axiom system $Ax(L_1^S)$ in L_1^S, which we will not specify, but such that we can assume that it suffices to derive a substantial part of mathematics (e.g., that part of mathematics that does not depend on the Axiom of Choice). Let us also assume that M is a model of axiom system. It should be intuitively clear – and is also an immediate consequence of the converse of semantic completeness – that two statements σ_1 and σ_2 of L_1^S that are interderivable in $Ax(L_1^S)$ should be either both true or both false in M. In other words, two closed sentences interderivable in $Ax(L_1^S)$ should have exactly the same models.

Now, since the case of sentences that talk about groups only the subdomain D_g of D that consists of groups needs to be taken into account to establish their truth or false hood – and not, e.g., the subdomain D_p of D that consists of partial orders –, such sentences are true in M if and only if they are true in the partial submodel M_g of M such that $M_g = <D_g, <F_i^n>_{i\in\mathbb{N}}>$ (or, to make it more uniform: $M_g = <D_g, <R_i^n>_{i\in\mathbb{N}}, <F_i^n>_{i\in\mathbb{N}}>$, where for all $i, n \in \mathbb{N}$, $R_i^n = \emptyset$)[21]. Thus, let us consider two partial submodels $M_1 = <D_1, <R_i^n>_{i\in\mathbb{N}}, <F_i^n>_{i\in\mathbb{N}}>$ and $M_2 = <D_2, <R_i^n>_{i\in\mathbb{N}}, <F_i^n>_{i\in\mathbb{N}}>$ of M, and let us assume that τ_1 and τ_2 are open sentences of L_1^S such that τ_1 assigns a property to entities in D_1 and τ_2 assigns a different property to entities in D_2, e.g., τ_1 is $P_i^1(x_k^1)$ and τ_2 is $P_j^1(x_n^2)$. It can be the case that a sequence s_p^1 in D_1 satisfies the open sentence τ_1 in M_1 if and only if the sequence s_q^2 in D_2 satisfies the open sentence τ_2 in M_2, (E.g., if τ_1 and τ_2 are both contradictory, then a sequence s_p^1 satisfies τ_1 in M_1 if and only if a sequence s_q^2 satisfies τ_2 in M_2, since no sequence satisfies either.) Moreover, in the case of closed sentences σ_1 and σ_2, for which, as Tarski has shown. Either all sequences in a model satisfy them or none does, it can be the case that σ_1 is true in

21 In the case of partial orders it is the operations F_i^n, for all $i, n \in \mathbb{N}$, that are $= \emptyset$. To simplify, we are using the same notation R_i^n and F_i^n for the partial models, although it is clear that not all of those relations and functions are non-empty in the partial submodels.

M_1 if and only if σ_2 is true in M_2. (Once more the simplest examples are those of closed sentences that are either both contradictory or both logically true.) Now, since the members of $D\backslash D_1$ are relevant for the truth or falsity of a sentence σ_1 that contains constants or quantified variables only of the sort 1, and, similary, the members of $D\backslash D_2$ are irrelevant for the truth or falsity of a sentence σ_2 that contains constants or quantified variables only of the sort 2, if σ_1 is true in M_1 if and only if σ_2 is true in M_2, then σ_1 is true in M if and only if σ_2 is true in M. We can consider the set of all (closed) sentences σ^* in $L_1{}^S$ that are equivalent to σ modulo M. Those sentences, in general, talk about objects of different sorts and ascribe to them different properties and relations, and, hence, clearly refer to different states of affairs. We can now obtain a first approximation to our envisaged notion of an abstract situation of affairs by defining a doubly relativized notion of an abstract situation of affairs corresponding in model M to a sentence σ of $L_1{}^S$ as the equivalence class of all states of affairs referred to in model M by sentences equivalent to σ modulo M. (It should be clear that there are only two such equivalence classes modulo M and, thus, this notion in itself does not have much interest.)

To eliminate one kind of relativization in the above definition, we can consider all models of $Ax(L_1{}^S)$ and sentences $\sigma_1, \sigma_2, \ldots, \sigma_r$ in $L_1{}^S$ that are true in exactly the same models as σ (and, thus, false in exactly the same models), although they ascribe different properties and relations to objects of, possibly, different sorts, and thus, refer, in general, to different states of affairs. Those sentences are intuitively semantically equivalent, but not relative to any particular model. We can, thus, define the notion of the abstract simulation of affairs that corresponds to a sentence σ in $L_1{}^S$ as the equivalence class of all states of affairs referred to by sentences true in exactly the same models as σ (and false in exactly the same models). In this way, all sentences of $L_1{}^S$ – and of any other language considered in this paper – can be partioned into such equivalence classes, even though some of these equivalence classes may be 'degenerate' in the sense of being unit classes. Moreover, it should be clear that, although the notion of an abstract situation of affairs modulo M is uninteresting and trivial, the more general and absolute notion is not only a significant refinement of the relative notion, but also far from uninteresting. Let us consider now two (closed) sentences σ and ρ that are interderivable in $Ax(L_1^S)$, and let Mod (σ) denote, as usual, the class of models of σ. Then we can prove following theorem:

Theorem 1: The sentences σ and ρ are interderivable in $Ax(L_1^S)$ if and only if Mod (σ) = Mod (ρ) if and only if the abstract situation of affairs that corresponds to σ is identical with the abstract situation of affairs that corresponds to ρ.

Proof: (Since the first equivalente is a trivial consequence of semantic completeness, we only prove that two sentences are interderivable if and only if they correspond to the same abstract situation of affairs, which, after all, is what we are interested in.)

(i) Let us assume that σ and ρ are interderivable. Let us also assume, however, that the abstract situation of affair that corresponds to σ is not identical to the abstract situation of affairs that corresponds to ρ. Since abstract situations of affairs are equivalence classes of states of affairs, the states of affairs referred to by σ in each and every model are not equivalent to those referred to by ρ in each and every model. Hence, there is some model M of $Ax(L_1^S)$ such that σ is true in M but ρ is false in M (or ρ is true in M but σ is false in M). Thus, σ and ρ could not be interderivable, contrary to what was assumed. Hence, if σ and ρ are interderivable, that abstract situation of affairs that corresponds to σ in L_1^S is identical to the abstract situation of affairs that corresponds to ρ in L_1^S.

(ii) If σ is not interderivable with ρ, let us say, σ does not derive ρ, then there is a model M of σ which is not a model of ρ. But then the state of affairs referred to by σ in M is not equivalent to that referred to by ρ in M. Thus, σ and ρ do not correspond to the same abstract situation of affairs modulo M, since the states of affairs denoted by them in M do not belong to the same equivalence class modulo M. Hence, the abstract situation of affairs that corresponds to σ in $L_1{}^S$ is not identical to the abstract situation of affairs that corresponds to ρ in $L_1{}^S$.[22]

§5. ABSTRACT SITUATIONS OF AFFAIRS FOR MANY-SORTED SECOND-ORDER LANGUAGES

Let us now consider a many-sorted second-order language $L_2{}^S$. This language contains all the symbols of $L_1{}^S$, but in addition to it, it contains relational variables $X_1{}^{n,k}$, $X_2{}^{n,k}$,..., where $1 \leq k \leq t$ represents the sort of the relation, and $n \in \mathbb{N}$ represents the arity of the relation, and function variables $F_1{}^{n,j}$, $F_2{}^{n,j}$,...., where $1 \leq j \leq r$ represents the sort of the function, and $n \in \mathbb{N}$ represents the arity of the function. Thus, $X_9{}^{p,d}$ is the ninth p-adic relational variable of sort d, and $F_1{}^{q,m}$, the first q-adic function variable of sort m. Since $L_2{}^S$ is a second-order language, quantifiers can be applied not only to individual variables but also to relational and function variables. Let us assume that $Ax(L_2{}^S)$ is an axiom system in $L_2{}^S$ adequate to derive a substantial part of mathematics (e.g., that part of mathematics that does not depend on the Axiom of Choice). A model for $Ax(L_2{}^S)$ is a structure $M_2 = \langle D, D^*, \langle R_i^{n,k}\rangle_{i,\, n\in\mathbb{N},\, 1\leq k\leq t}, \langle S_i^{n,j}\rangle_{i,\, n\in\mathbb{N},\, 1\leq j\leq r}\rangle$, where $D = D_1 \cup D_2 \cup \ldots \cup D_m$, $D^* = \langle D^{n,p} :_{i,\, n\in\mathbb{N},\, 1\leq n\leq t}\rangle$ where $D^{n,p}$ is the domain of n-ary relations of sort p, t is the number of different sorts of n-ary relations for any $n \in \mathbb{N}$ and r is the number of different sorts of

[22] From the proof of the theorem one might be prone to believe that abstract situations of affairs should, simply, be identified with the corresponding classes of models. This, however, would be a mistake, as we shall see below. Abstract situations of affairs are equivalence classes of states of affairs, whereas the Mod(σ) are classes of structures.

n-ary functions for any $n \in \mathbb{N}$. (To avoid further notational complications caused by the inclusion of additional domains for functions, we are presupposing that the number of sorts of functions does not exceed the number of sorts of relations. Hence, we can assume that n-ary functions of sort i are special n+1-ary relations and, thus, belong to the domain of n+1-ary relations of the corresponding sort i.) To obtain definitions of the notions of abstract situation of affairs modulo M_2 and of abstract situations of affairs for sentences of $L_2{}^S$ similar to those obtained for sentences of $L_1{}^S$, we proceed essentially as before, i.e., via partial models, using the fact that satisfiability of an open sentence τ of $L_2{}^S$ in a model M_2 reduces to satisfiability in the partial submodel of M_2 that contains exactly the appropriate sorts of individuals, and relations or operations, whereas the surplus structure of M_2 is completely irrelevant. The only difference with respect to the definitions for sentences of $L_1{}^S$ is that for the definition of satisfaction in the second and higher-order cases we make use of many-rowed sequences. Specifically, for any concrete case we have to consider both sequences of individuals of the appopriate sort and sequences of relations (or operations) of the appropriate sort and arity. Although there would be denumerably many such sequences to be considered, namely, h sequences of individuals of the h different sorts, and for any $n \in \mathbb{N}$, k sequences of n-ary relations and j sequences of n-ary operations, for each of the k sorts of n-ary relations and each of the j sorts of n-ary operations, respectively, in each case only a finite (very small) number of sequences are relevant. Once we have considered open sentences τ_1 and τ_2 of $L_2{}^S$ such that τ_1 is satisfied by a sequence s_1 in the partial submodel $M_1{}^2$ of M^2 if and only if τ_2 is satisfied by a sequence s_2 in the partial submodel $M_2{}^2$ of M^2, we consider closed sentences σ_1 and σ_2 of $L_2{}^S$ such that σ_1 is true in $M_1{}^2$ if and only if σ_2 is true in $M_2{}^2$, and immediatly obtain that σ_1 is true in M^2 if and only if σ_2 is true in M^2, since the surplus structure added by M^2 to $M_1{}^2$ is irrelevant for the truth or falsity of the sentences σ_1 and σ_2. (It should be clear that both here and in the first-order case we could proceeded by considering not two partial submodels M_1^2 and M_2^2 (in the first-order case, M_1 and M_2) but a partial submodel $M^* = D_1 \cup D_2, \langle R_i^{n,k} \rangle_{i,\, n \in \mathbb{N},\, 1 \leq k \leq t}, \langle S_i^{n,j} \rangle_{i,\, n \in \mathbb{N},\, 1 \leq j \leq r}$ – for the first-order case the symbols for the sorts of relations and functions are, of course, omitted – and would have obtained the same definitions and results.)

We can now consider the set of all sentences $\sigma_1, \sigma_2, \ldots, \sigma_r$ of L_2^S equivalent modulo M^2 to a (closed) sentence σ, but referring to a different states of affairs – since, in general, they talk about objects of different sorts and ascribe to them different properties and relations of corresponding different sorts –, and then define the abstract situation of affairs modulo M^2 that corresponds to σ as the equivalence class of all states of affairs referred to in M^2 by sentences equivalent to σ modulo M^2. (It should be clear that once more we only have two different equivalence classes modulo M^2.) Moreover, we can consider all models of $Ax(L_2^S)$ and sentences $\sigma_1, \sigma_2, \ldots, \sigma_r$ that are true in exactly the same models as a (closed) sentence σ (and, thus, false in exactly the same models), although they refer, in general, to different states of affairs – since they talk about entities of different sorts and ascribe to them properties and relations of corresponding different sorts –, and then define the notion of the abstract situation of affairs that correspond to a (closed) sentence σ as the equivalence class of all states of affairs referred to by sentences true in exactly the same models as σ (and false in exactly the same models). Once again we have a partition of the set of sentences of the language – in this case L_2^S – and this more absolute notion represents a substantial refinement and a non-trivial improvement of the relative and trivial notion.

It is appropriate to ask if we have for L_2^S a theorem similar to that obtained for L_1^S, namely, that sentences σ and ρ are interderivable in an axiom system $Ax(L_2^S)$ for L_2^S if and only if Mod $(\sigma) = (\rho)$ if and only if the abstract situation of affairs corresponds to σ is identical to the abstract situation of affairs that corresponds to ρ. At first sight, some caution seems pertinent, since when we talk about the class of models of a sentence σ in a second-order language we have in mind the full second-order structures, with as many relations and operations as possible. For such models there is no second-order semantic completeness and thus, although if σ and ρ are interderivable sentences in *Ax*(L_2^S), then Mod (σ) = Mod (ρ), it is not the case if Mod (σ) = Mod (ρ), then σ and ρ are interderivable in *Ax*(L_2^S). However, since for the definition of the notion of the abstract situation of affairs that corresponds to a given sentence σ only partial models were essentially used, whereas all surplus structure was irrelevant, no matter whether the added structure amounts to a full (many-sorted) second-order structure or to a (many-sorted) second-order Henkin structure, the difference between taking into account every (many-sorted) second-order structure that is a potential model of (i.e., a structure adequate for) the relevant sentences and taking into account exclusively all the full (many-sorted) second-order structures is itself irrelevant for the discussion of abstract situations of affairs[23]. Thus, we have the following theorem:

Theorem 2: The sentences σ and ρ are interderivable in *Ax*(L_2^S) if and only if the abstract situation of affairs that corresponds to σ is identical to the abstract situation of affairs that corresponds to ρ.

---

[23] It should be pointed out that we do not consider Henkin's semantics for second-order logic as a legitimate rival of standard semantics, but as a deviant semantics that in fact reduces second-order logic to first-order logic. It is simply that for our purposes, since satisfiability in a many-sorted second-order model reduces essentially to satisfiability in some partial models, it is irrelevant if the surplus structure amounts or not to a full model, and this fact allows us to use Henkin's Semantic Completeness theorem in the proof of the following theorem.

Proof:

(i) Let us assume that σ and ρ are interderivable. Let us also assume, however, that the abstract situation of affairs that corresponds to σ is not identical to the abstract situation of affairs that corresponds to ρ. Thus, since abstract situations of affairs are equivalence classes of states of affairs, the states of affairs referred to by σ in each and every model are not equivalent to those referred to by ρ in each and every model. Hence, there is some model M^2 of $Ax(L_2{}^S)$ such that σ is true in M^2 but ρ is false in M^2 (or ρ is true in M^2 but σ is false in M^2). But then σ and ρ could not be interderivable, contrary to what was assumed. Thus, if σ and ρ are interderivable, the abstract situation of affairs that corresponds to σ in $L_2{}^S$ is identical to the abstract situation of affairs that corresponds to ρ in $L_2{}^S$.

(ii) Suppose now that σ and ρ are not interderivable. Specifically, let us assume that σ does not derive ρ. Then there is a partial model M^* for $L_2{}^S$ which has the relevant structure to interpret both σ and ρ and is such that it is a model of σ but is not a model of ρ. We can expand M^* to a model M^2 of $Ax(L_2{}^S)$ since, if this were not so, $Ax(L_2{}^S) \cup \{\sigma\}$ would be inconsistent and, thus, would derive ρ, contrary to what was assumed. But since ρ is false in M^2, the state of affairs referred to by σ in M^2 is not equivalent to that referred to by ρ in M^2. Hence, the abstract situation of affairs that corresponds to σ in M^2 is not identical to the abstract situation of affair that corresponds to ρ in M^2. Thus, the abstract situation of affairs that corresponds to σ in $L_2{}^S$ is not identical to the abstract situation of affairs that corresponds to ρ in $L_2{}^S$.

§6. ABSOLUTE ABSTRACT SITUATIONS OF AFFAIRS

There is still a sort of relativization in our considerations that we would like to eliminate, namely, the relativization to the language $L_2{}^S$ (or $L_1{}^S$) in which the axiom system is expressed. This is really an inconvenience at least for the following reasons: (i) when we go from a language L_n^S to a language $L_n^S{}_{+1}$ it is clearly possible that the added expressive power will allow us to express new statements interderivable with a given sentence σ, e.g., with the Axiom of Choice, in some axiom system in a language L_{n+m}^S for some $m > 1$; and (ii) as it is well known, although in ZF^1 the Axiom of Choice is interderivable with the Löwenheim-Skolem-Tarski Theorems, in ZF^2 those theorems are false and, thus, no longer interderivable with the Axiom of Choice. (This is, of course, one of the main reasons for the inadequacy of first-order languages for the present discussion.)

However, putting aside those 'circunstantial' equivalents of the Axiom of Choice, which essentially depend on a kind of 'degeneracy' of first-order logic, the natural tendency should be that as we strengthen the language (and the corresponding axiom system), e.g., from L_2^S to L_n^S for $n > 2$, the sets of interderivable statements in the corresponding axiom systems will either augment or remain constant, forming, thus, a non-decreasing monotone (although not strictly monotone) sequence. Since all those languages are denumerable and the cardinality of the sets of interderivable statements cannot exceed the cardinality of the corresponding language, for any $n \in \mathbb{N}$, the cardinality of the set of interderivable statements is at most denumerable. Hence, the sequence $Ax(L_2^S)$, $Ax(L_3^S)$,..., $Ax(L_n^S)$,..., $Ax(L_\omega^S)$,..., where $L_\omega^S = \cup L_n^S$ and $Ax(L_\omega^S) = \cup Ax(L_n^S)$, cannot strictly increase indefinitely and there must exist a language L_k^S such that for no $j > k$ is the set of interderivable statements in $Ax(L_j^S)$ greater than the set of interderivable statements in $Ax(L_k^S)$. Thus, there is no sentence expressible in L_k^S such that it is a member of the set of interderivable statements in $Ax(L_j^S)$ without being a member of the corresponding set of interderivable statements in $Ax(L_k^S)$. Hence, L_k^S is a sort of fixed point, and this result can be seen as a fixed point theorem. Thus, we have obtained the following result:[24]

[24] Although we are considering, as usual, languages with a denumerable vocabulary, such a restriction is inessential. What is essential is that the set L of all those languages is an inductive coset and since the obvious mapping from L

Theorem 3: There is a least language L_k^S in the sequence L_2^S, L_3^S,..., L_n^S,... such that all interderivability results are theorems of $Ax(L_k^S)$, i.e., for any sentences σ and ρ expressible in any of the languages in the sequence and, thus, for any two such sentences expressible in L_k^S, σ and ρ are interderivable in some $Ax(L_n^S)$ if and only if they are interderivable in $Ax(L_k^S)$.

Hence, we can now finally obtain the absolute notion of an abstract situation of affairs we had been looking for. Thus, the abstract situation of affairs that corresponds to a (closed) sentence σ – in symbols: $<\sigma>$ – is the abstract situation of affairs that corresponds to σ in $Ax(L_k^S)$.

§7. CONCLUDING REMARKS

Finally, it should be mentioned that the equivalence relation of corresponding to the same abstract situation of affairs in this absolute sense induces a partition in the set of all sentences of the fixed point language L_k^S. Moreover, the abstract situations of affairs originate a Boolean algebra of propositions, thus, a so-called Lindenbaum algebra on L_k^S. To see this, one can introduce a relation '≡' between (closed) sentences such that $\sigma \equiv \rho$ if and only if $<\sigma> = <\rho>$, i.e., if and only if they correspond to the same abstract situation of affairs. One can also define a relation of implication ↠ between abstract situations of affairs, and write $<\sigma> \twoheadrightarrow <\rho>$, if and only if $\sigma \vdash \rho$ but ρ does not derive σ, inducing, thus, a partial order between abstract situations of affairs, which is essentially the inverse of the relation of implication. We define $<\sigma> = 1$ if and only if $Ax(L_k^S) \vdash \sigma$ and $<\sigma> = 0$ if and only if $Ax(L_k^S) \vdash \neg\sigma$. These definitions allow us to prove that the set of

to L is clearly monotone, we can apply the Least Fixed Point Theorem to obtain the desired fixed point. For an elegant treatment of fixed point theorems and related issues, see Moschovakis (1994).

equivalence classes of statements induced by the abstract situations of affairs form a Boolean algebra of propositions, i.e., a so-called Lindenbaum algebra. In this way, we can express the relation obtaining between, e.g., the Axiom of Choice and its many equivalents and the Ultrafilter Theorem and its many equivalents. As it is well known, the Axiom of Choice implies the Ultrafilter Theorem, but it is not implied by it. Thus, <The Axiom of Choice> $\twoheadrightarrow$ <The Ultrafilter Theorem>. Moreover, the class of all sentences derivable in $Ax(L_k^S)$ from a sentence σ, i.e., Con $\vdash^{(\sigma)}$, is the class of all sentences ρ such that $\langle\sigma\rangle \twoheadrightarrow \langle\rho\rangle$.[25]

REFERENCES

BARWISE, J., FEFERMAN, S. (eds.). *Model Theoretic Logics.* New York/ Berlin: Springer, 1985.

CHANG, C. C., KEISLER, H. J. *Model Theory,* third edition. Amsterdam: North Holland, 1990.

FREGE, G. "Über Sinn und Bedeutung". *Zeitschrift für Philosophie und Philosophische Kritik,* vol. 100, pp. 25-50, 1982.

_____. "Der Gedanke". *Beiträge zur Philosophie des deutschen Idealismus* I, pp. 58-77, 1918.

_____. *Grundgesetze der Arithmetik* [1893-1903]. Repr. Hildesheim: Olms, 1962.

_____. *Begriffsschrift* [1879]. Repr. Hildesheim: Olms, 1964.

_____. *Wissenschaftlicher Briefwechsel.* Edited by G. Gabriel, H. Hermes, F. Kambartel, C. Thiel and A. Veraart. Hamburg: Felix Meiner, 1976.

---

[25] Part of a first draft of this paper, under the title "Abstract Situations of Affairs for the Semantics of Mathematics", was read at the 10th International Congress of Logic, Methodology and Philosophy of Science in Florence in August 1995.

______. *Die Grundlagen der Arithmetik* [1984]. Centernarausgabe. Edited by Christian Thiel. Hamburg: Meiner, 1986.

______. *Kleine Schriften*. Edited by U. Angelelli, second edition. Hildesheim: Olms, 1911.

HUSSERL, E. *Logische Untersuchungen* [1900-1901]. Husserliana, vol. XVIII and XIX. The Hague: Kluwer, 1975 and 1984a.

______. *Erfahrung und Urteil* [1939], fifth edition. Hamburg: Felix Meiner, 1976.

______. *Einleitung in die Logik und Erkenntnistheorie*. Husserliana, vol. XXIV. Dordrecht: Kluwer, 1984b.

______. *Vorlesungen über Bedeutungslehre*. Husserliana, vol. XXVI. Dordrecht: Kluwer, 1987.

MOORE, G. H. *Zermelo's Axiom of Choice*. New York: Springer, 1982.

MOSCHOVAKIS, Y. N. *Notes on Set Theory*. New York: Springer, 1994.

ROSADO HADDOCK, G. E. "Remarks on Sense and Reference in Frege and Husserl". *Kant-Studien*, 74, nº 4, pp. 425-439, 1982.

______. "On Frege's Two Notions of Sense". *History and Philosophy of Logic*, 7, nº 1, pp. 31-41, 1986.

______. "On Husserl's Distinction between State of Affairs *(Sachverhalt)* and Situation of Affairs *(Sachlage)*". In: Seehbohm et al. (eds.) (1991), pp. 31-48.

______. "Interderivability of Seemingly Unrelated Mathematial Statements and the Philosophy of Mathematics". *Diálogos*, 59, pp. 121-134, 1992.

RUBIN, H., RUBIN, J. E. *Equivalents of the Axiom of Choice* II. Amsterdam: North Holland, 1985.

SEEBOHM, T. S. et al. (eds.). *Phenomenology and the Formal Sciences.* Dordrecht: Kluwer, 1991.

SHAPIRO, S. *Foundations without Foundationalism.* Oxford: Oxford University Press, 1991.

TARSKI, A. *Pojecie prawdy w jezykach nauk dedukcyjnych.* Warsaw, 1933.

_______. "Der Wahrheitsbegriff in den formalisierten Sprachen". *Studia Philosophica*, I, pp. 261-405, 1935.

_______ "The Concept of Truth in Formalized Languages". In: A. Tarski *Logic, Semantics, Metamathematics,* second edition. Edited by John Corcoran. Indianapolis: Hackett, 1983.

_______. "Truth and Proof" [1969]. Repr. in Tarski (1986), pp. 399-423.

_______. *Collected Works,* vol. II. Boston: Birkhäuser, 1986.

VON KUTSCHERA, F. *Gotlob Frege.* Berlin: De Gruyter, 1989.