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MODULATED LOGIC, MODAL LOGIC AND TRANSLATIONS BETWEEN LOGICS

HÉRCULES DE A. FEITOSA

Mathematic Department
São Paulo State University (UNESP)
Campus of Bauru
17033-360 BAURU, SP
BRAZIL

haf@fc.unesp.br

FRANK THOMAS SAUTTER

Philosophy Department
Federal University of Santa Maria
Campus Universitário, km9, Camobi
97105-900 SANTA MARIA, RS
BRAZIL

sautter@terra.com.br

Abstract: From generalized quantifiers we move to modulated logic. So with this motivation we show ways for the construction of some modal logics. With the translations between logics we show some inter-relations between modal logics. Finally, we introduce some opportune concepts for a type of classification of deontic logic.

Key-words: Modulated logic. Modal logic. Deontic logic. Logical translations.

INTRODUCTION

In this moment that we have the opportunity to write an article to homage professor Itala Maria Loffredo D'Ottaviano, we try to mention aspects of logic of her interest and as her students we share the same interest. Another aspect to be mentioned is the fact that this work is motivated by a Brazilian production in logic. In Section 1, we comment about the logic with generalized quantifiers, that although it does not have appeared and expanded in Brazil, has our attention directed to recent works elaborated in Brazil. Particularly, we detach the modulated logic shown in Section 2, which has taken us to several and constant reflections. In Section 3, we show how motivation in modulated logic takes us to the analysis and construction of some modal logics. In Section 4, we deal with translations between logics, which is a subject of professor Ítala's interest. In the last section, we use the translations to show some interrelations between modal logics and we introduce some concepts considered opportune for a type of classification of modal logic, particularly, deontic logic.

1. LOGIC WITH GENERALIZED QUANTIFIERS

The first order logic considers mathematical structures with certain algebraic particularities, where it is chosen an arbitrary set as the domain of individuals for a structure $\mathcal{B}$ and relations and functions, which serve as interpretations for the predicate symbols and functional symbols. Besides, expressions constructed from the operators "not", "and", "or", "if ..., then", and from the quantifiers "all" and "exist", or others that can be expressed from these are relevant.

The basic idea that crosses the inquiries in first order logic is that it can establish a relation between some mathematical structures and some collections of expressions given in a language to describe properties of such structure.

The notion of satisfaction in a structure $\mathcal{B}$, denoted by $\mathcal{B} \vDash \varphi$, means that the expression φ is true in $\mathcal{B}$ or it is satisfied in $\mathcal{B}$.

However, besides the universal character of logic, we must acknowledge that several mathematical concepts as so many expressions of natural language cannot be treated in this theoretical context.

Particularly we are interested in the concepts that cannot be investigated from the universal ($\forall$) and existential ($\exists$) logical quantifiers. This leads us to the creation of new quantifiers different from the usual, which can be used as two sources: (i) the characterization of specific mathematical aspects or (ii) the analysis of quantifiers used in natural language that cannot be defined from the logical ones as: many, some, almost all, almost none, the majority, the minority, amongst others.

The motivation for the first source is the question "What is the logic of specific mathematical concepts?". Or directly, given a particular mathematical property as: good order, infinitude, continuity, etc ..., what would be the implicit logic in the use of this property? What type of structure isolates this property more naturally? What type of language better mirrors the mathematical idea given by the property? Which reasoning is legitimated?

Thus, the theory of extended models takes the basic idea and expands it in some ways, producing richer mathematical structures or structures with greater power of expression in its language or both. In this scope, logic consists of a collection of mathematical structures, a collection of formal expressions and the satisfaction relation among them, or, in other words, logic is something constructed to study the coherence of parts of mathematics.

The first clear proposal to investigate the extensions of the first order logic through methods of model theory is attributed to Mostowski (1957). Since concepts as finitude (infinitude) and denumerability, important for the modern mathematics, are not defined in the first order logic, the author suggested the following syntactical rule:

• if $\varphi(x)$ is a formula, then $Qx\varphi(x)$ is also a formula, which x does not occur free in this new formula.

This rule of formation is increased to make the interaction with "and", "or", "not", "if ..., then", "all" and "exists" possible.

The meaning of Q depends on a new semantical rule that, in general, works this way:

• given a cardinal $\aleph_\alpha$, we get a logic $\mathcal{L}(Q_\alpha)$ defined by the following semantial rule:

$$\mathcal{B} \vDash Qx\varphi(x) \text{ if there is at least } \aleph_\alpha \text{ elements } b\text{, such that } \mathcal{B} \vDash \varphi(b).$$

The logic $\mathcal{L}(Q_0)$ considers the distinction finite/infinite not contemplated by first order logic; the logic $\mathcal{L}(Q_1)$, on the other hand, contemplates the distinction countable/uncountable, also absent in first order logic.

Mostowski left many open problems on its logic with generalized quantifiers, particularly in establishing a formal calculus in which it is possible to prove all the sentences that involve the new introduced quantifier.

Lindström (1966) redefined the concept and studied the extended logic by quantifiers in semantical structures a little more general, but still had not obtained complete success in introducing axioms and structure that allowed a proof of completeness for $\mathcal{L}(Q)$.

Other properties of the deductive systems are in the context of the extended logic.

The compacity is defined by:

Strong compacity: if Δ is a set of logical sentences and if every finite subset of Δ has a model, then Δ has a model;

Countable compacity: if Δ is a set of logical sentences and if every countable subset of Δ has a model, then Δ has a model.

There are two reasons for the interest in these results. The first reason is because it is nearly related to the problem of completeness. The completeness theorem usually establishes that when φ is a logical consequence of some set (possibly countable) Δ of hypotheses, then φ is derivable from some finite subset of Δ. In particular, if Δ is inconsistent, that is, it does not have models, then some contradictory sentence is a consequence of Δ and, in this in case, some finite sub set of Δ is inconsistent. This way, the compacity, in the two versions, agrees to the completeness if it exists.

The second reason is that in a first order model theory, the compacity theorem is a very important tool and it is used almost all the time. So, when present, a lot of the first order model theory can be recouped.

Fuhrken (1964) proved that the compacity theorem holds in $\mathcal{L}(Q)$. Vaught (1964) proved that the set of valid formulas of $\mathcal{L}(Q)$ is recursively denumerable and, later, supplied a test of the completeness with a short but complex collection of axioms. These results had been refined, but Keisler (1970) proved correction and completeness for $\mathcal{L}(Q)$ in a sufficiently natural and intuitive system.

This way stimulated many research works about logic with generalized quantifiers to deal with certain mathematical structures. We can mention the research work on topological logic of Sgro (1977) and Garavaglia (1978).

Before mentioning the other source, we imagine that we must mention another result of Lindstrom (1969) that imposes new reflections on the extended logic.

To establish what becomes natural an extended logic or what allows us to find useful and interesting logic, the experience has shown that some of the next three principles must be followed:

(i) to construct in natural semantic a language that discloses the important notions of some particular domain of mathematical activity;

(ii) to keep the semantic restricted at the field of relevant ideas and implicit concepts of notions;

(iii) to find a syntax in which the basic ideas can naturally be expressed.

In any way, when it is left the target of the first order logic with the objective to prevent some difficulty, some results are lost. In particular, theorems on non-definable concepts in first order logic are defective in the extended theories.

Lindstrom proved in 1969 a theorem that opened new ways for the study of logic. This result affirms that in any attempt of constructing a more expressive logic than the first order logic, one of the analogous theorems of Countable Compacity or Löwenheim will not be satisfied.

With this, the first order logic can be characterized as a strong logic, which satisfies the two following properties:

Countable Compacity Property: if a countable set of sentences Δ does not have a model, then some finite subset of Δ does not have a model;

Löwenheim property: if a sentence has an infinite model, so it has a countable model.

The Compacity and Löwenheim (-Skolem) theorems are two of the most remarkable ones results in the first order model theory and they are quite used tools for investigations of the subject. The Lindstrom's characterization theorem is in a certain way discouraging, because it affirms that in the interest for extensions of first order logic, one of two results must be abandoned. Fortunately, some success was reached in the extended theories.

The second source deals with the relation between natural language and quantifiers. Montague (1974) presented a theory that unifies or identifies substantive expressions of the English language, such as “all the birds”, “Maria”, “it”, to the notion of generalized quantifiers.

Barwise and Cooper (1981), following Montague, deal with the identification between this syntactical category (noun expressions) of the natural language and the generalized quantifiers of logic, contributing for a re-approaching between logic and natural language, which attracted the interest of linguists and computers.

Barwise and Cooper had argued that the quantifiers of the classical first order logic are inadequate to deal with the quantified sentences of the natural language in at least two aspects:

- in natural language there are quantified sentences that cannot be expressed through the quantifiers $\exists$ and $\forall$. In this way, a semantical theory for the natural language cannot be based on a context restricted to classical predicate logic.
- the syntactical structure of quantified sentences in natural languages is very different from the syntactical structure of quantified sentences in the classical first order logic.

The central idea of Barwise and Cooper is that the non logical quantifiers correspond to the noun phrases of the natural language.

Thus, a quantifier has the syntactical form $D\eta$, where D is a determiner and η is a set of terms, or yet, a set of individuals.

More specifically, let’s consider a model $\mathcal{M} = (E, ||\ ||)$, where E is the domain and $||\ ||$ is a function of attribution. In this case, η denotes a set of individuals given by subsets X of E, whose the expression $D\eta(X)$ holds.

Some of the most usual determiners as every, some, many and exactly one are given by:

$||\text{every } \eta|| = \{X \subseteq E \mid ||\eta|| \subseteq X\}$

$||\text{some } \eta|| = \{X \subseteq E \mid ||\eta|| \cap X \neq \varnothing\}$

$||\text{many } \eta|| = \{X \subseteq E \mid ||\eta|| \cap X > |\, ||\eta|| - X|\}$

$||\text{exactly one } \eta|| = \{X \subseteq E \mid ||\eta|| \cap X = 1\}$.

Since $||D\eta|| = ||D||(||\eta||)$, so the determiners denote functions from $\mathcal{P}(E)$ into $\mathcal{PP}(E)$, as in the case of $||$ none$||$:

$||$none$||(Y) = \{X \subseteq E \ / \ Y \cap X = \varnothing\}$.

The other generalized quantifiers can be defined in the same acceptance, as Mostowski's quantifier Q_1 or Rescher's quantifier of majority. The notion of "majority", whose formalization appeared at the first time in Rescher (1962), is centered in a generalized quantifier semantically interpreted by a collection of subsets of the universe whose cardinal numbers are bigger than their complements:

$||Q_1 \eta|| = \{X \subseteq E \ / \ ||\eta|| = \omega\}$

$||$the majority $\eta|| = \{X \subseteq E \ / \ ||\eta|| > ||E - X||\}$.

Elements of this theory which were started with Barwise and Cooper (1981) are in several research works about generalized quantifiers as in Westerstahl (1984 and 1985).

Applications of the generalized quantifiers in computational contexts had led many investigators to imagine that the best environment for its treatment would be from non monotonic logic, as in Reiter (1980).

Further to this context, some interesting contributions of Brazilian thinkers had appeared. Sette, Carnielli and Veloso (1999) introduced a monotonic logic in which could be interpreted the natural quantifiers almost always (almost all) and almost never (almost none) through filters (ideal) prime and argued that this was a more productive alternative than in the non monotonic contexts.

From these perspectives, Grácio (1999) introduced an ample family of logical systems, the family of modulated logic, that will be more detailed in the next section. Each modulated logic is syntactically characterized for the inclusion of new generalized quantifiers, the modulated quantifiers, in the language of first order logic. Semantically, such quantifiers are interpreted by subsets of the set of the parts of the universe.

Grácio introduced two new monotonic logical systems, indicated by the formalization of the following notions of quantity: "many" and "for a 'good' part", semantically interpreted, respectively, by structures of *superior closed set* and *reduced topology*.

In this Brazilian environment we have directed our research in partnership with professor Maria Claudia Cabrini Grácio.

2. MODULATED LOGIC

The family of *modulated logic* is characterized by the inclusion of a generalized quantifier Q at first order logic language, named modulated quantifier, which must be semantically interpreted by a set $\mathbf{Q}$ of set of parts of universe. Intuitively, this subset $\mathbf{Q}$ represents an arbitrary set of propositions justified by evidences, into a basis of knowledge. So we present a formalization of modulated logic denoted by $\mathcal{L}(Q)$.

Let L be the first order language of type τ, with symbol for predicates, functions and constants, closed to the logical operators $\wedge$, $\vee$, $\rightarrow$, $\neg$ and also to the quantifiers $\exists$ and $\forall$.

The extension of first order logic $\mathcal{L}$ obtained by the inclusion of a generalized quantifier Q, named *modulated quantifier*, is denoted by $\mathcal{L}(Q)$.

The formulas (and sentences) of $\mathcal{L}(Q)$ are those of $\mathcal{L}$ increased by formulas generated by the following clause:

• if φ is a formula in $\mathcal{L}(Q)$, so $(Qx)\varphi$ is a formula of $\mathcal{L}(Q)$.

The concepts of free and bounded variable in a formula are extended to the quantifier Q, that is, if x is free in φ, then x occurs bounded in $(Qx)\varphi$.

We denote by $\varphi(t/x)$ the result of replacing every free occurrences of the variable x by the term t in φ. By simplicity, where there is no problem, we write only $\varphi(t)$ in the place of $\varphi(t/x)$.

The semantics of modulated logic is defined as it follows.

Let $\mathcal{A}$ be a classical structure of first order with domain A, and $\mathbf{Q}$ a set of subsets of A, such that $\emptyset \notin \mathbf{Q}$, that is, $\mathbf{Q} \subseteq \mathcal{P}(A) - \{\emptyset\}$. The *modulated structure* for $\mathcal{L}(Q)$, denoted by $\mathcal{A}^{\mathbf{Q}}$, is determined by the pair $(\mathcal{A}, \mathbf{Q})$.

Veloso (1998) named this type of structure of *complex structure* and Keisler (1970) of *weak structure*.

The interpretation of relational symbols, functional symbols and individual constants of $\mathcal{L}(Q)$ is the same of $\mathcal{L}$ in $\mathcal{A}$.

The *satisfaction* of a formula of $\mathcal{L}(Q)$ in a structure $\mathcal{A}^{\mathbf{Q}}$ is defined recursively as usual by the inclusion of the clause:

• let φ a formula whose set of free variable is included in $\{x\} \cup \{y_1, ..., y_n\}$ and consider a sequence $\bar{a} = (a_1, ..., a_n)$ in $\mathcal{A}$. Then:

$$\mathcal{A}^{\mathbf{Q}} \vDash (Qx)\varphi[x, \bar{a}] \text{ iff } \{ b \in A \,/\, \mathcal{A}^{\mathbf{Q}} \vDash \varphi[b, \bar{a}] \} \in \mathbf{Q}.$$

In this case, $\mathcal{A}^{\mathbf{Q}} \vDash \psi[\bar{e}]$ denotes that $\mathcal{A}^{\mathbf{Q}} \vDash_s \psi$, when the free variable of the formula ψ occurs in the set $\{z_1, ..., z_n\}$, $s(z_i) = e_i$ and $\bar{e} = (e_1, ..., e_n)$. Since $A \neq \emptyset$, if x does not occur free in φ, then $\mathcal{A}^{\mathbf{Q}} \vDash (Qx)\varphi[\bar{a}]$ iff $\mathcal{A}^{\mathbf{Q}} \vDash \varphi[\bar{a}]$. In particular, for a sentence $(Qx)\sigma(x)$, we have $\mathcal{A}^{\mathbf{Q}} \vDash (Qx)\sigma(x)$ iff $\{a \in A \,/\, \mathcal{A}^{\mathbf{Q}} \vDash \sigma(a)\} \in \mathbf{Q}$.

As mentioned, when identifying the set **Q** with mathematical structures, Grácio (1999) formalized propositions of the type "majority", "many" e "for a 'good' part". The logic of ultra filters, introduced by Sette, Carnielli and Veloso (1999) and Carnielli and Veloso (1997), formalizes propositions of the type "almost all" or "generally", and must also be considered as a particular case of modulated logic.

As we can observe, the notions of true and false, associated to the modulated quantifiers do not depend only on the underlying logic, but of which measure (quantification) we are using and that "[...] must be included as part of the model before the sentences have any true value whatsoever" (Barwise, Cooper 1981, p. 163).

The usual semantical notions such as model, validity, logical consequence, etc., for these systems, can appropriately be adapted.

The axioms of $\mathcal{L}(Q)$ are the same of $\mathcal{L}$, with the identity axioms, plus the following axioms destined to the quantifier Q:

(Ax_1) $(\forall x)\varphi(x) \rightarrow (Qx)\varphi(x)$

(Ax_2) $(Qx)\varphi(x) \rightarrow (\exists x)\varphi(x)$

(Ax_3) $(\forall x)(\varphi(x) \leftrightarrow \psi(x)) \rightarrow ((Qx)\varphi(x) \leftrightarrow (Qx)\psi(x))$

(Ax_4) $(Qx)\varphi(x) \leftrightarrow (Qy)\varphi(y)$.

Intuitively, given an interpretation whose universe is A and the formulas φ and ψ, with exactly one free variable x, then for the sets $[\varphi] = \{a \in A \,/\, \varphi[a]\}$ and $[\psi] = \{a \in A \,/\, \psi[a]\}$, the axiom ($Ax_1$) affirms that if all the individuals satisfy φ, then the universe A belongs to the family of subsets **Q** of A that interprets the quantifier Q. The axiom (Ax_2) affirms that the empty set does not belong to **Q**. The axiom (Ax_3) affirms that if [φ] and [ψ] are identical, then one of them belongs to family **Q** that interprets Q if, and only if, the other also belongs to **Q**.

The rules of modulated logic are the usual rules of: *Modus Ponens* (MP) e Generalization (Gen).

Other axioms must be introduced to characterize specific modulated logic as we can see in (Grácio, 1999).

The usual syntactical notions for $\mathcal{L}(Q)$ as sentence, proof, theorem, logical consequence, consistence and others are defined from an analogous way to the classical logic.

2.1 SOME STRUCTURES FOR Q

In this subsection we introduce some mathematical structures which are relevant for the next discussions.

Let E be a universe of discourse. For each subset $B \subseteq E$, we denote its complement by B^C.

We define following a hierarchy of structures showing *up*, for the all E (superior closed set, filter, prime filter). Dually, we can define another hierarchy pointing *down*, for the empty set (inferior closed set, ideal, prime ideal), but due to their similarity we define only one of the hierarchies.

Let E be a non empty set. A superior closed set in E is a collection $\Theta \subseteq \mathcal{P}(E)$ for which:

(i) if $A \cap B \in \Theta$, them $A, B \in \Theta$[1];

(ii) $E \in \Theta$;

(iii) $\varnothing \notin \Theta$.

[1] This condition is equivalent to (i'): if $A \in \Theta$ and $A \subseteq B$, them $B \in \Theta$.

A *filter*[2] in E is a collection Θ such that Θ is a superior closed set and holds:

(iv) if A, $B \in \Theta$, them $A \cap B \in \Theta$.

A *prime filter* or *ultra filter* is a filter Θ such that:

(v) if $A \in \Theta$ or $B \in \Theta$, them $A \cup B \in \Theta$[3].

3. GENERALIZED MODALITIES

In this section, following motivations originated in modulated predicate logic, we introduce a new kind of modal logic.

3.1. STRUCTURES, DOMAINS AND "LOGIC"

We saw, in previous sections, how it is possible to interpret generalized quantifiers through aggregation of mathematical structures to the domain of individuals. These structures supply measures to the domain of individuals (Barwise, Cooper, 1981). We also saw how logic associated to these generalized quantifiers can be constructed. Such logic had been initially designated "generalized logic", and in the research work of Maria Cláudia Cabrini Grácio (1999) a particular class of such logic had been designated "modulated logic", for which we had reserved a special look. The same procedure can be applied, *mutatis mutandis*, to the achievement of generalized modalities and, with these, to the construction of modulated modal logic. The adaptation consists of substituting the domain of individuals for the domain of possible worlds. Thus, instead of aggregating mathematical structures to the domain of individuals, one aggregates structures to the domain of possible worlds or, alternatively, for each possible world we aggregate specific structures to the set of its accessible possible worlds.

A proposal of semantics for this last case will be exemplified in the next section. In the rest of this section, we make two brief comments on

[2] This definition correspond to usually known *proper filter* in which the empty set $\varnothing$ does not belong to Θ.

[3] In Boolean structures, the item (v) is equivalent to (v'): for each $A \in \mathcal{P}(E)$, exactly one um among A and A^C is in Θ.

what we consider a non critical reception of some notions employed in the Brazilian tradition of research in logic, that is, notions in relation to which our tradition adopts an eminently pragmatic position.

However, it conveys before to stand out that the use of the term "world" is not necessarily associated to any specific ontological thesis. We can, for instance, simply adopt it as a technical term, like Kant in the *Inaugural dissertation* of 1770. In that research work, Kant compares the notion of world with the notion of simple in the following way: "(...) just as analyses does not come to an end until a part is reached which is not a whole, that is to say a SIMPLE, so likewise synthesis does not come to an end until we reach a whole which is not a part, that is to say a WORLD" (Kant, 1770). Here, still, the notion of *possible* world makes sense, since we need a criterion, in that case the joint possibility (co-possibility), to guideline the synthesis process and to indicate its limits.

The first comment is regarding the attribution of the expression "logical" to the constructions obtained by aggregation of mathematical structures to the domains, which are domains of individuals as well as domains of possible worlds. Barwise and Cooper (1981) suggested that, when aggregating such structures to the domains, we lose the characteristic notes of a logic, that is, its generality, its universal applicability, its application to arbitrary domains.

The second comment, also suggested by the reading of (Barwise, Cooper, 1981), regards the adoption of the generalized operators (quantifiers and/or modalities) as monadic operators. Barwise and Cooper suggested that the universal and the existential quantifiers are atypical cases of quantifiers of the ordinary language, because they are monadic operators. According to Barwise and Cooper, the majority of the quantifiers of the ordinary language are dyadic operators, including those formalized as generalized quantifiers. The same could be said of the corresponding modalities.

It is not up to us, at this moment, to make a value judgment on such comments of Barwise and Cooper. However, we consider important to mention them, because the technical development is not always followed in the same degree of sophistication, with critical examination of the notions obtained this way.

3.2. EXTENDED STANDARD MODELS

Sautter (2000) gives an alternative to the semantical treatment of modulated modal logic. The author examines a family of multi-modal logic in which, simultaneously to the study of arbitrary normal modalities, also associated modalities with filters and modalities associated with ideals are examined, that is, generalized modalities are examined. The unified semantical treatment of these types of modalities is developed with the help of the following extension of the notion of standard model:

$M = \langle W, R, P, F, I\rangle$ is an extended standard model if:

(i) W is a non-empty set of possible worlds;

(ii) R is a binary relation of accessibility in W;

(iii) P is a mapping of natural numbers in subsets of W. $P(n) = W'$ establishes that the atomic sentence P_n is true in all the possible worlds pertaining to W' and only in these;

(iv) F is a mapping of W in subsets of $\mathcal{P}(W)$ such that, for each $w \in W$, $F(w)$ is a filter on $R[w]$, that is, there is a filter for each possible world;

(v) I is a mapping of W in subsets of $\mathcal{P}(W)$ such that, for each $w \in W$, $I(w)$ is an ideal on $R[w]$, that is, there is an ideal for each possible world;

(vi) For all $w \in W$, $F(w) \cap I(w) = \varnothing$.

The truth conditions of the normal modalities are the usual ones and the truth conditions of the generalized modalities are the supplied ones by the following definitions:

(a) $\models^{M}_{w} \nabla A$ if $\{w' \in R(w) \ / \models^{M}_{w'} A\} \in F(w)$, where ∇ is a generalized modality whose meaning is associated to filters and that has a non-logical necessity character.

(b) $\models^{M}_{w} \Delta A$ if $\{w' \in R(w) \ / \models^{M}_{w'} A\} \in I(w)$, where Δ is a generalized modality whose meaning is associated to ideals and that has a non-logical impossibility character.

Here the association to filters and to ideals is obviously not essential: definitions can be easily adjusted for generalized modalities associated to any other mathematical structures.

4. TRANSLATIONS BETWEEN LOGIC

Our advisor Itala, some of her fellows, and some of her students have a great interest in translations between logics (Da Silva, D'Ottaviano and Sette 1999, D'Ottaviano and Feitosa 1999, Feitosa and D'Otaviano 2001). It is subject matter of their interests: the characterization of useful notion of translations between logics; a taxonomy for these translations; model-theoretical, algebraic and category treatments of translations; elements of a Theory of Translations Between Logics; as well as the construction or proof of existence of diverse translations between logics. In (Feitosa, 1997) we have a complete exhibition of these contributions.

Afterwards, we will mention two case studies, one of which uses generalized modalities, the importance of the 'theory of translations between logics' as an instrument for philosophical analysis. Before, however, we will introduce the notion of conservative translation between logics, essential to the rest of the research work.

Let $\mathbf{L}_1 = \langle L_1, C_1\rangle$ and $\mathbf{L}_2 = \langle \mathbf{L}_2, C_2\rangle$ be logics, such that $\mathbf{L}_1$ and $\mathbf{L}_2$ are sets of formulas and C_1 and $\mathbf{C}_2$ are consequence operators of the respective logic. A *translation* from $\mathbf{L}_1$ into $\mathbf{L}_2$ is a mapping t: $\mathbf{L}_1 \to \mathbf{L}_2$ such that, for all subset $\Gamma\cup\alpha$ of $\mathbf{L}_1$, we have:

$\alpha \in C_1(\Gamma) \Rightarrow t(\alpha) \in C_2(t(\Gamma))$.

A *conservative translation* of $\mathbf{L}_1$ into $\mathbf{L}_2$ is a mapping t: $\mathbf{L}_1 \to \mathbf{L}_2$ such that, for all subset $\Gamma\cup\alpha$ of $\mathbf{L}_1$, we have:

$\alpha \in C_1(\Gamma) \Leftrightarrow \mathrm{t}(\alpha) \in C_2(t(\Gamma))$.

5. MODAL LOGIC AND TRANSLATIONS

In this section we introduce some concrete modal logics having as a motivation the structures associated to the modulated predicate logic and we use the translations as an instrument for analysis of the interrelations between diverse modal logics, with emphasis on deontic logic.

5.1 A HIERARCHY OF MODULATED MODAL LOGIC

We initially introduce some concrete modal logics and show, with the help of conservative translations, their connections with more known logics in literature.

The clauses related to the notion of a family of superior closed sets (see Section 2 for an examination of this mathematical structure and other structures used in this section) are formalized by the following modal axioms schemes:

(i) $\nabla(\alpha\wedge\beta) \to (\nabla\alpha\wedge\nabla\beta)$,

corresponds to the clause of closure for supersets,

(ii) $\nabla(\alpha\vee\neg\alpha)$,

corresponds to the clause that affirms that the unitary set is a superior closed set, and

(iii) $\neg\nabla(\alpha\wedge\neg\alpha)$,

corresponds to the clause that affirms that the empty set is not a superiority closed set.

Adding these axioms schemes and the deduction rule:

(iv) $\alpha \leftrightarrow \beta \ / \ \nabla\alpha \leftrightarrow \nabla\beta$

to the classical propositional logic we get the logic of the superior closed sets.

The clauses related to the notion of a family of inferior closed sets are formalized by the following modal axioms schemes:

(i) $\Delta(\alpha\vee\beta) \to (\Delta\alpha\wedge\Delta\beta)$

that corresponds to the clause of closure for subsets,

(ii) $\Delta(\alpha\wedge\neg\alpha)$,

corresponds to the clause that affirms that the empty set is an inferior closed set, and

(iii) $\neg\Delta(\alpha\vee\neg\alpha)$,

corresponds to the clause that affirms that the unitary set is not an inferior closed set.

Adding these axioms schemes and the deduction rule:

(iv) $\alpha \leftrightarrow \beta \;/\; \Delta\alpha \leftrightarrow \Delta\beta$

to the classical propositional logic we get the logic of inferior closed sets.

Adjoining the logic of superior closed sets and inferior closed sets and adding the axioms scheme:

(v) $\neg(\nabla\alpha \wedge \Delta\alpha)$,

we get the logic of superior closed sets and inferior closed sets.

If we interpret the symbol ∇ as an operator of necessity and the symbol Δ as an operator of impossibility, this set of axioms schemes corresponds to the set of axioms schemes characteristic of the normal deontic logic (Chellas, 1980, p. 224).

The filter logic results from adding to the logic of superior closed sets the axioms scheme:

(vi) $(\nabla\alpha \wedge \nabla\beta) \rightarrow \nabla(\alpha \wedge \beta)$,

that corresponds to the clause of closure for finite intersections.

The interpretation of ∇ as a necessary operator supplies a conservative translation of the filter logic in the modal logic *K*, to be precise, in these conditions, the systems coincide (Chellas, 1980, p. 115).

The ideal logic results from adding to the logic of inferior closed sets the axioms scheme:

(vi) $(\Delta\alpha \wedge \Delta\beta) \rightarrow \Delta(\alpha \vee \beta)$,

that corresponds to the clause of closure for finite unions.

The interpretation of Δ as an impossible operator also supplies a conservative translation of the ideal logic in the modal logic *K*, therefore, in these conditions, the systems coincide.

Adjoining the ideal and filter logic and adding the axioms scheme:

(vii) $\neg(\nabla\alpha \wedge \Delta\alpha)$,

we get the filter and ideal logic.

If we interpret the symbol ∇ as a necessary operator and the symbol Δ as an impossible operator we have a conservative translation of the filter and ideal logic in the modal logic *KD*, the basic normal logic of deontic modalities.

The prime filter (ultra filter) logic results from adding to the filter logic the axioms scheme:

(viii) $\nabla\alpha\vee\nabla\neg\alpha$,

which corresponds to the clause of complementing the filter.

If we interpret ∇ as a necessary operator we have a conservative translation of the prime filter logic into the modal logic *KDc* (Hughes, Cresswell, 2003, p. 123).

The prime ideal logic results from adding to the ideal logic the axioms scheme:

(ix) $\Delta\alpha\vee\Delta\neg\alpha$,

which corresponds to the clause of complementing the ideal.

If we interpret Δ as an impossible operator it still supplies in time, a conservative translation of the prime ideal logic into the modal logic *KD*.

Adjoining the prime filter and prime ideal logic and adding the axioms scheme:

(x) $\neg(\nabla\alpha\wedge\Delta\alpha)$,

we get the prime filter and prime ideal logic.

If we interpret ∇ as a necessary operator and Δ as an impossible operator we have a conservative translation of the prime filter and prime ideal logic into the modal logic *KD*! (Chellas, 1980, p. 93).

The principal filter logic results from adding to the filter logic the axioms scheme:

(xi) $((\nabla\alpha\wedge(\beta\rightarrow\alpha) \rightarrow \neg\nabla\beta) \wedge (\nabla\chi\wedge(\delta\rightarrow\chi) \rightarrow \neg\nabla\delta)) \rightarrow (\alpha\leftrightarrow\chi)$,

which corresponds to the clause of the existence of a base for the filter.

The principal ideal logic results from adding to the ideal logic the axioms scheme:

(xii) $((\Delta\alpha\wedge(\alpha\rightarrow\beta) \rightarrow \neg\Delta\beta) \wedge (\Delta\chi\wedge(\chi\rightarrow\delta) \rightarrow \neg\Delta\delta)) \rightarrow (\alpha\leftrightarrow\chi)$

which corresponds to the clause of the existence of base for the ideal.

Adjoining the principal filter and principal ideal logic and adding the axioms scheme:

(xiii) $\neg(\nabla\alpha\wedge\Delta\alpha)$,

we get the principal filter and principal ideal logic.

Finally, the clauses associated to the notion of ubiquity are formalized with help of the following modal axioms schemes:

(xiv) $(\bullet\alpha\wedge\bullet\beta) \rightarrow \bullet(\alpha\wedge\beta)$,

which corresponds to the clause of closure for supersets, and

(xv) $(\bullet\alpha\wedge\bullet\beta) \rightarrow \bullet\ (\alpha\vee\beta)$,

which corresponds to the clause of closure for subsets).

Adjoining these axioms schemes and the rule of inference

(xvi) $\alpha\leftrightarrow\beta$ / $\bullet\alpha\leftrightarrow\bullet\beta$

to the classical propositional logic we get the ubiquity logic.

The existence of simple conservative translations between logic constructed by means of the mere use of very general mathematical structures and the logic *KD* only stands out the incapacity of this last logic in capturing any significant deontic content. Rigorously, the axioms scheme for *D*, characteristic of the normal logic of deontic modalities, simply affirms the non-vacuity of true mandatory propositions, that is, the proposition according to which α is mandatory in a possible world *w* only if it is a possible world w' (not necessarily distinct to *w*) accessible to *w* such that α is true in w'.

5.2 KANTIAN DEONTIC LOGIC

Modulated modal logic is not the only one that, thanks to the use of conservative translations, discloses an unsuspected relation with deontic notions (or notions considered deontic). Kielkopf (1975) considers a conservative translation of the standard deontic logic in one subsystem of the temporal logic *K*1 resulting from the restriction to the following language:

(1) if *p* is a propositional constant, then *p* is a formula.

(2) if α and β are formulas, then $\neg\alpha$, $\alpha\supset\beta$, $\Box\Diamond\alpha$ and $\Diamond\Box\alpha$ are formulas.

(3) formulas are only those which result by the application of steps (1) e (2).

Kielkopf considers the following function of translation *t*, where $\circ$ it is the modality of deontic necessity, *P* is the modality of deontic possibility,

◊ is the modality of temporal possibility and □ is the modality of temporal necessity:

(i) $t(\alpha) = \alpha$, if α is a propositional constant.

(ii) $t(\neg\alpha) = \neg t(\alpha)$.

(iii) $t(\alpha\supset\beta) = t(\alpha)\supset t(\beta)$.

(iv) $t(\circ\alpha) = \Diamond\Box t(\alpha)$.

(v) $t(P\alpha) = \Box\Diamond t(\alpha)$.

This conservative translation has the peculiar property (peculiar because it does not normally consider the inquiry of the characteristics preserved by the translation function) of preserving the irreducible modalities, that is, *K* and SDL are the same ones, modulo translation function. This example gives us a way to continue the development of the 'theory of translations between logics', to point general conditions to be satisfied by the functions of translations such that some relevant properties are preserved.

This conservative translation of Kielkopf has also the merit to light up the approach proposed by Kant, when considering the Categorical Imperative in its diverse formulations, particularly that formulation which puts in analogy the Kingdom of Nature and the Kingdom of Ends. Kielkopf suggests that, far from making us incur into the Naturalistic Fallacy, this conservative translation shows us that the modalities of *K*1 are morally tinged.

Even if we disagree with the reading of Kielkopf, his proposal will be able to show us how much the 'theory of translations between logics', independently of its mathematical value, can be an useful instrument of philosophical analysis.

5.3 PROTO AND HYPOMODALITIES

The existence of simple conservative translations between the standard deontic logic and the temporal logic *K*1 on one side, and between the standard deontic logic and modal modulated logic on the other side, is an indication that the way adopted in the formalization of the deontic reasoning has not succeeded.

The standard deontic logic is a subsystem of the modal logic **S5**, whose modalities are, traditionally, adopted as the canonical interpretation of the ontic (alethic) modalities used by the philosophers. This seems to be an error, for this reason an example of the history of philosophy will help us to clarify such error.

Let us consider, on one hand, the logical necessity and the logical possibility, that is, a pair of dual logical modalities, and, on the other hand, the real necessity and the real possibility, that is, a pair of dual real modalities. There are the following relations between these pairs of modalities:

(i) if a proposition is logically necessary, then it also has real necessity, but the inverse is not always the case.

(ii) if a proposition has real possibility, then it is also logically possible, but the inverse is not always the case.

Disregarding, for argumentative purposes, the problem proposed by Hume, Moore and others concerning the existence of conceptual and inferential barriers between the ontic and the deontic, the relations between ontic and deontic modalities, in case they are permissible, are analogous to that which subsists between the logical modalities and the real modalities:

(i) if a proposition is ontically necessary, then it also is deontically necessary (obligatory), but the inverse is not always the case.

(ii) if a proposition is deontically possible (permissible), then it also is ontically possible, but the inverse is not always the case.

This similarity gives us the opportunity to propose the following distinction:

Let $\mathbf{L}_1 = \langle L_1, C_1 \rangle$ and $\mathbf{L}_2 = \langle L_2, C_2 \rangle$ be logics, M_1 be a modality of $\mathbf{L}_1$, M^*_1 be another modality of $\mathbf{L}_1$ dual of M_1, M_2 be a modality of $\mathbf{L}_2$ and M^*_2 be another modality of $\mathbf{L}_2$ dual of M_2. At least two types of correlation are allowed between the pairs of modalities $\langle M_1, M^*_1 \rangle$ and $\langle M_2, M^*_2 \rangle$:

(i) M_1 is a *prototype* of M_2 (and M_2 a *deuterotype* of M_1) if $M_1\alpha \in C_1(\varnothing)$, them $M_2\alpha \in C_2(\varnothing)$, and if $M^*_1\alpha \in C_1(\varnothing)$,them $M^*_2\alpha \in C_2(\varnothing)$.

(ii) M_1 is a *hypotype* of M_2 (and M_2 is a *hypertype* of M_1) if $M_1\alpha \in C_1(\emptyset)$, them $M_2\alpha \in C_2(\emptyset)$, and if $M^*_2\alpha \in C_2(\emptyset)$, them $M^*_1\alpha \in C_1(\emptyset)$.

Translated to this terminology, our thesis assumes the following way: although we usually formalize the deontic necessity as a prototype of the ontic necessity, the correct formalization is the one in which that is a hypotype of this. This would point out the deontic logic as an incomparable logic with the most traditional systems of modal logic (**S5** and **K**, for example), which does not seem to be out of question, considering the differences between the 'is' being reality and the 'ought' being reality.

FINAL CONSIDERATIONS

In this research work we started with two sufficient general results of logical analysis: (i) the generalized quantifiers, viewed from the perspective of modulated logic, a Brazilian logical contribution and (ii) translations between logics, subject matter investigated by our Brazilian colleagues and our advisor, to make a comparative analysis of diverse modal systems. Based on these reflections we suggested a definition that allows us a conceptual separation of the ontic and of the deontic in the modal logical contexts.

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