

A PHENOMENOLOGICAL INQUIRY INTO THE CONCEPT OF SET*

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Abstract: The main concern of this paper is the justification of the axioms of Zermelo-Fraenkel set theory, either as true statements about a concept of set (if we consider set theory as a conceptual theory) or, alternatively, as true statements about abstract objects (considering ZFC as an objectual theory). I want to argue here that, in either case, set theory can be seen as a body of knowledge largely built on intuitive foundations (rather than an instrumental theory conceived mainly for pragmatic purposes, the needs of mathematics in particular). I call this inquiry “phenomenological” for it approaches its subject from the perspective of the intentional acts that originate sets as doubly dependent objects (of other objects – their elements – and of a subject – taken here simply as the abstract form of a real subject – who collects these elements into a set). Such an inquiry, I believe, brings to light the essential characters of sets as objects or, alternatively, the concept of set, which the axioms of the theory (or at least most of them) express.

* This paper further elaborates and (hopefully) improves some points in da Silva 2002. It is dedicated to Itala Maria Loffredo D’Ottaviano in her sixtieth birthday as a token of my friendship and admiration.

Mathematical theories, or at least *interpreted* theories, are *about* something¹. Since set theory – as axiomatized, for instance, in ZFC – is a mathematical theory (maybe the most fundamental one), the question is pertinent: what is it about? The obvious answer – set theory is about sets – is not satisfactory, for it is not clear what we mean by “set”, an object (the realm of which the theory of sets describes) or a concept (whose essential aspects the theory unveils). The first alternative induces further questions: are sets given independently of the theory or exclusively determined by it? What comes first, the theory or the domain of objects it is about? Clearly, the theory can only have the primacy if it is the theory of *something* that has the power to determine a domain of objects. It seems obvious that this something must be a concept (whose extension constitutes precisely the domain of sets). So, in the end, we have only two alternatives, either set theory describes a domain that is independent of the theory, or a concept, and derivatively the extension of this concept. Depending on which alternative we choose, the strategy to justify the axioms of ZFC as true assertions about sets will vary. In the first case we must ascend to the realm of sets *independently of the theory* (and explain how we did it); in the second we must bring to light the concept of set the theory is about in order to reveal the essential aspects in terms of which the axioms of the theory are justified (and also explain how this could be done).

I believe that the theory of sets expressed in ZFC is about a concept (and derivatively a domain of abstract objects) and, moreover, that this concept arises naturally when we consider sets as the outputs of ordinary acts of collecting. In order to justify this belief I present here the concept of set I think ZFC describes and explicitly derive from its main features some guiding principles in terms of which most (maybe all) axioms of this theory can be justified.

Ontological realists (or Platonists) with respect to objects – who believe that sets are independent abstract objects and that set theory is about

¹ One may argue that even non-interpreted, purely formal theories, are also about something, namely, formal *structures*.

them – are notoriously evasive when the issue of a direct access to the realm of sets – in terms of which we can justify the basic assumptions of the theory (its axioms) – is brought up. How do we know, for instance, that *all* the subsets of *any* given set exist? In terms of what insight into the domain of sets can we justify *this*? Without an appropriate account of the intuition of sets, how can the truth of the axioms of set theory be verified or justified? But conceptual ontological Platonists – who believe that set theory describes an independently existing concept (like Gödel) – are not much better off; the problem regarding the access to an independent concept still presses for an answer².

Platonists with respect to objects (like P. Maddy before her “naturalistic” turn) typically take an indirect path to ascend to the directly inaccessible parts of the domain of sets (although she thinks some reasonably small impure sets can still be directly accessed by the *senses*). The strategy is *first* to justify the *theory* in terms of its indispensability to mathematics, and *then* all the sets, properties of sets and set constructions that the theory requires in order to do its work. In short, for them, anything that *must* exist or be the case (for theoretical purposes) *does* indeed exist and is the case. Clearly this strategy is more akin to pragmatism than realism proper and more often than not develops into full-blown pragmatism, in which justification depends on necessity, sometimes embellished by other desiderata such as beauty, elegance and the like.

Now, supposing the axioms of our best-known set theory – the Zermelo-Fraenkel theory – are conceptual truths, are they *constitutive* or *descriptive*?³ In other words, are these axioms more or less arbitrary

² Some conceptual Platonists make a visible effort to solve this problem. Gödel, for instance, believed that Husserl’s notion of conceptual intuition could be useful in providing an acceptable account of set intuition. As will be clear below, I will follow more or less along this path in order to bring to light certain basic properties of the concept of set and the domain of objects associated with it.

³ Feferman (2000) uses the distinction between *foundational* and *structural* axioms more or less to the same purpose. Structural axioms are definitional by nature (hence constitutive); they are simply definitions of kinds of structures. Foundational axioms are (descriptive, I suppose) axioms for fundamental mathematical concepts

statements, selected by their role in providing a manageable and useful theory, which constitute, when put together, a conception of set specific to mathematics, devised in order to deal with mathematical problems – and maybe with problems in correlated scientific fields as well –, or are they descriptions of self evident aspects of a pre-theoretical concept, maybe borrowed from our common stock of mundane notions? Is Zermelo-Fraenkel set theory (which, as we know today, is incomplete) only the present stage in the history of the constitution of a concept – a work in progress –, or is it the (maybe necessarily) incomplete outcome of an intuitive grasp of a (maybe hopelessly vague) notion?

The history of the subject apparently suggests the first alternative. The only supposedly “natural” concept of set – a collection of arbitrary objects “held together” by an “intention” to keep them all together as a unity – revealed itself problematic (to say the least) when Frege reduced this “intention” to an arbitrary predicate (thus defining sets as extensions of predicates – the so-called *logical* conception of set). Cantor apparently (but only apparently) had a similar notion – for him too a set was simply a collection of objects that *could* be considered as a unity: many in one. But, in fact, as we will see later, Cantor’s conception differed substantially from Frege’s (which explains why the collapse of Frege’s theory of classes did not take Cantor’s along with it, despite the apparent contradictions that popped up in Cantor theory as well). The problem with the “natural” logical conception of set, of course, is that it is inconsistent. Cantor, on his turn, never presented a mathematically acceptable *definition* (or even a clear *explanation* of the concept) of set, and the historical development of set theory from Cantor’s original creation on was much more effectively

that underlie all mathematical theories (such as set, number and function). Potter (2004) considers two strategies of justification for the axioms of set theory, which he calls *regressive* and *intuitive*. The former takes for granted that it is the task of set theory to provide a suitable foundation for mathematics; hence any axiom that does the job is acceptable. The later enforces intuitiveness as the sole criterion for the acceptance of axioms. Regressive and intuitive strategies are linked respectively to constitutive and descriptive perspectives on the axioms of set theory.

conditioned by the resolution of mathematical problems⁴ than an increasingly clear insight into a given concept. Some philosophers of set theory⁵ saw this as an indication that the axioms of set theory are not *a priori* conceptual truths but rather basic presuppositions supported mostly by *extrinsic* rather than *intrinsic* justification, that is, by their consequences and usefulness rather than intuitiveness.

After all, the axioms of Zermelo-Fraenkel were not discovered all at the same time by the same person thinking hard about some pre-theoretical notion, but by different mathematicians, at different occasions, for different purposes. Nonetheless, this historical fact may be less of an overwhelming evidence for the constitutive approach than it seems. From a different perspective we may view the historical development of the set concept as the difficult unfolding of a conception so deeply entrenched in our consciousness, but yet so simple as to go unnoticed and demand a historical development in order to come out fully into the open. The axioms may be *obvious* conceptual truths, and still require some effort to be seen as such. Not everything that is obvious is *immediately* so. But if the descriptive account of the axioms of set theory is correct⁶, given that the only natural conception of set that we seem to have is inconsistent, what other notion is possibly being described by the axioms of set theory?

The point of view I want to substantiate here is that Cantor's original conception of set – nothing more than our mundane notion of a quantitatively determined collection of *given* objects taken as a unity – when properly scrutinized from a phenomenological perspective – that is, in terms

⁴ The problem concerning the well-ordering of arbitrary sets being, maybe, the most prominent. As we know, this was the question that launched the first axiomatization of set theory by Zermelo in 1908.

⁵ Maddy (1988), in particular.

⁶ Or at least partially correct. Different axioms of set theory can, of course, have different justifications; some can be intuitively true, others only pragmatically justified, and some, as I will argue below, justified in terms of the *nature* of conceptual mathematical theories – such as ZFC – instead of a direct insight into the relevant concept (which can be impotent to justify certain axioms, existential axioms in particular).

of the intentional acts involved⁷ – reveals fundamental conceptual truths and a picture of the extension of this conception – the universe of sets – robust enough to ground most, maybe all, the axioms of ZFC. I want to give here some substance to the following theses: 1) we do have a pregnant pre-theoretical concept of set; 2) this conception originates from a reflexive analysis of intentional acts of collecting; 3) this conception is the so-called *mathematical* (iterative) conception of set (according to which sets are disposed in a hierarchy of levels of dependence); 4) the relation of dependence between sets is of an ontological or metaphysical nature (the hierarchy of sets is not a temporal “construction”); and 5) we can justify most, maybe all, axioms of ZFC by an analysis of either this concept of set or the domain of sets generated by acts of collecting considered from a purely formal perspective⁸.

Stephen Pollard (Pollard 1990), although recognizing that there is a mundane notion of set, argues that the *mathematical* theory of sets is not founded and does not describe this commonsensical notion. He goes through the axioms of Zermelo-Fraenkel set theory one by one in order to show that each one of them is at odds with our (supposedly) only mundane conception of set. For instance, there cannot be an empty set, for it is a piece of common sense that a set disappears whenever its members vanish. Not to mention infinite sets, of course, obviously offensive to the common sense. But the argument Pollard prefers is that commonsensical set formation cannot be iterated. An army may be composed of divisions, but it

⁷ Phenomenological analysis can focus either on intentional acts themselves (*noetic* analysis) or their objectualities (*noematic* analysis). Here, noetic analyses focus on acts of collecting and noematic analyses on sets – as objects generated in acts of collecting. These analyses provide, respectively, the foundations for a higher-level act of formal abstraction (that gives us the formal structure of real acts of collecting) and a higher-level act of essential intuition (that gives us a concept of set).

⁸ I oppose this view, in particular, to those (represented, for instance, by Penelope Maddy in (Maddy 1988)), who believe that the axioms of ZFC do *not* “enjoy a preferred epistemological status” for supposedly following “directly from the concept of set” (id. *ibid.* p. 482).

is still made up of individual soldiers. In few words, according to the commonsensical notion, he argues, sets have the character of mereological assemblages, which have parts but not elements and are nothing over and above their constitutive parts.

All this is uncontroversial, of course, but what is dubitable is that this so-called commonsensical set concept is the only non-mathematical, pre-theoretical, intuitive conception of set that can be put at the basis of the mathematical theory of sets. Pollard – together with many other philosophers of mathematics – believe that the creation and development of a mathematical theory – set theory in particular – is exclusively induced by mathematical concerns, and that it is pointless to look for some non-mathematical notion out of which the corresponding mathematical theory develops.

Ironically, this point of view does not go uncontested even by mathematicians, including some of the greatest. Hermann Weyl, for instance, in his “The Continuum” says explicitly that analysis – that is, the theory of the mathematical continuum – must reflect and describe as accurately as possible the intuitive geometrical continuum as presented, for instance, in the flow of time. The mathematical continuum is a model, he says, whose right of citizenship in mathematics is not forever granted, and whose correctness must be evaluated by its ability to capture the *relevant* and *mathematically interesting* features of the intuitive continuum⁹.

What Pollard does not see is that in ordinary life our attention is drawn to the objects we collect, not to the *act* of collecting. This can only be a focus of interest by means of an act of reflection we usually do not perform in ordinary circumstances. He ignores also that there can be no set theory if we do not shift our attention from the objects that are collected to their sets, viewed now as *new objects*. As soon as we reflect on the act of collecting and its products, seen as *new objects*, we are on the way to an intuitive conception of set robust enough to provide intuitive grounding to a

⁹ As we can see, Weyl suggests a mixture of intuitiveness and pragmatism, regressive and intuitive strategies for the justification of the axioms of analysis.

mathematical notion of set and some relevant aspects of its theory. My claim is that mathematical theories of sets, in particular axiomatic theories such as Zermelo-Fraenkel, can indeed, for some extent at least, be seen as unfolding an intuitive, natural, pre-theoretical notion, *regardless of the actual historical development of the theory*.

Let me put into words what I see as the most fundamental aspects of the conception of set that arises from reflecting on acts of collecting: 1) sets are multiplicities of objects that can be seen *themselves* as objects – sets are many made into one; 2) (essential for the justification of the axiom of replacement) sets are *numerically determinate* multiplicities, as opposed to collections that are not themselves objects (by this I mean that sets – and sets only – have a well determined *quantity* of elements); 3) sets are *dependent* objects with regard to their elements. It is not hard to see how these aspects of the concept of set necessarily emerge as soon as we reflect on *acts of collecting*. Firstly, acts of collecting can be iterated; therefore its products – sets – can themselves be collected. Secondly, acts of collecting presupposes that we have something to collect; so sets presuppose their elements, or, in other words, sets *depend ontologically* on their elements. (Anyone not sympathetic with my approach would feel the urge at this point to observe that if the concept of set *really* originated from reflecting on the essential aspects of acts of collecting there would be no justification for an empty set¹⁰. But, as Husserl observed, the *intention* to perform such an act – the intention to collect – can be *frustrated*. The empty set is nothing but the outcome of a frustrated act of collecting¹¹. Incidentally, my approach is also very convenient to bring to light the difference between an object and its

¹⁰ It is interesting to notice that Dedekind, Husserl, Zermelo and Gödel, among others, showed different degrees of resistance to the postulation of the existence of an empty set. Dedekind, as well as Husserl in an early period of his philosophical development, took the empty set as a convenient fiction; Zermelo and Gödel to a lesser degree also endorsed this point of view.

¹¹ Usually in set theory the existence of an empty set can be demonstrated by specifying a subset of any existing set by means of an *absurd* property. What is this if not an intention to collect frustrated by the absurdness of the collecting property?

singleton: the singleton is the product of an act of collecting whereas the single object is not. An object and its singleton are two *phenomenologically distinct* intentional objectualities.) Thirdly, acts of collecting can only be performed on *given well-determined* totalities; hence, sets are *quantitatively* determined collections of objects, since it is obvious that no collection can be well determined without being also quantitatively or numerically determined.

But we must be careful here. I do *not* claim that our concept of set can only be instantiated by (and set theory can only admit the existence of) sets that can be “assembled” by *real* acts of collecting performed by *real* agents in *real* time. Mathematics, as usually understood, has nothing to do with real processes, but only with the *formal* structure of real processes; mathematics is a *formal* science. This means that set theory in particular, as a *mathematical* theory, is entitled to admit the existence of any set that can be assembled in a *formally correct* act of collecting, regardless of any formally irrelevant constraints imposed on it (such as physical or temporal limitations). From a strictly formal perspective an act of collecting is simply a well-ordered sequence of points (the pure form of an instant). I will express this equivalently by saying that a set exists if it can *in principle* exist. So, as a mathematical (hence, formal) theory, set theory is allowed to admit the existence of any set that can in principle exist.

We can also, alternatively, take set theory as a conceptual theory whose concept emerges from reflecting on acts of collecting. This amounts, as we have seen above, to a notion of sets as dependent, quantitatively well-determined objects (that is, collectable items themselves). Axiomatic set theory – such as Zermelo Frankel –, seen as a conceptual theory, is, at least in part, the spelling out of this idea into a set of conceptual truths – the axioms of the theory (which are then analytically true, or true of the concept). As we can see, from the phenomenological perspective, there are two strategies we can adopt in order to justify ZFC (these strategies are, of course, related, and both are descriptive in nature): to consider ZFC as the theory of a manifold of objects (an objectual theory) which exist provided only they are the outcome of acts of collecting possible *in principle*, that is,

that are *formally* correct, or as the theory of a concept of set brought to light by reflecting on the essential aspects of the processes by means of which sets come to life: acts of collecting. The justification for the axioms of ZFC that I present here will consider both alternatives.

A good start point for our enterprise is to see what Husserl has to say about these matters¹². And he has some interesting things to say both in *Philosophie der Arithmetik* (1891) and *Erfahrung und Urteil* (posthumous). Despite the time gap that separates these two works – and some minor differences in treatment – they share a common view on the nature of the concept of set: by their very *essence*, sets originate in second order reflexive acts of collecting (no matter how abstractly considered) that take the objects of first order acts of “plural assembling”: *pluralities*, and make them into singular objects: *sets* (Husserl 1954, pp. 293-294). Husserl also says that “sets can, on their turn, also be collected with other sets, constituting sets of a higher order”, but that, nonetheless, “*every set must be conceived as being a priori susceptible of being decomposed in its ultimate members, which are then no longer sets*” (Husserl 1954, pp. 294-295, emphasis in the original). Hence, according to Husserl: 1) there are no sets in the domain of pure passivity; in order to be constituted as higher level categorial objectivities sets demand the reflexive turning of consciousness to pluralities constituted in lower level acts of plural assembling to make them into unities; 2) so, sets are both many and one, depending on how consciousness seize the members of a multiplicity, as many in a first level act of assembling or as one in a second level act of unification; 3) being unities sets can be collected into other sets and 4) sets are, *by their very essence*, grounded, that is, well-founded; they are *a priori* decomposable into *urelemente*.

This last point in particular deserves some comments. Being categorial objectivities, as Husserl believes they are, sets require a

¹² Husserl was a close friend of Cantor’s during their years in Halle, Cantor even acted as a member in the committee of Husserl’s *Habilitation*, approving his thesis on the philosophy of arithmetic, which took for granted Cantor’s ideas on the nature of sets and cardinal numbers and the processes by means of which they are obtained.

foundation, a basis of *pre*-constituted pluralities on which they *depend*. So, since the series of relative dependence cannot retrocede forever, Husserl thinks, we cannot conceive sets that are not decomposable into ultimate components that are not sets. The fact is that, according to Husserl, higher order categorial objectivities *necessarily* depend on *relatively independent* lower order objectivities on which they are founded; but, considering that the foundational series refers back to a series of *acts* (of a subject in general), it cannot retrocede infinitely (for *any* series of acts is well ordered); hence, sets depend ultimately on non-sets (*no matter how abstractly sets are considered in mathematics*). Well-foundedness is then an *essential* aspect of the concept of set, according to Husserl. Summarizing and highlighting Husserl's conception of set: sets are objects that depend on their elements (which are independent with respect to the sets they belong) – they cannot exist unless their elements exist as a pre-constituted totality. The elements of sets can also be sets, but they all decompose into non-sets in a finite number of steps.

But, if we were Platonists, we might wonder why to tie necessarily *mathematical* sets to *acts* of collecting, since we believed that they exist independently of a subject and, consequently, are not “produced” in acts of collecting. The problem is that, as I have already mentioned, Platonists do not in general have a good account of how to *directly* access the platonic realm of sets, except for a few restricted types of sets (maybe only finite and denumerable infinite sets and the continuum¹³); all information we have about sets in general derive from the *theory* of sets, which leave us remarkably empty handed when a justification for some *existential axiom* is required: how do we know that some *particular* sets exist, or some *particular* set formation operations are allowed (the full power set axiom being one of the most dramatic cases), in the supposedly independent universe of sets, without appealing to the theory of sets? The Fregean strategy of reducing existence to definability as the extension of a predicate, which might tempt Platonists, proved disastrous.

¹³ Cf. Belaga (1988).

The constructivist alternative of reducing existence to *real* constructions, on top of denying the independent character of sets, if taken seriously cannot justify even the existence of a denumerable infinite set, let alone the power set of an arbitrary set. In order to be of any help, constructivist approaches to the justification of set theory must make sense of the notion of a super task, that is, a construction that involves infinitely many steps carried out in a finite stretch of time. But even if we could accept that super tasks can produce denumerable infinite sets in an arbitrarily small amount of time, it is still far from clear how to ascend to higher levels of infinity¹⁴. Real time constructions seem irredeemably restrictive. What Husserl offers us is something completely different, to consider the *abstract structure* of acts of collection – time-like constructions without the constraints imposed by *real* time – as the only constraint for the existence of sets. From this perspective a set exists provided it conforms to the following formal (purely objective) condition: all of its elements belong to a pre-constituted (given) multiplicity of objects (that, moreover, can ultimately be decomposed into non-sets). Husserl believed that this condition derived from the *essential* aspect of sets revealed in phenomenological analysis: sets are *second* level, *dependent* objects made up of other objects assembled into *given* multiplicities. The advantage of this approach is that it does justice to the dependent, second level, character of sets without imposing real constructibility restrictions on them. Set theory acquires then its true *mathematical* character of a theory of objects that could in principle exist (an aspect of a pure formal ontology, as Husserl believed it was).

Let us now take the concept of set we obtained, leaving aside for a moment considerations concerning acts of collecting, and see which properties belong *necessarily* to it. We immediately face here too the problem of existential assertions, which arises because no such an assertion can be *conceptually* justified: no concept can imply, only by its essential nature, that it

¹⁴ If a super task is taken to be a *temporal* process, since any amount of time has at most the power of the continuum, it is difficult to see how super tasks can generate sets with higher cardinalities.

applies to anything. So, we are compelled to investigate the role of existential claims in *conceptual* theories and how they can be justified.

Existential statements in mathematical theories

From a descriptive point of view, there are different types of mathematical theories, theories that describe determinate given domains of objects (which I will call *objectual* theories); theories that describe given formal structures (*structural* theories); and theories that describe given concepts (*conceptual* theories). (I do not claim that this is an exhaustive classification.) Arithmetic is a canonical example of either an objectual theory, if we believe (with Frege) that numbers are self-subsisting objects, or a structural theory, if we believe (with the structuralists) that all that arithmetic describes is an abstract structure, the ω -sequence – that is, the discrete linear sequence of arbitrary objects with a first, but no last element, in which any element has a successor, no two different elements have the same successor and any element can be obtained from the first by a finite iteration of the successor operation. The usual arithmetical axioms – the Dedekind-Peano axioms – are nothing but the description of what we intuitively apprehend by contemplating either numbers themselves or the ω -sequence. Theories such as group theory, on the other hand, are typical structural theories: they purport to describe structures (that of group, in this case) instead of objective domains (the axioms of group theory taken together amount, in fact, to a *structural* definition). The usual ZFC set theory can be seen either as a conceptual or an objectual theory. Moreover – and my main goal here is to argue for this – the concept this theory describes can be found by isolating the *essential* properties of sets as (noematic) correlates of (noetic) acts of collecting.

I call *domains of existence* those domains described by objectual theories, and *domains of meaning* those described by conceptual theories. The reason for this terminology is that domains of existence are determined independently of their theories, whereas domains of meaning are determined by the concepts their theories describe; the properties the objects of a domain of meaning have are exclusively derived from the

meaning attached to the concept that regulates this domain. Whether an object exists in a domain of existence is a matter of fact; in a domain of meaning, it is (or should be) a matter of the meaning associated with the relevant concept. This implies that existential statements have different truth conditions depending on which type of domain they refer to. Given an statement $S = (\exists x \in D) A(x)$, if D is a domain of existence the truth of S depends primarily on what things objectively exist in D , and only secondarily on whether S can be proved from the theory T of D (in the ideal situation in which T is categorical – hence complete – these conditions are equivalent); if D is a domain of meaning the truth of S depends only on its provability from T , since what exists in D is entirely determined by T .

A special case to be considered is when S is a candidate for an axiom of T . Of course, S must be true in order to be raised to the dignity of a foundational truth; moreover, ideally, it must also be *obviously* true. If T is an objectual theory, S is a suitable axiom for T if it is *intuitively true* that there is an object in D satisfying A . This may involve some sort of intuitive presentation of an object with this property. However, in the case T is a conceptual theory things change radically. In general the truth condition for S requires that it be derivable from T , but this does not apply if S is an axiom candidate for T . Conceptual theories are designed by inspecting the concept they are expected to faithfully describe (some form of *conceptual intuition* is involved here), so the truth of S – and hence its admissibility as an axiom – must be decided exclusively by inspection of the relevant concept.

This seems to require some sort of ontological argument: a proof that there is an object falling into a concept exclusively from an analysis of this concept. This is in general not possible; the analysis of the meaning of a concept does not in general (or ever) imply the existence of an instance of this concept. Should we then conclude that conceptual theories do not admit existential axioms? I believe this is too restrictive a conclusion and that there is a way of justifying existential axioms for conceptual theories.

If we adopt a purely phenomenological attitude with respect to a concept whose meaning we want to analyze, that is, if we take this concept simply as an intentional objectuality, we do not have to care whether it does

or does not exist objectively. Phenomenologically considered conceptual theories are indifferent to the ontological status of their concepts; all they care about are analyses of meaning. Also, we do not care whether this concept is or is not instantiated. Nothing prevents us from developing a theory of a concept that is not instantiated, or is even self-contradictory (this may very well be the case of many well established mathematical theories). But then, what condition must S satisfy in order to be accepted as an axiom of T ? Here is one that seems reasonable: we are allowed to admit the existence of any instance of the relevant concept short of manifest contradiction. I will call this the *principle of maximality*. Gödel (Gödel 1944) believed that “only a maximum property would harmonize with the concept of set”. It is not clear why he thought so, but we can offer some suggestions (without suggesting that any of them was actually considered by Gödel). 1) If there is no conceptual “proof” that a certain set must exist, but no “disproof” either, to outright deny its existence from the start would seem an unjustified restriction of the range of the concept. 2) Since group theory or any structural theory puts no *a priori* restriction on the size of the domains structured by them, there is no reason why set theory or any conceptual theory should put any restriction on the size of the domain of objects over which they rule; so, the domain of sets must be maximal short of inconsistency (some people call this “one step before disaster”). 3) Any theory strives for completeness, both logical and ontological; conceptual theories in particular must then incorporate (by logical completeness), short of inconsistency, any assertion that cannot be decided by an analysis of its concept so as to broaden the domain it determines (ontological completeness).

Of course, maximality is unacceptable if T is an objectual theory, since it imposes a totally unjustified inflationary condition on reality. In this case, maximality would be equivalent to saying that *as a matter of fact* anything whose non-existence is not provable exists. This is obviously too much to ask. On the other hand, if T is a conceptual theory, its goal is not to describe a previously existing domain, but the meaning (or essence) of a concept. If nothing in the concept implies the existence of a particular instance of the

concept, and nothing implies its non-existence, to admit its existence is a way of not arbitrarily restricting *a priori* its scope, and unjustifiably restricting its possible applications. This seems enough to justify the principle of maximality, I believe¹⁵.

As I have already stressed, I consider here two strategies of justification for set theory: (a) set theory is an objectual theory and sets are the outputs of acts of collecting considered formally; (b) set theory is a conceptual theory and sets are conceived as quantitatively determined multiplicities taken as unities, which, moreover, depend on their elements, are essentially grounded and exist whenever they can. Let us see which facts about sets and which set theoretical principles these strategies suggest

1. Sets are *collections* of objects and there is *no more* to a set than the objects it contains (a multiplicity and the set based on it differ only formally, not materially). If we agree that this is an *essential* characteristics of sets, then sets are extensional entities and the *principle of extensionality* is true of them: two sets are equal if, and only if, they have the same elements.

2. Sets are *objects*. This means that any set can be collected into a set. But if a set *can* be an element of another set, then, either by maximality or by an act of collecting, it *must* be an element of another set. That is, our understanding of the concept of set allows for any set – as the object it is – to be collected into a set (a set is a collectable item). Therefore, our theory of sets – as either a purely conceptual theory ruled by the principle of maximality, or a theory of entities obtained by acts of collecting considered formally – requires for a collection of objects to be a set that there is a set of which this collection is an element: $\text{Set}(x) \rightarrow (\exists y)(\text{Set}(y) \wedge x \in y)$ ¹⁶.

3. Sets are *dependent* objects. This, as already stressed, follows from the fact that sets are *second level* objectualities. A trivial fact about sets is that a

¹⁵ Potter (Potter 2004) separates this principle of maximality, or something akin to it, into two, called the first and second principles of plenitude. The first guarantees basically the existence of the full power set of any arbitrary set; the second, the existence of arbitrarily high levels in the hierarchy of sets.

¹⁶ In terms of the universe of sets \mathbf{V} this translates into $x \in \mathbf{V} \rightarrow (\exists y) (x \in y \rightarrow y \in \mathbf{V})$.

particular set ceases to be what it is if any of its members is removed from it. In fact, the principle of extensionality implies this much; the reduced set is *another* set, another object. On the other hand if the set character is removed from a set, that is, if its members are no longer seen as unified into a unitary whole, the being of these elements is not affected; each one of them preserves its individuality. The being of the elements of a set contributes to the being of the set, but the converse is not true, the being of a set does not contribute to the being of its elements. To be or not to be a member of a set does not alter the essence of an object, but to contain precisely the objects it contains is part of the *nature* of a set. I will express these facts by saying that a set *depends* on its elements, but the elements of a set are *independent* of it.

This is how Husserl defines the relation of ontological dependence between objects in general (*Logical Investigations* III, § 21):

A content of the species *A* is founded upon a content of the species *B*, if an *A* can by its essence (i.e. legally, in virtue of its specific nature) not exist unless a *B* also exists.

Then, for Husserl, the dependence of an object upon another (the fact that an object is founded on another) is relative to the categories these objects are taken to belong; the notion of ontological dependence is then an affair of categories, and the category *A* is said to depend on the category *B* if, by the very *essence* of being an *A*, no *A* can exist without a *B* existing as well. For instance, the category of colors depends on the category of extensions, for no color can exist (properly, in physical space) without an extension existing as well (any color needs an extension to be the color of). The *object* *x* depends on the *object* *y* if *x* belongs to the category *A*, *y* belongs to the category *B* and *A* depends on *B*. In symbols: *x* depends on *y* $\leftrightarrow x \in A \wedge y \in B \wedge \mathbf{N}_A [(\exists z \in A) \rightarrow (\exists z \in B)]$, that is, *x* is an *A*, *y* is a *B* and by the very essence (or nature) of *A*, if there is an *A*, then there is a *B*. For instance, *any* particular moment of color depends on *any* particular extension, for, by the nature of colors, if a color exists, then an extension must exist as well.

But there is a problem here. This definition implies that an object may depend on another that does not have anything to do with its *particular* (not generic) nature. But we want to be able to say that a *particular* color depends on the extension that *supports* it, not on *any* extension. Of course, if a color depends on any extension, it depends in particular on its extended support, but Husserl's definition of dependence gives us more than just the *particular* relation of dependence of a color on its support. We may adapt Husserl's definition in order to have what we want thus: x depends on y if, and only if, by the very nature, or essence of x , if x exists, y also exists. This is the account of dependence Kit Fine (Fine 1995) calls *essentialist-existential*¹⁷. The problem with this approach is that it emphasizes an *existential* dependence between x and y , although conditioned by the nature of x , and not, as desirable, a dependence between the *essential nature* of x and y . Our intuition tells us that x depends on y if, and only if, y contributes somehow to the nature of x . These are difficult problems, but since it is not my purpose here to develop a general theory of ontological dependence, I will be satisfied with a loose characterization of the relation of ontological dependence.

I will take the following, which I call the *principle of ontological dependence*, as a basic fact concerning sets: sets are dependent objects with respect to their elements, but the elements of a set are independent objects with respect to them. A consequence of this is that a *necessary* condition for a set to exist is that all of its elements also exist. If we are dealing exclusively with *pure* sets, that is, sets “built up” exclusively from other sets, this implies that the collection of all sets – the universe \mathbf{V} – is closed with respect to elementhood. In symbols: $x \in \mathbf{V} \rightarrow (\forall y) (y \in x \rightarrow y \in \mathbf{V})$ ¹⁸. But the converse is not necessarily true. All the elements of a collection may exist (or be in \mathbf{V} , if they are all sets) but not the collection itself; we may have $C \subseteq \mathbf{V}$, for a particular collection C , but $C \notin \mathbf{V}$. For instance, $\mathbf{V} \subseteq \mathbf{V}$, but $\mathbf{V} \notin \mathbf{V}$, as we will see below. Our notion of set is a specification of the more general

¹⁷ This is a variant of what Fine calls the *modal-existential* account: x depends on $y \leftrightarrow \Box(E(x) \rightarrow E(y))$ (that is, necessarily, if x exists, then y also exists).

¹⁸ In general, the *transitive closure* of a pure set is contained in \mathbf{V} .

notion of collection. Any multiplicity is a collection, but only those that are *also* objects – hence collectable items – are sets.

An immediate consequence of these considerations is that the \in -relation is anti-symmetrical: $x \in y \rightarrow y \notin x$. For otherwise x would be both dependent and independent on y , which cannot happen. This implies that no set can be an element of itself: $(\forall x) x \notin x$. In particular V is not a set, for if it were, it would contain itself as a member. This shows not only that a collection *needs* not to be a set; it shows that some collections are *definitely* not sets.

The notion of ontological dependence has a long history in philosophy (it appears already in Aristotle¹⁹, *Metaphysics* 1019a1 – 4), but I would like, in passing, to call your attention to one important methodological principle that involves a particular version of it, Poincaré's vicious circle principle: never to define an object in terms of a class that contains, involves or presupposes it. At first sight this principle has nothing to do with ontological matters, since it is concerned with definability, not being. Gödel focus his critique of the vicious circle principle precisely on this point: a definition does not create an object; it simply characterizes it. But for Poincaré a definition does create the object it defines, it constitutes the being of this object. Therefore, an object depends, on an ontological sense, on its definition. If a definition presupposed its object it would depend on it, thus generating a vicious circle where nothing would be properly defined, that is, created.

Poincaré's principle was aimed at avoiding paradoxes such as Russell's, which he considered the consequence of indulging into short circuitry of the type above. Russell's paradox was indeed the outcome of Frege's insufficient attention to ontological matters. By not making any ontological distinction between objects, Frege obliterated the distinction he so carefully drew between objects and concepts. If classes, as logical objects,

¹⁹ “There are things we call prior and posterior ... prior, in this sense, is anything that can exist independently of other things, while the others cannot exist without them”.

are not distinguished from other objects, it makes sense to say that they are – or are not – members of themselves, thus opening the doors to paradox.

4. The relation of ontological dependence between sets and their elements induces a partition of the universe of sets into levels ordered in terms of this relation. The basic fact is that a set, being dependent of its elements, which in turn are independent of their set, occupies a level that follows in the sequence all the levels where elements of the set occur. The universe of sets is then structured in a well-ordered sequence of mutually disjoint levels (a non-cumulative hierarchy that, for practical purposes, can be made into a cumulative one). The sequence of levels is determined by the iteration of the operations of set formation, or, which is the same, acts of collecting. Hence, it should begin with the multiplicity of all urelemente, which by maximality we can make into a set. But since urelemente are not sets, they cannot appear in the universe of sets. So the first level of the hierarchy contains only the set of all urelemente and the empty set²⁰, whose existence does not depend on any other set or non-set. If we restrict ourselves to *pure* sets it contains only the empty set (but we could also make the first level empty, and put the empty set in the following level only).

Now, given a level, by considerations of maximality or the fulfillment of the conditions for renewed acts of collecting, we have a level that *immediately follows* it containing *all* the sets whose elements belong to the given level and levels prior to it (a level then includes all the levels prior to it as elements; hence, levels are sets too). So, in fact, a set always belongs to the level immediately following the last level that contains elements of it. Moreover, there is no last level; otherwise the sets whose elements belong to this level would not find a place in the hierarchy. As is easily seen, a level contains all the (mutually independent) sets that depend *only* on elements of

²⁰ Remember that an empty set exists as the outcome of a frustrated act of collecting (or directly by maximality: an empty set exists as a subset of any existing set; by extensionality there is only one empty set). Maximality also imposes that *all* the subsets of any given set exist, since the elements of the given set constitute a *given* multiplicity of objects (thus providing the sufficient pre-condition for the existence of any subset of it) from which any part can be collected into a set.

prior levels. A level V is *prior* to a level V' (V' is *posterior* to V) when *all* the elements of V are *independent* of *any* element of V' . For convenience we can make this hierarchy cumulative, including in any level all the levels prior to it.

As our phenomenological inquiry revealed, an act of collecting, and consequently a set, requires a *given* multiplicity of objects as a material precondition of existence. In terms of the cumulative hierarchy of sets, this translates into the following condition: a multiplicity of sets is itself a set if it is contained in some level of the hierarchy: $\exists V: x \subseteq V \rightarrow \text{Set}(x)$. To be part of a level is then our reading of “given”. Hence, a multiplicity is *not* a set if it is cofinal with the sequence of all levels, that is, given any level, there is a level posterior to it that contains some element of this multiplicity.

This picture of the universe of sets is rich enough to immediately justify some axioms of ZFC, namely, the empty set axiom, separation, power set, choice, union and regularity²¹. Our problem now is to justify infinity and replacement. Let me carry to conclusion the task I imposed myself in this paper by providing a justification for these axioms based on the strategies I favored.

Replacement. In Shoenfield (1967) the axiom of replacement is justified in terms of what Shoenfield calls the *principle of cofinality*: given a set \mathcal{A} , there must be a level that follows all the levels at which an element of \mathcal{A} appears. The justification he presents for this principle is the following: since \mathcal{A} is a set, we can “visualize” it as a single object; therefore we can also visualize the collection of levels V_a in which an element a of \mathcal{A} appears as a single object. So there must be a level in which all the V_a 's are completed; this is the level that follows them all. For short, the principle of cofinality states that a multiplicity is a set only if it is *not* cofinal in \mathbf{V} . Shoenfield formulation of replacement is the following: if for each element x of a set w there is a set \mathfrak{z}_x of all y 's such that $\mathbf{A}(x, y)$, then there is a set containing all such y 's. And the justification for it goes like this: by the cofinality principle

²¹ The justification is essentially that presented in Shoenfield (1967), with the fundamental difference that I, contrary to Shoenfield, do not think of the sequence of levels in terms of a *temporal* sequence, acts of collecting as temporal acts or the relation of ontological dependence in terms of before and after.

there is a level V after all the levels at which some z_x is construed, since all the elements of z_x appear before it, all the elements of all z_x 's appear before V and can be collected into set at this level V . Of course, there is something missing in this argument. We must presuppose, for the application of the cofinality principle, that all the z_x 's form a *set*. But, of course, this is the axiom of replacement we wanted to justify in the first place. Cofinality is not enough to justify replacement.

Potter (Potter 2004) presents an appealing suggestion, especially for mathematicians, but for which he has no justification. The idea is that, to the same extent that group theoretical properties are preserved by group isomorphisms and topological properties by homeomorphisms, set theoretical properties – including the most fundamental one, being a set – are preserved by bijections. So the image of a set by a bijection is also a set.

As I see it, behind Potter's idea there is a fundamental intuition on the nature of sets, namely, that as well-determined objects composed of other objects, sets (as opposed to *absolutely* infinite multiplicities, which are not sets²²) are *quantitatively determined*. It is also evidently true, or so I think, that if there is a bijection from a multiplicity onto another, they are either both quantitatively determined or both quantitatively undetermined. It follows from these *conceptual truths* that the multiplicity formed by all the images of a bijection defined on a set is itself a set, which is a way of stating the axiom of replacement. In symbols: $\text{Set}(x) \wedge x \approx y \rightarrow \text{Set}(y)$.

The principle of limitation of size in set theory states that if a multiplicity has at most as many elements as a given set, it is itself a set. Replacement follows straightly from this principle (and so does separation). Obviously, the principle of limitation of size follows directly from the fact that quantitative determination is a distinctive feature of sets. Potter (Potter 2004) claims that the strategy of justification for set theoretical axioms based on the notion of ontological dependence is impotent to handle limitation of

²² We can *define*, I think, *absolutely infinite* multiplicities as those that cannot be numbered. According to Cantor, a cardinality is absolutely infinite if no collection of objects can have it as its cardinality. But I think this formulation is inappropriate; in fact, absolute infinity admits *no* cardinality.

size, and that only a combination of ontological dependence and limitation can do the job completely. Of course it is hard, or maybe impossible to establish a connection between dependence and limitation of size, but this is not what I claim. My point is that the principle of limitation *follows* from the fact that sets are numerically determined multiplicities, which I count as an essential aspect of the concept of set *together with* ontological dependence. Some people may find arguable that sets are numerically determinate. Cantor believed they are (and suggested some theological argument for it); I believe that the determinacy of the multiplicities on which sets are based includes *numerical* determinacy as a particular aspect, and hence that numerical determinacy is an *essential* aspect of our (and Cantor's) concept of set. In short, I do not claim that the axioms of ZFC are *exclusively* justified in terms of the notion of dependence, but that they (to a large extent) are true of a certain natural concept of set from which both numerical determinacy (hence limitation of size) and dependence derive.

Infinity: This is probably the most difficult axiom to justify and apparently does not follow from the picture we drew of the universe of sets. After all, it is perfectly possible that only a ω -sequence of levels exhausts \mathbf{V} . If we consider only pure sets, this initial segment is called the universe of *hereditarily finite* – **HF** – sets. The axiom of infinity does not follow solely from our concept of set either; nothing in our understanding of this concept seems to prevent $\mathbf{V} = \mathbf{HF}$. If this were the case, all the axioms of ZFC would be true, excepting, of course, the axiom of infinity. So, adopting this axiom appears to be a pure act of will, supported, of course, by the needs of mathematics (which seems to speak strongly for pragmatic, external justification in this case). But I think we have enough reasons to believe that **HF** is a completed (given) multiplicity – that can be made into a set from where the set formation operations can start afresh. In fact maximality seems to force us into this direction, namely, the admission of a limit level – **HF** itself²³. We have after all a clear picture (it seems) of this multiplicity

²³ Michael Potter (Potter 2004) states the axiom of infinity precisely in these terms: there is a limit level.

and, more to the point, that it does not exhaust the universe of sets, since we can easily figure out how to step out of it (and, moreover, have compelling reasons to do so²⁴). So, we can take **HF** as “given” (that is, as itself a level of **V**, the first limit level). Arbitrary acts of collecting performed on this multiplicity will give origin to infinite sets, thus justifying the axiom of infinity.

Now, what difference does it make how we justify the axioms of our theory of sets, provided that we have some justification for them? And, more importantly, the strategies of justification we used here can be of any utility for an extension of ZFC (in order to decide, for instance, the continuum hypothesis)? As for this last question, the answer is probably *no* (this is why people like Maddy favor more promising strategies). But I think the relevant *philosophical* problem concerns epistemology, not the *extension* of set theory (philosophy is not mathematics, and how best extend set theory is the job of mathematicians). And, *epistemologically*, to justify set theory conceptually, rather than pragmatically, is a better solution. But the fact that a conceptual analysis of the concept of set will not help us decide the continuum hypothesis is also philosophically relevant, since it shows that our concept of set is intrinsically incomplete and probably essentially vague. Finally, a conceptual justification of set theory counts in favor of a general view of the nature of mathematical knowledge as conceptual *a priori* knowledge.

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²⁴ A completely different situation develops with respect to **V** itself. We might ask: why not take it as given, make it into a set and move upward in the hierarchy? The problem is that we do *not* have a clear grasp of **V**, at least as clear a grasp as we have of **HF**, so as to consider it as “given”.

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