CDD: 511.3

ON THE MATHEMATICS OF LOGIC AND THE LOGIC OF MATHEMATICS

RODOLFO C. ERTOLA BIRABEN

Departamento de Filosofía
Facultad de Humanidades y Ciencias de la Educación
Universidad Nacional de La Plata
Calle 48 entre 6 y 7
1900 LA PLATA
ARGENTINA

rudolf@huma.fahce.unlp.edu.ar

À estimada Itala

Abstract: In this paper we deal with two different approaches to logic, the Boolean and the Fregean. In particular, we give some reasons to explain certain ignorance of the Boolean approach at some Philosophy Departments, particularly in the teaching of logic. After giving some reasons for the convenience of combining both approaches, we recommend material on the Boolean approach to be included in the mentioned contexts.

Key-words: Logic. Boole. Frege. Mathematics. Algebra.

INTRODUCTION

This paper deals with two different approaches to logic, both of which began in the nineteenth century (Leibniz will be ignored): the approach that started with Boole, which we shall call *Boolean* and the approach that started with Frege, which we shall call *Fregean*.

The contents may be useful to logicians interested in historical aspects of their discipline, to philosophers interested in basic or historical matters regarding logic and also to anyone interested in logic. We will avoid formulas as much as possible.

The reader is advised not to expect too much precision nor many details. In any case, the reader interested in more information is advised to take a look at the material included in the references.

After recalling briefly the general character of the two mentioned approaches we shall be concerned with examining certain ignorance of the Boolean approach at some Philosophy Departments, particularly in the basic teaching of logic.

By way of conclusion we shall suggest the inclusion of the Boolean approach in the introductory courses at the mentioned places where this approach is still excluded.

We shall not be concerned with approaches other than the two mentioned. For instance, nothing will be said about the Hilbert approach to logic. We shall also say nothing about first order predicate logic or second order logic, restricting ourselves to propositional logic.

1. THE BOOLEAN APPROACH

This approach was begun by the English mathematician George Boole (1815-1864) who published in 1847 a short book entitled *The Mathematical Analysis of Logic*, which was followed in 1854 by another treatise bearing the name *An Investigation of the Laws of Thought*. Later on, other English mathematicians such as Jevons (1835-1882) and De Morgan (1806-1871) and also the North-American mathematician Charles Peirce (1839-1914) and the German mathematician Ernst Schröder (1841-1902) made interesting contributions. In the twentieth century the Boolean approach continued to be developed, for instance, by the polish mathematicians Tarski and Lindenbaum. Today this approach is pursued in many countries including some places in Latin-America.

The Boolean approach is algebraic in nature and has the aim of examining and developing logic from a mathematical point of view, as the title of the already mentioned first book by Boole clearly suggests. This also explains our use of the phrase "the mathematics of logic" in the title of this paper. So, what we have here is a mathematician elaborating logic with algebraic tools.

The basic algebraic tool is the concept of an *algebra*, particularly the concept of a *Boolean algebra* in the case of the so called *classical* propositional logic. In the remains of this section we shall try to explain briefly what this means using current concepts and terminology, that is, we shall *not* employ Boole's or some other nineteenth century way of conceptualization, which would not be easy to understand. An *algebra* in its modern form is just a non empty set A with elements a, b, etc., called the *carrier* set, together with certain *operations*, that applied to the elements in A give other elements (this is the usual concept of *function* in mathematics). In order to obtain a particular algebra, these operations will be related to each other by certain *equations*. For instance, if we are interested in working some particular logic algebraically, we might define an algebra $(A, \wedge, \vee, \rightarrow, \neg)$, where the symbols represent certain desired operations and state e. g. the equation that $a \wedge a = a$, trying to reflect a property of the concept of *conjunction*, that is the concept involved in the use of "and" in mathematical discourse. We might also want to add more equations in order to make the operations $\wedge$, $\vee$, $\rightarrow$ and $\neg$ reflect some other properties of conjunction, disjunction, conditional and negation, respectively, as used in mathematical contexts. As a historical matter of fact, the equation given, currently called *idempotent law*, is exactly one of the "laws of thought" Boole stated as basic already in his first book. He also immediately noted that this equation does not hold in the case of numerical structures usual to a mathematician (see Boole (1951, Chap. III, 12)), as it is easy to see that it is not true in general that $a + a = a$ or that $a \times a = a$, e.g. if a is a natural number.

As a final remark for this section we add that it seems interesting from a historical point of view that in the case of the operation of disjunction (corresponding to "or" in mathematical contexts) Boole preferred *exclusive* instead of the now ubiquitous *inclusive* disjunction. This fact also had a not very convenient consequence from a mathematical point of view, to wit, that Boole´s system lacked the usual phenomenon of *duality* between conjunction and disjunction. But this defect was soon repaired by the already mentioned Jevons; in Schröder, for instance, the phenomenon of duality between conjunction and disjunction is to be found explicitly and systematically treated (see e.g. Schröder (1879)).

2. THE FREGEAN APPROACH

This approach began with the German mathematician Gottlob Frege (1848-1925), more precisely with a book published in 1879 entitled *Begriffsschrift*, word that can approximately be translated into English as *Conceptual Script* or *Conceptual Notation*. This approach was continued with other books by the same author and also by the monumental *Principia Mathematica* by the English mathematicians Alfred Whitehead (1861-1947) and Bertrand Russell (1872-1970). It is to be especially noted that Boole's first book was published one year *before* Frege's birth!

The Fregean approach is logico-philosophical in nature as the aim of the mentioned authors was to develop what has been called *The Logicist Program*, that is, a program to develop mathematics from logic (where logic includes what is now called *set theory*), as the title of the mentioned work by Whitehead and Russell partially suggests. This also explains our use of the phrase "the logic of mathematics" in the title of this paper. So, what we have here is *philosophically minded* mathematicians pursuing to base mathematics on logic.

The Fregean approach may be summarized as follows: we begin with a set of propositional letters *a*, *b*, etc. that combined with symbols representing the usual concepts of conjunction, disjunction, conditional and negation (we may refer to the mentioned symbols in the same way we referred to operations in the Boolean approach, that is, using $\wedge$, $\vee$, $\rightarrow$ and $\neg$, so here $\wedge$, for example, is a *symbol*, not an operation) will allow us to achieve the concept of *formula*. Then selecting some special *formulae* as *axioms* and using an *inference rule* we are able to define the concept of *derivable formula* (usually and not very conveniently called *theorem* ; this use is most probably a vestige of logicism). For instance, we might have as axioms the *formulae* $(a \wedge a) \rightarrow a$ and $a \rightarrow (a \wedge a)$. But note that even having the given axioms, the *formulae* $a \wedge a$ and a are *different* objects, that is, it would be false to say that $a \wedge a$ equals a, even though that *is* true in the Boolean approach!

As a final remark for this section we add that it seems interesting from a historical point of view, that Frege developed his system just using the symbols for the conditional and the negation. This is possible because the mentioned symbols constitute what is currently called a *complete* set of connectives. We also note that Frege and Whitehead-Russell used *implicitly* a second rule of inference for *substitution* of *formulae*.

3. PREDOMINANCE OF THE FREGEAN APPROACH

We have experimented in Philosophy Departments interested in logic a tendency to almost completely ignore the Boolean approach. By way of example we mention the emphasis of Quine in many introductory courses to the subject. Specifically, as a very illustrating example, let us recall the first sentence of the preface of perhaps his most influential text, that is, *Methods of Logic*: "Logic is an old subject, and since 1879 it has been a great one" (see Quine (1950) and other editions). Quine, of course, is referring to the date of publication of the first book by Frege, as if nothing important in logic had happened before in the nineteenth century, that is, completely ignoring Boole and his followers.

It is important to add that the text referred to had *many* editions and that it was not just another text to be used in Philosophy Departments, but that it was *very* influential in some of those contexts, where students usually do not have the possibility of comparing different approaches to logic by themselves, not being trained mathematically or ignoring some possibly relevant bibliography.

It is no surprise that algebraic aspects of logic are almost completely ignored in the mentioned text. Fortunately, for instance, the North-American mathematician and philosopher Hilary Putnam has called attention to this omission (see Putnam, 1982).

It is natural to ask oneself *why* has this happened, that is, why the Boolean approach or the algebraic aspects of logic have been ignored in the contexts already mentioned? The only answer seems to be that Frege and Russell's attitude was more interesting philosophically even if the logicist program would turn out to be a very bad idea philosophically speaking. We should also add that Quine maybe was the most prestigious and influential North-American philosopher in the second half of the twentieth century, and so, people working at a Philosophy Department with no or poor mathematical training perhaps gave to his opinion much more weight than it really deserved, with regretful consequences.

4. CONVENIENCE OF COMBINING BOTH APPROACHES

In this section we shall consider some reasons that favor the combination of both approaches in introductory courses directed to almost any public, without distinguishing, at least with respect to the contents, philosophy students from mathematics or computer science students.

One reason why the exclusive Fregean approach seems unreasonable is that it was developed in the context of the logicist program, a program that has lost interest during the twentieth century. On the other hand, the algebra of logic has experienced enormous growth.

Secondly, there is an example of a very simple situation in logic where the convenience of combining both approaches appears very clearly. Let us consider the usual introduction and elimination rules for disjunction in the Gentzen natural deduction system. If algebra is ignored, then it will also be ignored the fact that the above mentioned rules express that the disjunction of two propositions *A* and *B* is nothing less, nothing more, the *supremum* of *A* and *B*! And it will also be ignored that the logical concept of disjunction is *analogous* to the concept of *minimum common multiple* in the set of natural numbers ordered by the divisibility relation! This last statement is also useful to see that knowing the algebraic aspects of logic is also useful to establish relationships between logic and arithmetical knowledge obtained at an early educational stage. But, of course, a logicist logician may not like to know that disjunction is so similar to such a mathematical concept as the minimum common multiple! Whoever is interested in more details as regards the contents of this paragraph may very fruitfully see Raggio (1979).

Thirdly, we call the reader's attention to a historical aspect. According to Putnam (1982, p. 298), Whitehead learned quantification through Peirce and his disciples, that is, one of the participants in the Fregean approach came to his knowledge of such a basic and important concept as quantification through members of the Boolean approach. This can also be useful to prove that both approaches were not developed totally independent from each other.

Ending this section, we will briefly mention one way to connect both approaches. The mechanism was devised by the already mentioned Tarski and Lindenbaum in the thirties of the twentieth century. Starting with an axiomatic system such as the one described in the section dedicated to the Fregean approach it is possible to obtain an algebra in the following way: define a binary relation $\equiv$ between *formulae* by the stipulation that, for any *formulae* φ and ψ, $\varphi \equiv \psi$ if and only if both $\varphi \to \psi$ and $\psi \to \varphi$ are derivable *formulae* (it is easy to see that this is an equivalence relation). Then form an algebra called the *Lindenbaum algebra* taking as carrier set the quotient set of the relation $\equiv$ (the members of this set are the so called *equivalence classes*, which are noted $[\varphi]$ for any formula φ and correspond to Boole's propositions) and defining operations for every connective in the logic as follows, for example, in the case of conjunction: $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$ (this is possible because if $\varphi \equiv \varphi'$ and $\psi \equiv \psi'$, then $\varphi \wedge \psi \equiv \varphi' \wedge \psi'$). Note our abuse of notation in the given equation, using the same notation "$\wedge$" to refer on the left hand side to the operation in the Lindenbaum algebra and on the right hand side to the symbol in the language of the logic. It is easy to see that the *Lindenbaum algebra* satisfies the equations defining a Boolean algebra. In particular, it will follow that $[\varphi \wedge \varphi] = [\varphi]$. The reader interested in more details is advised to see e.g. Dunn and Hardegree (2001), a book that includes lots of material in algebraic logic and has been a partial motivation for the writing of this paper.

CONCLUSION

By way of conclusion, let us ask: What is better, to teach logic keeping to pretensions, logicist, linguistic or other, which might just be fashionable and turn out to be wrong or generate endless or useless discussions, or to teach logic as any area of mathematics, making it play, *vis-à-vis* Philosophy, a role similar to the one Plato attributed to geometry? We think the second is the only alternative. A tentative list of algebraic material to be included in such courses is the following: Partitions and equivalence relations. Equivalence classes and quotient sets. Order relations. Algebras, Lindenbaum algebras. Homomorphisms. The reader interested in details is advised again to see e.g. Dunn and Hardegree (2001).

REFERENCES

BOOLE, G. *The Mathematical Analysis of Logic*. Reprinted. Oxford: Blackwell, 1948.

————. *An Investigation of the Laws of Thought*. Reprinted. New York: Dover, 1951.

DOFFI, M., MORENO, A. *Escritos Completos*. Buenos Aires: Eudeba, 2002.

DUNN, J. M., HARDEGREE, G. M. *Algebraic Methods in Philosophical Logic*. Oxford: Clarendon Press, 2001.

FREGE, G. *Begriffsschrift, eine der aritmetischen nachgebildete Formelsprache des reinen Denkens*. Halle: Louis Nebert, 1879.

PUTNAM, H. "Peirce the Logician". *Historia Mathematica*, 9, pp. 290-301, 1982.

QUINE, W. *Methods of Logic*. First edition. New York: Holt, 1950.

RAGGIO, A. "Processus Cognitifs Logiques". In: M. Doffi, and A. Moreno (eds.) (2002), pp. 287-292.

SCHRÖDER, E. *Der Operationskreis des Logikkalküls*. Leipzig, 1877.