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A SHORT NOTE ON INTUITIONISTIC PROPOSITIONAL LOGIC WITH MULTIPLE CONCLUSIONS

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Abstract: A common misconception among logicians is to think that intuitionism is necessarily tied-up with single conclusion (sequent or Natural Deduction) calculi. Single conclusion calculi can be used to model intuitionism and they are convenient, but by no means are they necessary. This has been shown by such influential textbook authors as Kleene, Takeuti and Dummett, to cite only three. If single conclusions are not necessary, how do we guarantee that only intuitionistic derivations are allowed? Traditionally one insists on restrictions on particular rules: implication right, negation right and universal quantification right are required to be single conclusion rules. In this note we show that instead of a cardinality restriction such as one-conclusion-only, we can use a notion of *dependency* between formulae to enforce the constructive character of derivations. The system we obtain, called FIL for full intuitionistic logic, satisfies basic properties such as soundness, completeness and cut elimination. We present two motivating applications of FIL and discuss some future work.

Key-words: Intuitionistic Logic. Multiple-conclusion systems. Dependency relations.

INTRODUCTION

Since Gentzen's pioneering work it has been traditional to associate intuitionism with a single-conclusion sequent calculus or natural deduction system. Gentzen's own sequent calculus presentation of intuitionistic logic, the famous system LJ, is obtained from his classical system LK by means of a cardinality restriction imposed on the succedent of every sequent. It is well-known that Gentzen's formulation of the system LK uses sequents expressions of the form $\Gamma \Rightarrow \Delta$, where both Γ and Δ may contain several formula occurrences. The intuition is that the conjunction of the formulas in Γ entails the disjunction of the formulas in Δ. In Gentzen's calculus LJ for intuitionistic logic, sequents are restricted to succedents with *at most* one formula occurrence. This is convenient, but by no means necessary. Since at least Maehara's work in the fifties (see Maehara 1954) it has been known that intuitionistic logic can be presented via multiple-conclusion systems. Maehara's system is described in Takeuti's influential book (Takeuti 1975), who calls it LJ'. Also Kleene in his monograph (Kleene 1952) presents systems which constitute multiple-conclusion versions of intuitionistic logic. But while both of these (classes of) systems stick to the idea that sequents can have *multiple conclusions*, they still keep some form of (local) cardinality restriction on succedents: the rules for implication right, negation right and universal quantification right must be modified in that they can only be performed if there is a single formula in the succedent of the premiss to which these rules are applied. If we don't impose these (local) restrictions, the systems collapse back into classical logic.

In the early nineties the authors devised a sequent calculus for (propositional) intuitionistic logic, the system *FIL*, where all rules may have multiple succedents, just as they do in classical logic. To make sure that classical inferences would not go through, we considered a relation of dependency between formulas and added a side-condition to the implication right rule based on these dependencies. In this way, cardinality restrictions (local or global) were replaced by restrictions based on control mechanisms over dependency relations. The original motivation for the work on FIL stemmed from Linear Logic. Martin Hyland and Valeria de Paiva were at that time working with full intuitionistic linear logic (FILL), that is, linear logic with the full class of additive and multiplicative operators, and in order to obtain sequent rules for an intuitionistic multiplicative disjunction, they needed multiple conclusion systems (Hyland and de Paiva, 1993). Their first attempt was to use Maehara's LJ', but this attempt was soon abandoned after the discovery of counter examples to cut-elimination (see Schellinx

1991)[1]. The idea of using dependency relations was given to the authors by Martin Hyland and it was used in different formulations for FILL. While working with FIL, de Paiva and Pereira realized that the intuitionistic system FIL could be used to solve the (old) problem of finding a cut-free system for the logic of constant domains. The logic of constant domains is an extension of intuitionistic first order logic (an *intermediary logic*) obtained by the addition of the axiom scheme $(\forall x(A \vee B(x)) \rightarrow (A \vee \forall xB(x))$, with x not free in A. The authors discovered afterwards that a FIL-like system had been independently worked on and published by Ryo Kashima and Tatsuya Shimura. Kashisma and Shimura (1994) is in fact the first published proof of the cut-elimination theorem for the logic of constant domains (see also Kashisma 1991), although they did not describe their system as based on a multiple succedent system for intuitionistic propositional logic itself.

Another nice application of the basic intuition of FIL was found by Torben Bräuner, who presented (as far as we know, for the first time) a cut-free system for the modal logic S5 (see Bräuner 2000).

A natural deduction version of FIL, the system NFIL, was defined and studied by Ludmilla Franklin, who proved the equivalence between FIL and NFIL, and the normalization theorem for NFIL (Franklin 2000).

The remaining part of this short note is organized as follows. We will give a rough description of the system FIL and its cut-elimination strategy. We then show in more detail how to obtain the two applications we mentioned above, to wit, the logic of constant domains and full intuitionistic linear logic. In the final part of the note, we discuss some possible extensions of the basic intuition of FIL.

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[1] The following counter-example was independently found by L.C.Pereira.

$$\dfrac{\dfrac{p \Rightarrow p}{p \Rightarrow p, \bot} \qquad \dfrac{\bot, 0 \Rightarrow q}{\bot \Rightarrow (0 \text{ -o } q)}}{p \Rightarrow (0 \text{ -o } q), p}$$

1. THE SYSTEM FIL (FULL INTUITIONISTIC LOGIC)

In this section we present the sequent calculus FIL (for Full Intuitionistic Logic). FIL is a multiple-succedent intuitionistic system, where an indexing device allows us to keep track of dependency relations between formulas in the antecedent and in the succedent of a sequent. Dependency relations determine the restriction in the formulation of the rule for introducing implication on the right ($\Rightarrow\rightarrow$), that guarantees that only intuitionistic valid formulas are derived.

First we introduce some conventions and terminology we will be using throughout this paper. Formulas (ranged over by A,B) are built in the usual way from propositional variables, propositional connectives for conjunction $\wedge$, disjunction $\vee$ and implication $\rightarrow$, and the constant $\perp$ for absurdity. As usual, negation $\neg A$ is defined as $(A \rightarrow \perp)$.

Definition: A (decorated) *sequent* is an expression of the form

$$A_1(n_1),\ldots,A_k(n_k) \Rightarrow B_1/S_1,\ldots,B_m/S_m$$

where
- A_i for $(1 \leq i \leq k)$ and B_j for $(1 \leq j \leq m)$ are formula;
- n_i for $(1 \leq i \leq k)$ is a natural number and $\forall i,j(1\leq i,j\leq k)$, $n_i \neq n_j$. We say that n_i is the *index* of the formula A_i;
- S_j for $(1 \leq j \leq m)$ is a finite set of natural numbers. We call S_j the *dependency set* of the formula B_j.

Now, our decorated sequents have an index for each formula in the antecedent and a dependency set for each formula in the consequent. The main intuition is that the set of natural numbers S_j records which formulas in the antecedent the succedent formula B_j depends on. This extended notion of sequent can also be seen as a simplification of a term assignment judgement. Capital Greek letters like Γ and Δ denote sequences of indexed formulas either in the antecedent or in the succedent.

To describe the inference rules of our sequent calculus we need some notational conventions. Assume that the set of formulas Δ consists of $B_1 /S_1,\ldots,B_m /S_m$,

- If S is any finite set of natural numbers, Δ[k|S] denotes the result of replacing each S_j in Δ such that $k \in S_j$ by $(S_j - \{k\}) \cup S$;
- Δ[k,S'] denotes the result of replacing each S in Δ such that $k \in S$ by $S \cup S'$.

The system FIL is given by the axioms and rules of inference below. We assume that in the case of the rules for conjunction on the right (⇒∧), disjunction on the left (∨⇒), implication on the left (→⇒) and Cut, the upper sequents of the premises have no index in common. This is in fact no strong restriction, since we can always rename the indices.

The System FIL

1. Initial axiom: $A(n) \Rightarrow A/\{n\}$

2. ⊥-axiom: $\bot(n) \Rightarrow A_1/\{n\}, \ldots, A_k/\{n\}$

3. Cut-rule:

$$\frac{\Gamma \Rightarrow \Delta, A/S \qquad A(n), \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'^{*}}$$

4.

$$\frac{\Gamma, A(n), B(m), \Gamma' \Rightarrow \Delta}{\Gamma, B(m), A(n), \Gamma' \Rightarrow \Delta}\text{(perm}\Rightarrow\text{)} \qquad \frac{\Gamma \Rightarrow \Delta, A/S, B/S', \Delta'}{\Gamma \Rightarrow \Delta, B/S', A/S, \Delta'}\text{(}\Rightarrow\text{perm)}$$

5.

$$\frac{\Gamma \Rightarrow \Delta}{A(n), \Gamma \Rightarrow \Delta^{*}}\text{(weak}\Rightarrow\text{)} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A/\{\}}\text{(}\Rightarrow\text{weak)}$$

6.

$$\frac{\Gamma, A(n), A(m) \Rightarrow \Delta}{\Gamma, A(k) \Rightarrow \Delta^{*}}\text{(cont}\Rightarrow\text{)} \qquad \frac{\Gamma \Rightarrow \Delta, A/S, A/S'}{\Gamma \Rightarrow \Delta, A/S \cup S'}\text{(}\Rightarrow\text{cont)}$$

7. $$\frac{\Gamma, A(n) \Rightarrow \Delta \qquad \Gamma', B(m) \Rightarrow \Delta'}{\Gamma, \Gamma', (A \vee B)(k) \Rightarrow \Delta^*, \Delta'^*} (\vee\Rightarrow) \qquad \frac{\Gamma \Rightarrow \Delta, A/S, B/S'}{\Gamma \Rightarrow \Delta, (A \vee B)/S \cup S'} (\Rightarrow\vee)$$

8. $$\frac{\Gamma, A(n), A(m) \Rightarrow \Delta}{\Gamma, (A \wedge B)(k) \Rightarrow \Delta^*} (\wedge\Rightarrow) \qquad \frac{\Gamma \Rightarrow \Delta, A/S \qquad \Gamma \Rightarrow \Delta, B/S'}{\Gamma \Rightarrow \Delta, (A \wedge B)/S \cup S'} (\wedge\Rightarrow)$$

9. $$\frac{\Gamma \Rightarrow \Delta, A/S \qquad B(n), \Gamma' \Rightarrow \Delta'}{(A \rightarrow B)(n), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'^*} (\rightarrow\Rightarrow) \qquad \frac{\Gamma, A(n) \Rightarrow \Delta, B/S}{\Gamma \Rightarrow \Delta, (A \rightarrow B)/S - \{n\}} (\Rightarrow\rightarrow)$$

Comments on the rules:

1) In the cut-rule, $\Delta'^* = \Delta'[n|S]$.
2) In the rule ($\Rightarrow$weak), *n* is a new index and Δ^* is obtained from Δ through the introduction of *n* in at least one S in Δ.
3) In the rule (cont$\Rightarrow$), k = min(n,m), and $\Delta^* = \Delta[\max(n,m)|\{k\}]$.
4) In the rule ($\vee\Rightarrow$), k is a new index, $\Delta^* = \Delta[n|\{k\}]$ and $\Delta'^* = \Delta'[m|\{k\}]$.
5) In the rule ($\wedge\Rightarrow$), k = min(n,m) and $\Delta^* = \Delta[\max(n,m)|\{k\})$.
6) In the rule ($\rightarrow\Rightarrow$), $\Delta^* = \Delta'[n,S]$.
7) Finally, and most importantly, in the rule ($\Rightarrow\rightarrow$) we have the restriction that $n \in S$ and that for every other S' in Δ, $n \notin S'$. If the restriction is satisfied, it means that no other formula in Δ depends on the indicated occurrence of A.

Let us illustrate how the system works with two examples of derivations.

Example 1:

$$\cfrac{\cfrac{\cfrac{A(1) \Rightarrow A/\{1\}}{B(2), A(1) \Rightarrow A/\{1,2\}}}{A(1) \Rightarrow (B \rightarrow A)/\{1\}}}{\Rightarrow (A \rightarrow (B \rightarrow A))/\{\}}$$

Example 2:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(1) \Rightarrow A/\{1\} \qquad B(2) \Rightarrow B/\{2\}}{(A \vee B)(3) \Rightarrow A/(3), B/(3)} \qquad C(4) \Rightarrow C/\{4\}}{(A \vee B)(3), (B \rightarrow C)(4) \Rightarrow A/\{3\}, C/\{3,4\}}}{(A \vee B)(3), (B \rightarrow C)(4) \Rightarrow (A \vee C)/\{3\}, C/\{3,4\}}}{(A \vee B)(3), (B \rightarrow C)(4) \Rightarrow (A \vee C)/\{3\}, (A \vee C)/\{3,4\}}}{(A \vee B)(3), (B \rightarrow C)(4) \Rightarrow (A \vee C)/\{3,4\}}}{(A \vee B)(3) \Rightarrow ((B \rightarrow C) \rightarrow (A \vee C))/\{3\}}$$

Consider now the following attempt to construct a proof of the law of the excluded middle:

$$\dfrac{\dfrac{A(1) \Rightarrow A/\{1\}}{A(1) \Rightarrow A/\{1\}, \bot/\{\}}}{\Rightarrow A/\{1\}, (A \rightarrow \bot)}$$

The last inference clearly doesn't satisfy the restriction imposed on (⇒→).

The system FIL is sound and complete with respect to LJ. The proof of completeness is completely straightforward: every proof in LJ can be easily "decorated" with labels and transformed into a proof in FIL. The proof of soundness is certainly more involved and can be obtained, for example, from the proof of Kashima and Shimura (for the case of Constant Domains) if we leave out of the proof the first order apparatus.

The proof of the cut-elimination theorem for FIL is quite standard. As usual the rule of contraction presents some difficulties that are dealt with through the use and eliminability of a generalized form of cut; instead of eliminating simple cuts, we will show how to eliminate *indexed cuts* (see Schellinx 1991). An indexed cut is defined as follows:

$$\frac{\Gamma \Rightarrow \Delta \quad \Theta \Rightarrow \Lambda}{\Gamma, \Theta' \Rightarrow \Delta', \Lambda^*} \; (A;n_1, \ldots ,n_k;m_1, \ldots m_j)$$

The information standing on the right of the inference line indicates that the cut-formula is A, and that Δ' (Θ') is obtained through the deletion of the cut-formula A in positions $n_1,...,n_k$ ($m_1,...,m_j$) in Δ (Θ).

It is routine matter to prove that the indexed-cut rule is equivalent to the simple cut rule. Trivially, an application of the cut rule is an (unary) application of indexed-cut. The other side of the equivalence can be proved with the use of permutations, contractions and the cut-rule. As in Gentenz's original proof of the *Hauptsatz*, we prove the following basic lemma:

Lemma: Let Π be a derivation $\Gamma \Rightarrow \Delta$ in FIL such that:

1) The last rule applied in Π is an indexed cut.
2) There is no other application of indexed-cut in Π.

Then, Π can be transformed into an indexed cut-free derivation Π' of $\Gamma \Rightarrow \Delta$.

Proof: By a routine induction over (lexicographically ordered) pairs (α,β) where α is the degree of the cut-formula and β its rank. In fact, we are not really working with Gentzen's *rank*, but rather with the sum of the longest *cluster sequences* for the cut-formulas. We shall show one case where the left-rank is equal to 1, the right-rank is greater than 1, and the right upper sequent is obtained by an application of $(\Rightarrow\rightarrow)$.

$$\frac{\dfrac{\Gamma \Rightarrow \Delta, A/S}{\Gamma \Rightarrow \Delta, A'/S'} \qquad \dfrac{\Gamma_1, C(n) \Rightarrow \Delta_1, D/S''}{\Gamma_1 \Rightarrow \Delta_1, (C \rightarrow D)/S'' - \{n\}}}{\Gamma, \Gamma_1' \Rightarrow \Delta', \Delta_1^*, (C \rightarrow D)(S'' - \{n\})^*} \; (A;n_1, \ldots,n_k;m_1, \ldots m_j)$$

This derivation can be transformed into the following derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, A/S}{\Gamma \Rightarrow \Delta, A'/S'} \qquad \Gamma_1, C(n) \Rightarrow \Delta_1, D/S''}{\Gamma, \Gamma_1', C(n) \Rightarrow \Delta', \Delta_1^*, D/(S'')^*}\ (A;n_1, \ldots, n_k;m_1, \ldots m_j)}{\Gamma', \Gamma_1^* \Rightarrow \Delta', \Delta_1^*, (C \rightarrow D)/(S'')^* - \{n\}}$$

The result now follows directly from the side-condition on the implication rule, as n is not an element of Δ and S', and $(S'')^* - \{n\} = (S'' - \{n\})^*$.

Theorem (Cut-elimination): If Π is a derivation of $\Gamma \Rightarrow \Delta$ in FIL, then Π can be transformed into a cut-free proof Π' of $\Gamma \Rightarrow \Delta$.
Proof: Directly from the basic lemma

2. TWO APPLICATIONS OF FIL

The two original applications of the system FIL were related to Linear Logic and to the Logic of Constant Domains. The system FIL was introduced in connection with a variant of Linear Logic, due to Martin Hyland and Valeria de Paiva, called Full Intuitionistic Linear Logic. On the one hand, the intuitionistic nature of this logic required that the logical operators were *not* defined in terms of each other, and hence one could not use the duality to define the multiplicative *or* (℘) in terms of tensor, say. On the other hand, the multiplicative *or* (℘) demanded a multiple succedent in the formulation of (⇒℘): the condition for introducing (A ℘ B) is that we have "disjunctively" both A and B, the classical rule for (⇒℘) being

$$\dfrac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, (A \wp B)}$$

The system FIL can be seen as a natural intuitionistic basis for the formulation of a system FILL for Full Intuitionistic Logic; after all FIL has multiple succedents, no cardinality restrictions, and a fairly standard cut-elimination strategy. The natural rule for ($\Rightarrow \wp$) in the system FILL is a simply a linear version of the rule for ($\Rightarrow\vee$) in the system FIL:

$$\frac{\Gamma \Rightarrow \Delta, A/S, B/S'}{\Gamma \Rightarrow \Delta, (A \wp B)/S \cup S'}$$

The cut-elimination theorem holds for FILL and a nice proof of it is presented in Braüner and de Paiva (1996). Other formulations of Full Intuitionistic Linear Logic can be found in Bellin (1997) and Bierman (1996).

In the early eighties it was conjectured that the Logic of Constant Domains (CD) would not admit a "complete, cut-free, sequent axiomatization" (see Lópes-Escobar 1983), where cut-free axiomatizations should satisfy the following two criteria:

1) The axioms should contain only atomic formulas; and
2) The operational rules should introduce a single logical operator.

The problem of finding a cut-free system for CD was solved by Ryo Kashima and Tatsuya Shimura in 1994 (see Kashima 1991 and Kashima and Shimura 1994) through the use of a FIL-like system CLD. The main idea of these authors in the definition of CLD was to use a binary relation, called by them "connection", described as expressing "a dependency between formulas in the antecedent and formulas in the succedent of a sequent". In order to obtain a cut-free sequent calculus for the Logic of Constant Domains we simply add the first order apparatus on the top of FIL, with dependencies being inherited in the quantifier rules, as in the example below:

$$\frac{A(t)(n), \Gamma \Rightarrow \Delta}{\forall x A(x)(n), \Gamma \Rightarrow \Delta}\ (\forall\text{-R})$$

CONCLUDING REMARKS

This short note described the multiple-conclusion system FIL for intuitionistic propositional logic and recounted some of its original applications. We are convinced that FIL is *not* simply a trick that logicians can play to reformulate a traditional system. Dependency relations and discharging functions play a fundamental role in the proof theory of natural deduction systems. Crude discharging strategies for several well-known natural deduction systems entail that good proof-theoretical properties are simply lost (strong-normalization, uniqueness of normal form, generality). From a very general conceptual point of view, FIL shows that classical logic can be distinguished from intuitionistic logic by the way dependency relations and discharging functions are handled in both systems. The introduction of multiple conclusions in intuitionistic logic imposes some constraints on which assumptions are available for discharge at each conclusion. In the case of propositional logic this means that in a multiple-conclusion system, it is the rule for implication introduction that distinguishes classical logic from intuitionistic logic: every assumption in classical logic is *global*, while some assumptions in intuitionistic logic may be *local*.

The idea that the distinction between intuitionistic logic and classical logic lies in the rule for implication introduction also suggests that there might be interesting translations from classical logic into intuitionistic logic which are not based on negation, but rather on implication. We plan to investigate these prospective translations in future work.

As for other future work, the system FIL is a propositional system and we know that the simply addition of the first order apparatus to FIL does not produce Intuitonistic First order Logic, but rather the Logic of Constant Domains. Quantifier rules require a different kind of dependency relation (connection) that has not so far been defined. The same applies to some intuitionistic modal logics: if we add the modal apparatus of S4 to FIL, we obtain a kind of "intermediary" modal logic S4, lying between classical S4 and (one of the several proposals for) intuitionistic S4. The proof theory and the model theory of this intermediary S4 have been studied (de

Medeiros 2001), but no multiple-conclusion (strictly) intuitionistic S4 has so far been defined.

And finally we should mention that although a natural deduction version NFIL of the system FIL has already been defined, see Franklin (2000), we do not have yet a corresponding more "liberal" semantic tableaux that incorporates the idea of dependency relations into its rules. Traditional intutionistic tableaux have the strong restriction that on each branch there can occur at most one *f* formula, and the use of dependency relations might produce more flexible tableaux, where multiple *f*-formulas are allowed to occur on the same branch.

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