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SETS, CLASSES AND THE PROPOSITIONAL CALCULUS

E.G.K. LOPEZ-ESCOBAR

University of Maryland
College Park
USA

egkle@math.umd.edu

In honour of Professor I.M.L. D'Ottaviano

Abstract: The propositional calculus **AoC**, "Algebra of Classes", and the extended propositional calculus **EAC**, "Extended Algebra of Classes" are introduced in this paper. They are extensions, by additional propositional functions which are *not* invariant under the biconditional, of the corresponding classical propositional systems. Their origin lies in an analysis, motivated by Cantor's concept of the cardinal numbers, of A. P. Morse's impredicative, polysynthetic set theory.

INTRODUCTION

In the Foreword to Boyer's "History of Mathematics" (Boyer 1989), Isaac Asimov makes the following observation:

> Now we can see what makes [the history of] mathematics unique. Only in mathematics is there no significant correction—only extension. Once the Greeks had developed the deductive method, they were correct in what they did, correct for all time. Euclid was incomplete and his work has been extended enormously, but it has not had

> to be corrected. ... Each great mathematician adds to what came previously, but nothing needs to be uprooted. ... and is as functional now as it was when Thales worked out the first geometrical theorems nearly 26 centuries ago.
>
> Nothing pertaining to humanity becomes us so well as mathematics. There, and only there, do we touch the human mind at its peak.

Now although I agree in principle with Asimov's remark, nevertheless I find that there have been exceptions. A particularly interesting exception concerns the work of George Boole (Boole 1847), which together with (Boole 1854), is generally recognized as the starting point of *Mathematical Logic*. Perhaps the following quotation from (Burris 1998), page 32, expresses the current sentiment towards Boole's methods:

> Needless to say Boole's book is not used as an introduction to logic, except perhaps to study how so many good ideas could be mixed up with so much confusion.

One of the consequences of this article is a logical resolution (of some) of Boole's 'confusions'. Of course this comes as no surprise; Bertrand Russell stated it in Russell (1945), p. 52:

> I have put the argument here to remind the reader that philosophical theories, if they are important, can generally be revived in a new form after being refuted as originally stated. Refutations are seldom final; in most cases, they are only a prelude to further refinements.

PART I

The first goal of this paper is to examine the following methods used by G. Boole:

1. The use of **0** as both[1] the empty set and the truth value '*False*'.
2. The use of **1** as both the class of all things and as the truth value '*True*'.

[1]Boole also considers **0** as a real number.

3. The use of variables both as sets (that is members, or perhaps subsets, of **1**) and as propositions.

0.1. THE ALGEBRA OF LOGIC

At the time the above identifications were not considered to be problematic and in fact C. S. Peirce started using the same symbol for the conditional and for the subset relation; then Ernst Schröder in his monumental compendium *Vorlesungen über die Algebra der Logik* took Peirce's lead and many other symbols were given dual interpretations, for example, the universal quantifier was often interpreted as intersection. The ambiguities in the operator symbols led to an economy of notation; but a price had to be paid for the economy, namely it made expressions more difficult to understand.

0.1.1. POLYSYNTHETIC LANGUAGES

As a consequence of the identification Schröder, in (Schröder 1890), obtained expressions that would be considered nonsense by the (late) XXth Century standards. A typical example consists of the following[2]:

$$(A = \mathbf{0}) = (A_{|} = \mathbf{1}) = A_{|}$$

Consider the expression: $(A = \mathbf{0}) = A_{|}$ included in the one above. From the current viewpoint, that is, as an expression of first-order logic, it is not a *well formed expression* since it equates an expression which is has a truth value, namely: $(A = \mathbf{0})$, with $A_{|}$, which represents a class. To many the above sentence would be as ridiculous as the English phrase:

Snow is white is equal to the set of algebraic numbers.

A probable reason for the unacceptability of the above English expression is that the English language requires that the *words* of the language belong to separate categories; noun, verb, adjective *etc.*, from which one can then construct 'noun phrases' *etc.* which in turn belong to different categories.

[2]Schröder uses **0** for the empty class, **1** for the universal class and $A_{|}$ for the complement (or the negation) of A.

And yet there are other natural languages, such as the polysynthetic[3] Yup'ik Inuit, in which the *words* of the language can not be so easily characterized[4].

Since in Schröder (1890) the 'words' of the formal (symbolic) language may belong to more than one semantical category, I propose to call it a **polysynthetic formal language**[5].

0.2. HILBERT'S *BEWEISTHEORIE*

In the early 1900's David Hilbert introduced his *Beweistheorie* with its clinically unambiguous, and thus simpler to understand, formal language with the result that the Boole-Schröder polysynthetic language was essentially forgotten. Furthermore Hilbert's Proof Theory allowed one to state as combinatorial problems some very interesting mathematical questions about Cantor's Set Theory; for example the Continuum Hypothesis. Since the problems could now be stated in a combinatorial form, the euphoria of those times led people (including Hilbert) to believe that it was only a matter of time before they would be resolved.

Since Cantor's Set Theory was much more mathematically interesting than the logic of classes, research in the foundations of mathematics veered away from the Algebra of Logic. It had a partial revival in the Henkin/Tarski's *Cylindric Algebras*; however the formal language used

[3]**Synthetic**: (Form, language) in which grammatical distinctions are realized by inflections. Opp. *analytic*: e.g. a possive construction is realized analytically in Italian (*la casa a Cesare,* lit. 'the house of Cesar'), but was realized synthetically with a genetive inflection, in Latin *domus Cesaris* 'house-nomsg Caesar-gensg'. **Polysynthetic**:(Type of language) in which there is a pattern of incorporation or in which, in general affixes realize a range of semantic categories beyond those of synthetic languages in e.g. Europe. *The Concise Oxford Dictionary of Linguistics*, Oxford University Press 1997.

[4]For example the single word: *tuntussuqatarniksaitengqiggtuq,* built up as follows: *tuntu ssur qatar ni ksaite ngqiggte uq,* and whose interpretation is *'He had not yet said again that he was going to hunt reindeer'.*
Source //www.sil.org/linguistics/GlossaryOfLinguisticTerms... and Eliza Orr, cited by Payne, T. 1997a.

[5]As far as I know, the term *polysynthetic* has not been used in the context of formal or symbolic languages; *polymorphic* has been used but with quite a different interpretation.

was modelled on Hilbert's language for mathematical logic and not on the Boole-Schröder polysynthetic one.

0.3. MORSE'S POLYSYNTHETIC LANGUAGE

A. P. Morse (1911-1984) developed an (almost) universal tool for mathematics. 'Universal' in a *practical* sense rather than in a foundational sense. Although at that time (middle 1900's) personal computers were only in Science-Fiction stories, Morse's language was in many ways like a computer language in that there were a few reserved symbols[6] and then it was up to the individual how to generate additional expressions[7]. In that way mathematicians could use their own familiar symbols, and Morse did in fact use it for his research in Functional Analysis.

Morse considered that all the mathematical 'objects' of interest were either sets or collections of sets so it is not surprising that the fundamental basis of his system is a set (class) theory. The requirements that he placed on the theory were that (i) it be powerful and at the same time avoided the known pitfalls, (ii) it follow as close as possible the mathematicians' definitions for specific sets and (iii) be as economical as possible.

Cantor's (unrestricted) set theory was inconsistent, Zermelo's was more suitable for logicians and philosophers, Frege's was flawed and Boole's 'Logic of Classes' barely got off the ground! The one that he found most suitable was that one developed by J. von Neumann[8] in which there were *classes* and *sets* (that is classes which are members of other classes)

Unlike Hilbert who carefully distinguished what belongs to *Logic*

[6]The most important reserved symbols in Morse's system are: $\bigwedge$, $\rightarrow$, $\in$, and the *definor*: $\equiv$.

[7]As long as they adhered to some very general directives, such as: '*a* mark *is a more or less connected inscription*' and '*a* symbol *is a mark which is not a quotation mark*'. It is clear that A. P. Morse was influenced by St. Leśniewski, perhaps through Tarski.

[8]Although often ignored, Mirimanoff also deserves much of the credit, see (Mirimanoff 1917), (Mirimanoff 1917a), (Mirimanoff 1920) and (Specker 2001). The reader is warned that Morse called '*sets*' what nowadays are '*classes*' and '*points*' what now are called '*sets*'.

and what belongs to *Mathematics*, Morse had a more holistic approach and did not find any advantage in separating logic from mathematics[9]. In particular he eschewed the separation of the well formed expressions of the Hilbertian languages into the **terms**, which represent mathematical 'objects' and the **formulae** which correspond to 'truth values'. Thus in Morse's Set Theory one finds well formed expressions of the form:

$$x \leftrightarrow (\mathbf{0} \in x) \ \ .$$

0.4. MORSE'S SET THEORY

Although for many years Morse had been **using** his Set Theory in the courses he presented at Berkeley, few persons outside of the Bay area knew of its existence. In the middle fifties, John L. Kelley, also at Berkeley, published the book (Kelley 1955) whose appendix contains a formulation, in the first order language of Hilbert's mathematical logic, the synthetic part of Morse's Set Theory. It became known throughout the research community as '**KM**' (Kelley/Morse Impredicative Set Theory).

Finally in 1965 Morse published the monograph '*A Theory of Sets*' which not only contains his (polysynthetic) impredicative class theory, but also the directives for constructing one's own 'Calculus Ratiocinator'.

Morse's book[10] has the dubious distinction that it is always mentioned in the bibliography of just about every article concerning impredicative set theory and yet the set theory actually used (or mentioned) in the paper itself is **KM** rather than Morse's!

[9]A view shared by L. E. J. Brouwer, but for quite different reasons. It is interesting to note that Hao Wang, in (Wang 1996), p. 266, reports that "On June 1971 I asked Gödel about the scope of logic and, specifically, about the view that logic should be identified with predicate logic. He had told me earlier that for him, logic included set theory and concept theory. On this occasion, however, he expanded on the relation between logic and predicate logic: 8.4.11 The propositional calculus is about language or deals with the original notion of language: truth, falsity, inference. We include the quantifiers because language is about something—we take propositions as talking about objects. They would not be necessary if we did not talk about objects; but we cannot imagine this. Even though predicate logic is "distinguished" there are also other notions, such as *many*. . . ."

[10]Reprinted in 1985 (Morse 1985).

0.5. MORSE'S POLYSYNTHETIC AXIOMS

The inferential apparatus for Morse (1965) consists basically of modus ponens and the rule for universal generalization. Following Leśniewski, he allows definitions as theorems and in this way he is able to introduce, as part of the formal language and not as external definitions, both additional symbols for logical connectives and quantifiers as well as special classes, for example: 0 (for the empty class) and U (for the class of all sets).

Morse liked to present the most compact axioms possible and thus in the axioms for sets he often made use of the polysynthetic nature of the language. However one may rewrite those axioms in the language of **KM** without affecting the theory.

The axioms which establish the polysynthetic nature of Morse's Class Theory, and which could not be rewritten in the first order language of Set Theory are basically the following:

2.5.0 $\quad x \longleftrightarrow (0 \in x)$

2.5.1 $\quad (t \in \mathsf{U}) \longrightarrow ((t \in (x \in y)) \longleftrightarrow (x \in y))$

2.5.2 $\quad (t \in a) \longrightarrow ((t \in (p \longrightarrow q)) \longleftrightarrow ((t \in p) \longrightarrow (t \in q)))$

2.5.3 $\quad (t \in \bigwedge x\underline{u}x) \longleftrightarrow \bigwedge x(t \in \underline{u}x)$

$\underline{u}x$ is Morse's way of presenting a schema of formulae.

Although at first sight the above axioms may appear strange, I propose to show that they have at least two plausible interpretations.

1. THE CARDINAL NUMBERS OF CANTOR

It is fairly well recognized that for Cantor, cardinals and ordinals were not sets although sets could be constructed from them, see (Wang 1974, p. 222); in effect they were *urelementen*. Since Cantor claimed that to every set A there corresponded a unique cardinal number $\mathfrak{C}A$ (Cantor's notation was: $\overline{\overline{A}}$), if one were to try to express Cantor's ideas in a first order language then in addition to the primitive term $\in$ one would require a primitive term to represent the 'extraction' of the cardinal number of a set; for the sake of definiteness I will use: $\mathfrak{C}$.

Let us use '$(x \cong y)$' as an abbreviation for the (formal) set theoretic statement that the sets x and y are *equinumerous*, *i.e.* that there exist a bijection between them and let us use '$\equiv$' for the biconditional. Then the following would be the kind of axiom required:

$$\bigwedge_x \bigwedge_y (\ (\mathfrak{C}x = \mathfrak{C}y) \ \equiv \ (x \cong y)\ ). \qquad \text{(AxCard)}$$

In the early 1900's Mirimanoff (and later, von Neumann) defined the *ordinal numbers* as specific well-ordered sets. The $\aleph$-cardinal numbers were then defined as the initial ordinal numbers. However in the absence of the axiom of choice, it was not clear whether all cardinal numbers were in fact $\aleph$-cardinals; hence it was not known whether one could consider all of Cantor's cardinals to be sets rather than urelementen.

It was not until much later that Richard Montague (1930-1971) and Dana Scott defined $\mathfrak{C}A$ as being the set of all those sets *of minimal rank* that are equinumerous with A and thus showing[11] that Cantor's set theory was just that, *i.e.* 'a theory of sets' and not 'a theory of sets and cardinal numbers'; and not only that, but also that *pure sets and classes* (*i.e.* without any urelementen) suffices for mathematics.

No one disputes that the iterative concept of pure sets has been extremely rewarding for the understanding of *sets*. Nevertheless it is worthwhile to remember that Cantor never discarded the view that the cardinal numbers had an independent existence and appeared to be ambivalent concerning the definition of sets; for example—as reported in (Wang 1974, p. 188):

> One feels vaguely that the iterative concept corresponds pretty well to Cantor's 1895 'genetic' definition of set:[12] 'By a "set" we shall understand any collection into a whole M of definite, distinct objects m (which will be called "elements" of M) of our intuition or our thought'.

[11]This definition clearly does not require the axiom of choice, however it does require well-foundedness. Wang states, in (Wang 1974, p. 212), that Cantor operated under the assumption that all sets were well-founded.

[12]'Unter einer "Menge" verstehen wir jede Zusammenfassung M von bestimmten wohlunterschieden Objecten m unserer anschauung oder unseres Denkens (welche die "Elemente" von M gennanten werden) zu einem Ganzen'.

> In 1882, Cantor explains that a set of elements is *well defined*, if by its definition and by the logical principle of excluded middle we must recognize as internally determined whether any object of the right kind belongs to the set or not[13]. One is inclined to think that the concept of set implicit in this context is closer to the logical concept rather than the mathematical one.

1.0.1. RELATION BETWEEN A SET AND ITS CARDINAL NUMBER

Note that even in a thoroughly Platonistic (or Theistic) interpretation of Set Theory, we still have the problem of understanding the relation between a set S and its Cantorian cardinal number $\mathfrak{C}(S)$.

The Cantorian 1882 logical concept of 'set' only considers its 'members' and thus leaves open the question of the cardinality of the set.

The 1895 definition, which can be read as an iterative 'construction', says that a set is a

> 'collection into a whole M of definite, distinct objects m (which will be called "elements" of M) of our intuition or our thought'.

This seems to suggest that the set M comes into 'existence' with the act of 'collecting the distinct objects m into a whole'; however the act of collecting into a whole could not be considered complete until M has been assigned its cardinal number $\mathfrak{C}(M)$.

Of course this puts quite a different slant to Cantor's conception of '*collecting into a whole*'! On the other hand, I do not think that Cantor's idea is far fetched; for suppose that a, b and c are distinct pebbles and then we collect them into the whole $\{a, b, c\}$[14]; in doing so we seem to be instantly aware that

$$\mathfrak{C}(\{a, b, c\}) = 3.$$

[13]Eine Mannigfaltigkeit (ein Inbegriff, eine Menge) von Elementen, die irgendwelcher Begriffssphäre angehören, nenne ich *wohldefiniert*, wenn auf Grund ihrer Definition und infolge des logischen Prinzips vom ausgesscholssenen Dritten es als *intern bestimmt* angesehen werden muss, *sowohl* ob irgendein derselben Begriffssphären angehöriges Objekt zu der gedachten Manningfaltigkeit als Element gehört oder nicht, *wie auch*, ob zwei zur Mengen gehörige Objeckte , trotz formaler Unterschiede in der Art des Gegebenseins eindander sind oder nicht.

[14]The set $\{a, b, c\}$ is an abstract object which goes beyond the *heap* of stones consisting of a, b and c.

Furthermore in Cantor's view it was not our responsibility to create the sets, cardinal numbers, *etc.*; they were created long before *homo sapiens* made its appearance and the Demiurge would have had no difficulty in assigning the cardinal numbers of larger sets, for example: $\mathfrak{c}$ to the set $\mathbb{R}$ of reals[15].

This brings up another interesting point, which although may have been obvious to many, I have not seen it in print; namely that in a **Platonic Class Theory** an iterative conception of sets is in effect what nowadays would be called *reverse engineering*. In other words, it is a useful tool for determining the structure of the available sets and not (necessarily) a way of creating them. Naturally in a constructive interpretation—or even in **ZF**—one often interprets the *iteration* as a way of *constructing* the sets[16].

2. CLASSES WITH ATTRIBUTES

The view that the cardinal numbers are *urelementen* has the advantage that one need not specify what they are. It has the disadvantage that each set must then be correlated with a unique urelemente (namely its cardinal number) and how this correlation is obtained is not at all clear.

Now although the works of Mirimanoff *et al.*, in which the cardinal numbers are pure sets, eliminated the *urelementen* and all their associated problems, one should not forget that having the cardinal numbers as attributes enabled Cantor to lay the foundations upon which later philosophers and mathematicians developed into the fully fledged modern set theory.

Hilbert may have hoped that Cantor's Set Theory would turn out to be a **TOEM** (Theory of Everything Mathematical), it is now known (thanks to Kurt Gödel) that it could never be. Furthermore Cantor's original question about the size of the real number line is still unknown and thanks to Paul Cohen it is known that **ZFC**[17] will not decide it. Consequently there is a search for *reasonable* axioms that may decide

[15] It would be nice to know if He had also assigned $\aleph_1$ (or even $\aleph_2$) to $\mathbb{R}$!

[16] Of course this brings up the question of when does, or did, the 'construction' take place.

[17] Nor **KM**.

it; and not only new axioms but also new formalizations, *e.g.* Gödel's idea of incorporating the *theory of concepts*, or Lawvere's categorial sets (Lawvere 1964).

One idea that seems to be missing is to return to Cantor's original conception of **sets with attributes**[18].

2.1. MORSE'S CLASS THEORY AS A THEORY OF CLASSES WITH ATTRIBUTES

A. P. Morse was the quintessential Set Theorist; to him every mathematical thing is a set[class] and even *statements* about mathematics were to him 'mathematical things' and thus had to be classes. His view is succinctly expressed in page 63 of (Morse 1985) [recall that for Morse: 'set' = 'class']:

> We believe every (mathematical) thing is a set. We believe there is no difference between the conjunction of two or more things and their intersection. We believe there is no difference between the disjunction of two things and their union. We believe there is no difference between the negation of a thing and its complement. We have come to believe a thing if and only if the empty set is a member of the thing. We believe $(x \in y)$ if and only if x is a member of y. We believe $(x \in y)$ if and only if $(x \in y)$ is the universe. We disbelieve $(x \in y)$ if and only if $(x \in y)$ is the empty set.

Now it is irrelevant whether we accept or disregard Morse's dogma, all that need be said is that it may have been relevant to Morse in his visualization of sets[19].

Assuming that to: "*believe that x is a member of y*" is tantamount to accepting that: "*the sentence 'x is a member of y' has the truth value* True," I propose to show that if one lets the classes have a *truth value attribute*, instead of a *cardinality attribute*, then one obtains a plausible interpretation of Morse's Polysynthetic Impredicative Class Theory.

[18]For some intuitionists the natural numbers carry along (as an attribute?) the proof that they are natural numbers.

[19]Just as Cantor's belief that the *transfinitum* resided in the Mind of God may have helped him formulate properties of the cardinal numbers.

2.2. TRUTH [BOOLEAN] ATTRIBUTES

One difference between Cantor's and Morse's systems is that in Morse's there are no *urelementen*. Furthermore, Cantor was able to take the cardinal urelementen and use them to form classes of the theory so that the cardinal numbers became part and parcel of his Set Theory. Morse accomplishes a similar feat by stipulating[20] that the truth value attributes are to be classes of the theory.

Why not sets? Because then we would have the class T of all truth attributes, and then T would also have a truth attribute, and we would be on the borderline of a Burali-Forti type of paradox. In addition Morse, perhaps following Boole, required that the universe—that is , the class of all sets—should also be the truth value 'True'.

Now the underlying 'logic' in Morse's system is traditional classical logic and thanks to the works of Boole *et al.* we know that the classical truth values are epitomized by Boolean algebras, or more specifically: *fields of sets*. However since the members of any field of sets are sets, Morse's truth attributes can only be required to be closed under unions, intersections and complements; we shall abbreviate the latter by saying that Morse's truth attributes form a **Boolean collection of classes**.

Since the English word '*truth*' is laced with emotional denotations I shall replace it by the word '*Boolean*' when referring to the class truth attributes.

In Morse's system (and in just about every class/set theory) one has that for any class A of the theory:

$$\mathbf{0} \subseteq A \subseteq \mathsf{U}\,. \qquad (*)$$

Let us, *temporarily* denote by $\mathfrak{b}(A)$ the Boolean attribute of the class A; note that the following, which is not part of Morse's polysynthetic language, holds:

$$\mathbf{0} \subseteq \mathfrak{b}(A) \subseteq \mathsf{U}\,.$$

As in any set theory, the set-theoretic structure of a class A is analyzed through the use of the *copula* $\in$. The novel idea of Morse is

[20] I should emphasize that I have no idea if Morse viewed his sets as having attributes, although it is pretty clear that he must have been aware of Cantor's view of the cardinals as urelementen.

to analyze the Boolean attributes of classes by the use of the logical operators on the classes themselves. In Cantor's Theory two sets had the same cardinal number attribute if and only if there was a bijection between the sets. In Morse's Theory, the equality of the Boolean attributes of two classes is determined by the biconditional:

$$\mathfrak{b}(A) = \mathfrak{b}(B) \quad \textit{if and only if} \quad (A \equiv B)$$

The polysynthetic nature of Morse's language and the above observation allows us to avoid the need for **explicitly** introducing the Boolean operator $\mathfrak{b}()$. In particular, if A and B are classes then the 'output' of '$(A \rightarrow B)$', is the class whose Boolean attribute is:

$$(\mathsf{U} - \mathfrak{b}(A)) \cup \mathfrak{b}(B).$$

Clearly the universe of sets: U, and the empty set: $\mathbf{0}$ correspond to the *top* and *bot*, respectively, of the Boolean collection of classes and thus it is reasonable to assume:

$$\mathfrak{b}(\mathbf{0}) = \mathbf{0}.$$

$$\mathfrak{b}(\mathsf{U}) = \mathsf{U}.$$

Consequently for every class A the following should be the case:

$$\mathbf{0} \rightarrow A \rightarrow \mathsf{U}. \qquad (**)$$

In Peirce's work, (*) and (**) were synonymous however in Morse's system (*) conveys more information than (**).

2.3. OBTAINING THE BOOLEAN ATTRIBUTE OF A CLASS

In Morse's polysynthetic language there is no distinction between the what traditionally would have been called *terms* and the *formulae*. Following Morse let us call **all the well formed expressions** of the polysynthetic language: **formulae**. Thus the underlying assumption is that all the formulae represent classes and that the classes have Boolean attributes (that are also classes).

Let us now consider the (traditional) formula: $(a \in A)$. What class should correspond to it? Now it is clear that—at least in a given context— $(a \in A)$ has a truth value and thus we should assign to the formula $(a \in A)$ a class whose Boolean attribute is the one represented by $(a \in A)$[21].

Next consider the (polysynthetic) formula: A. It obviously represents—again, in a given context—a class. The question this time is what is its Boolean attribute? Furthermore the correspondence should be as uniform as possible. Since we have a fairly good understanding of the truth value of a *traditional formula*, Morse's idea was to choose a traditional formula $\mathcal{F}_A$, whose only free variable was A and then add the polysynthetic axiom:

$$(\ A \equiv \mathcal{F}_A \ ),$$

which is the way to guarantee that:

$$\mathfrak{b}(A) = \mathfrak{b}(\mathcal{F}_A).$$

Since this is one of the most fundamental axioms—in the sense that it is used over and over to generate additional logical and set theoretical concepts—it is clear that the (traditional) formula $\mathcal{F}_A$ must be as simple as possible. We are thus led to two possible cases: (1) $(A \in \tau)$ or (2) $(\tau \in A)$, where τ is a defined constant.

Since the polysynthetic axiom must also apply to proper classes it is clear that (1) is not suitable. Thus the required formula must be of the form (2), that is:

$$(\ \tau \in A \ ).$$

The only classes that can be shown to exist, without the use of fairly complicated set theoretical axioms, are: $\mathbf{0}$ and U; furthermore U is a *proper* class and $(\mathbf{0} \in \mathsf{U})$.

Also, U representing the Boolean value of 'True' should be provable in the system. And $\mathbf{0}$ is also the *empty set* and thus the Boolean attribute of '$(\mathbf{0} \in \mathbf{0})$' should be $\mathbf{0}$. Combining all these observations one obtains that an obvious candidate for 'τ' is '$\mathbf{0}$'.

[21]It is of interest to note that Morse had developed his polysynthetic set theory well before the Boolean valued models of first order logic of Rasiowa/Sikorski and before the Scott/Solovay Boolean valued models of set theory.

Thus the 'Boolean attributes' interpretation has led us to Morse's axiom:

$$\bigwedge_x (\ x \equiv (\mathbf{0} \in x)\ ).$$

2.4. THE CLASSES CORRESPONDING TO TRADITIONAL ATOMIC FORMULAE

If, in a given context, the set a is indeed a member of the class A, then the truth value of $(a \in A)$ is True, that is: U. But U is a class with Boolean attribute True, so there should be an axiom whose interpretation is:

$$(a \in A) \equiv [(a \in A) = \mathsf{U}].$$

In order to avoid introducing equality at this early stage the above formula can be rendered as Morse's second polysynthetic axiom:

$$\bigwedge_x \bigwedge_a \bigwedge_A [\ (x \in \mathsf{U}) \ \rightarrow\ ((x \in (a \in A)) \equiv (a \in A))\ ].$$

Since the primitive logical atoms used by Morse were $\rightarrow$ and $\bigwedge$, the remaining two polysynthetic axioms mentioned in Subsection 0.5 express the idea that the logical operations are the traditional set theoretic operations on the collection of Boolean attributes.

3. A RELATIVE CONSISTENCY RESULT FOR MORSE'S POLYSYNTHETIC SYSTEM

Since the interpretation that I gave to Morse's Polysynthetic System[22] makes fundamental use of the Boolean operations, one would expect that the Boolean valued models for **ZF** of D. Scott and R. Solovay could be extended to Morse's system. And that is indeed the case.

An informal outline of the method is as follows: Start with an ε-model $\mathfrak{M}$ of **ZFCI** (Zermelo Fraenkel, with Choice and with an inaccessible cardinal κ). Then let B be a Boolean algebra of $\mathfrak{M}$ of cardinality strictly smaller than κ.

[22]Which, as I have already mentioned, may or may not be the one that Morse had in mind when he developed the system.

Iterating the B-valued universes through the ordinal $\kappa + 1$, one obtains a set of $\mathfrak{M}$ which can then be used to interpret Morse's system. For the traditional formulae $(a \in b)$, $(a = b)$, one uses the method of Scott/Solovay. The polysynthetic axioms can then be used as directives on how to extend the interpretation to the remainder of the polysynthetic formulae.

4. AN ALTERNATE VIEW OF MORSE'S SYSTEM

As already mentioned, the polysynthetic axioms, for example:

$$\bigwedge\nolimits_x (x \equiv (\mathbf{0} \in x)),$$

appear strange when one uses the traditional first-order languages of **ZF** or **KM** as a reference; the Boolean attributes interpretation provided both a way to understand its content as well as to provide a justification for it.

On the other hand, there is a much simpler *reading* of the polysynthetic axioms; namely as sentences of an Applied Extended Classical Propositional Calculus[23].

There are some interesting byproducts of considering Morse's system as an Extended Propositional Calculus—specially when presented as a Natural Deduction System for the logical atoms—for example: (i) it clearly shows that the impredicativeness of Morse's system is caused by the logical rules of inference for $\bigwedge$ and (ii) the powerful axiom of choice used by Morse is a restricted version of Hilbert's *epsilon symbol* and in fact naturally suggests an even stronger version of Choice.

Another byproduct, which I very much doubt that Morse would have appreciated, is that it could be argued that '**every mathematical thing is a proposition**'.

[23]'*Extended*' because of quantifications involving propositional variables, '*Applied*' because of the binary propositional function symbol: $\in$. Instead of the 'Extended Propositional Calculus' one often finds the '*Second Order Propositional Calculus*'; however that would make the usual Propositional Calculus the '*First Order Propositional Calculus*'. Thus we prefer the name used by Russell, Łukasiewicz *et al.*

PART II

5. EXTENDED ALGEBRA OF CLASSES

By the '*Extended Algebra of Classes*', **EAC**, I understand the theory obtained from Morse's Polysynthetic Class/Set Theory by eliminating most of the axioms that guarantee the existence of *sets* (that is, classes that are members of other classes). Note that this theory is much weaker than *Tarski's General Class Theory* (Chuaqui 1981), since, for instance, the singleton of a set is not guaranteed to be a set; consequently ordered pair of sets need not be sets and thus relations, functions, *etc.* are not available. One could even argue that the '**C**' in **EAC** should be read as '*Concept*' rather than '*Class*', specially since **EAC** can be considered as an Extended Propositional Calculus.

The second goal is to develop **EAC** so that eventually one can determine[24] the status of **M**orse's **P**ropositional **C**alculus:

$$\mathbf{MPC} = \{\, \theta \,:\, \theta \text{ a quantifier free theorem of } \mathbf{EAC}\},$$

and specially its relation to the works of Boole.

6. THE LANGUAGE OF EAC

From a purely syntactical viewpoint, the Polysynthetic language of Morse can be considered as an Extended Propositional Calculus with symbols for additional propositional functions[25]. Consequently there is no need to describe in detail the syntax, except that, perhaps, some details should be given about the Leśniewskian/Morse definitional axioms.

6.0.1. RESERVED SYMBOLS

The (non-parsing) reserved symbols are to be:

$$\rightarrow,\ \bigwedge,\ \in,\quad \textit{and} \quad \mathrel{\overset{\circ}{=}}\ .$$

' $\overset{\circ}{=}$ ' is my symbol for Morse's *definor*.

[24]In a future article.

[25]I reserve the name of '*propositional connective*' to those propositional functions which are invariant under the logical biconditional '$\equiv$'.

6.0.2. PRIMARY DEFINITIONS

By *primary definitions* I understand those definitions of the *defined constants* which allow us to simplify the sentences expressing the axioms. They are (the universal closure of) the following formulae:

$$
\begin{array}{lrcl}
D1. & \mathbf{0} & \overset{\circ}{=} & \bigwedge_x x \\
D2. & \sim p & \overset{\circ}{=} & (p \rightarrow \mathbf{0}) \\
D3. & \mathsf{U} & \overset{\circ}{=} & (\mathbf{0} \rightarrow \mathbf{0}) \\
D4. & (p \wedge q) & \overset{\circ}{=} & \sim(p \rightarrow \sim q) \\
D5. & (p \equiv q) & \overset{\circ}{=} & ((p \rightarrow q) \wedge (q \rightarrow p)) \\
D6. & (x \subseteq y) & \overset{\circ}{=} & \bigwedge_z (z \in x \rightarrow z \in y) \\
D7. & (x \overset{e}{=} y) & \overset{\circ}{=} & \bigwedge_z (z \in x \equiv z \in y) \\
D8. & (x \overset{l}{=} y) & \overset{\circ}{=} & \bigwedge_z (x \in z \rightarrow y \in z)
\end{array}
$$

'$\overset{e}{=}$' corresponds to *extensional equality*, '$\overset{\circ}{=}$' to *definitional (intensional) equality* and '$\overset{l}{=}$' to *Liebnizian equality*[26].

6.1. AXIOMS OF EAC

For the axioms of **EAC** we take the universal closure of the following formulae—$\mathcal{F}$ being an arbitrary formula—:

A1. $(x \overset{\circ}{=} y) \overset{e}{=} (x \overset{e}{=} y)$

A2. $x \equiv (\mathbf{0} \in x)$

A3. $t \in A \rightarrow [t \in (x \in y) \equiv (x \in y)]$

A4. $t \in A \rightarrow [t \in (p \rightarrow q) \equiv ((t \in p) \rightarrow (t \in q))]$

A5. $(t \in \bigwedge_x \mathcal{F}) \equiv \bigwedge_x (t \in \mathcal{F})$

A6. $x \in \mathsf{U} \rightarrow [(x \overset{e}{=} y) \equiv (x \overset{l}{=} y)]$

[26]Note that, when considering the system as a set theory, $\equiv$ is also a kind of equality (of Boolean attributes) and if ordered pairs are definable (and hence so are the functions), then $\cong$ is yet another kind of equality—this time, of cardinality—. Also, not surprisingly, some 'equalities' are more equal than others.

A7. $B \in \mathsf{U} \;\rightarrow\; (A \subseteq B \;\rightarrow\; A \in \mathsf{U})$

A8. $(\mathbf{0} \in x) \vee \sim(\mathbf{0} \in x)$[27]

The following axioms of Choice and of Well-foundedness, which are also included in Morse's system, will not be included in **EAC**.

Ch. $z \in A \;\rightarrow\; \mathsf{mel}(A) \in A$

We. $\bigwedge_y \sim(y \subseteq A \;\wedge\; y \in \sim A) \;\rightarrow\; A \overset{e}{=} \mathsf{U}$

Axiom (A1.) is **not a definitional axiom**; it claims that the class $(x \overset{\circ}{=} y)$ is extensionally equal to the class $(x \overset{e}{=} y)$. Note also that (5.) is the only schema.

6.2. LOGICAL RULES OF INFERENCE OF EAC

For '$\rightarrow$' and '$\bigwedge$' we take their Intuitionistic Natural Deduction rules of inference and as usual we split them into the *Introduction* and *Elimination* rules. The form of the $\bigwedge$-Elimination is as follows:

$$\frac{\begin{array}{c}\vdots\\ \bigwedge_x \mathcal{F}^y_x\end{array}}{\mathcal{F}^y_{\mathcal{G}}}$$

where $\mathcal{F}^y_x$ is the result of properly substituting all free occurrences of the variable y by the variable x and correspondingly for $\mathcal{F}^y_{\mathcal{G}}$ where $\mathcal{G}$ is an arbitrary formula. It is precisely this 'logical' rule of inference that gives to Morse's System its impredicative power.

6.3. DEFINITIONAL RULES OF INFERENCE OF EAC

Although the only time that '$\overset{\circ}{=}$' is used is in the definitional axioms of *new* operators—and consequently subject to all the rules and regulations concerning definitions—nevertheless $(a \overset{\circ}{=} b)$ is a well

[27] This axiom had to be added because the rules of inference that I chose are the intuitionistic ones. Note that this axiom brings out a requirement that Cantor placed in his 1882 definition of 'set'. Through the use of the universal quantifier I could have chosen the primary definitions to have an intuitionistic flavour, but at the moment there is no particular advantage in so doing.

formed formula of the language and thus represents (in a given context) a class with a Boolean attribute[28].

The first rule for $\mathrel{\mathring{=}}$ does not have any premises (*i.e.* it is an axiom schema):

$$\frac{}{(\mathcal{F} \mathrel{\mathring{=}} \mathcal{F})}$$

And the second one is:

$$\frac{(\mathcal{A} \mathrel{\mathring{=}} \mathcal{B}) \qquad \mathcal{F}^y_{\mathcal{A}}}{\mathcal{F}^y_{\mathcal{B}}}$$

I prefer to use rules of inference for $\mathrel{\mathring{=}}$, rather than listing equivalent axioms, because I like to consider the rules of inference as being *logical rules of inference* and they are to be dependent solely on the logical form of the formulae; the *axioms* are to be much more dependent on $\in$.

7. THEOREMS OF EAC

Since the formalization presented here is technically (but hopefully, not conceptually) different from the one given by Morse, I shall give a sketch of how to obtain the basic theorems of (Morse 1965) so as to be able to use Morse's book as a reference for other theorems.

Derivibality '$\vdash$' is to be understood with respect to **EAC**.

[28]If one is more interested in the metatheory of **EAC** than in the actual theorems of **EAC** then one could completely avoid $\mathrel{\mathring{=}}$ and its rules and axioms by the tried and true way of treating the definitions as temporary typographical abbreviations which are introduced merely as a courtesy to the reader and are not even part of the system.

Lemma 7.1

1. $\vdash p \equiv p$
2. $\vdash x \in A \equiv x \in A$
3. $\vdash A \stackrel{e}{=} A$
4. $\vdash A \stackrel{e}{=} B \rightarrow B \stackrel{e}{=} A$
5. $A \stackrel{e}{=} B, B \stackrel{e}{=} C \vdash A \stackrel{e}{=} C$
6. $A \subseteq B, A \vdash B$
7. $A \stackrel{e}{=} B, A \vdash B$
8. $A \stackrel{\circ}{=} B \vdash A \stackrel{e}{=} B.$
9. $A \stackrel{e}{=} B \vdash A \stackrel{\circ}{=} B$
10. $A \stackrel{e}{=} B, \mathcal{F}^y_A \vdash \mathcal{F}^y_B$
11. $A \stackrel{e}{=} B \vdash \mathcal{F}^y_A \equiv \mathcal{F}^y_B$
12. $A \stackrel{e}{=} B, x \in \mathcal{F}^y_A \vdash x \in \mathcal{F}^y_B$
13. $A \stackrel{e}{=} B \vdash \mathcal{F}^y_A \stackrel{e}{=} \mathcal{F}^y_B$
14. $\vdash \mathbf{0} \stackrel{e}{=} \bigwedge_p p$
15. $\vdash \mathsf{U} \stackrel{e}{=} (\mathbf{0} \rightarrow \mathbf{0})$
16. $\vdash \mathsf{U}$
17. $\vdash \mathbf{0} \in \mathsf{U}$
18. $\vdash (p \vee \sim p)$
19. $\vdash (A \vee \sim A) \stackrel{e}{=} \mathsf{U}$

20. $\vdash (A \subseteq B) \ \rightarrow \ (A \rightarrow B)$

21. $\vdash A \stackrel{e}{=} B \ \rightarrow \ A \equiv B$

Proofs:

The first one to make essential use of the axioms of **EAC** is (6.):

$$\begin{array}{rll} A \subseteq B, A & \vdash & \mathbf{0} \in A \\ & \vdash & \mathbf{0} \in B \\ & \vdash & B \end{array}$$

□

Note that (13.) corresponds to (Morse 1965) axiom (schema) 2.5.5.

Next it must be shown that $\mathbf{0}$ is both *falsum* and the empty class (actually a *set* since $\vdash \mathbf{0} \in \mathsf{U}$):

Lemma 7.2

1. $\vdash \mathbf{0} \ \rightarrow \ A$
2. $\vdash \mathbf{0} \in \mathbf{0} \ \equiv \ \mathbf{0}$
3. $x \in \mathbf{0} \vdash \ \mathbf{0}$
4. $\vdash \bigwedge_x \sim (x \in \mathbf{0})$
5. $\vdash \mathbf{0} \subseteq A$

Proof: Of (3.)

$$\begin{array}{rll} x \in \mathbf{0} & \vdash & x \in \bigwedge_p p \\ & \vdash & \bigwedge_p (x \in p) \\ & \vdash & x \in (\mathbf{0} \ \rightarrow \ (\mathbf{0} \in \mathbf{0})) \\ & \vdash & x \in \mathbf{0} \ \rightarrow \ x \in (\mathbf{0} \in \mathbf{0}) \\ & \vdash & x \in (\mathbf{0} \in \mathbf{0}) \\ & \vdash & (\mathbf{0} \in \mathbf{0}) \\ & \vdash & \mathbf{0} \end{array}$$

Before giving the corresponding properties for U, let us introduce the existential quantifier $\bigvee$ by a definitional axiom schema:

For every formula $\mathcal{F}$ the universal closure of the following formula is a definitional axiom:

$$D9. \qquad [\ \bigvee_x \mathcal{F} \ \mathrel{\mathring{=}} \sim\bigwedge_x \sim \mathcal{F}\ ].$$

It is a routine matter to show that $\bigvee$ satisfies all the standard properties of the existential quantifier in the Classical Extended Propositional Calculus[29].

Lemma 7.3

1. $\vdash A \to \mathsf{U}$
2. $x \in A \vdash x \in \mathsf{U}$
3. $\vdash A \subseteq \mathsf{U}$
4. $\vdash x \in \mathsf{U} \ \equiv \ \bigvee_A (x \in A)$

Proof: Of (2.)

$$\begin{aligned} x \in A \ \vdash & \ \ (x \in \mathbf{0}) \ \to \ (x \in \mathbf{0}) \\ \vdash & \ \ x \in (\mathbf{0} \to \mathbf{0}) \\ \vdash & \ \ x \in \mathsf{U}. \end{aligned}$$

□

7.1. CLASS BUILDING THEOREMS

Following (Morse 1965) let us introduce the following definitional axioms and axiom schema:

$$\begin{aligned} &D10. \quad \mathsf{sng}\, x \quad \mathrel{\mathring{=}} \quad \bigwedge_y (y \ \to \ (x \in y)) \\ &D11. \quad \mathsf{E}_x \mathcal{F}_x \quad \mathrel{\mathring{=}} \quad \bigvee_x ((\mathbf{0} \in \mathcal{F}_x) \wedge \ \mathsf{sng}\, x) \end{aligned}$$

Read '$\mathsf{E}_x \mathcal{F}_x$' as '*the class of sets x such that $\mathcal{F}_x$ (holds)*'. '$\mathsf{sng}\, x$' should be read as '*singleton x*'.

[29] One could have just as well used the Intuitionistic definition:

$$[\ \bigvee_x \mathcal{F} \ \mathrel{\mathring{=}} \ \bigwedge_y ((\bigwedge_x \mathcal{F} \to y) \ \to \ y)\ ].$$

7.1.1. PROPERTIES OF THE SINGLETONS

Lemma 7.4

1. $\vdash x \in \mathsf{U} \ \rightarrow\ (x \in \mathsf{sng}\, a) \equiv (x \stackrel{e}{=} a)$
2. $\vdash (x \in \mathsf{sng}\, a)\ \equiv\ (x \in \mathsf{U}\ \wedge\ (x \stackrel{e}{=} a))$
3. $\vdash x \in \mathsf{U} \ \rightarrow\ x \in \mathsf{sng}\, x$
4. $\vdash x \in \mathsf{U} \ \rightarrow\ (x \in A \equiv (\mathsf{sng}\, x \subseteq A))$
5. $\vdash \sim(x \in \mathsf{U}) \ \rightarrow\ (\mathsf{sng}\, x \stackrel{e}{=} \mathbf{0})$

Note that the axioms of **EAC** are not strong enough to obtain:

$$x \in \mathsf{U} \ \rightarrow\ \mathsf{sng}\, x \in \mathsf{U}\,.$$

7.1.2. THE CLASSIFIER

The system **EAC** is nevertheless powerful enough to prove the basic property of the classifier E :

Theorem 7.1

$$a \in \mathsf{E}_x \mathcal{F}_x\ \equiv\ (a \in \mathsf{U}\ \wedge\ \mathcal{F}_a).$$

The proof is given as a sequence of biconditionals:

$$\begin{aligned}
a \in \mathsf{E}_x \mathcal{F}_x \ &\equiv\ a \in \bigvee_x((\mathbf{0} \in \mathcal{F}_x)\ \wedge\ \mathsf{sng}\, x) \\
&\equiv\ \bigvee_x(a \in (\mathbf{0} \in \mathcal{F}_x) \wedge a \in \mathsf{sng}\, x) \\
&\equiv\ \bigvee_x((\mathbf{0} \in \mathcal{F}_x) \wedge a \in \mathsf{sng}\, x) \\
&\equiv\ \bigvee_x(\mathcal{F}_x \wedge a \in \mathsf{U} \wedge (a \stackrel{e}{=} x)) \\
&\equiv\ \bigvee_x(a \in \mathsf{U} \wedge \mathcal{F}_x \wedge (a \stackrel{e}{=} x)) \\
&\equiv\ a \in \mathsf{U} \wedge \bigvee_x(\mathcal{F}_x \wedge (a \stackrel{e}{=} x)) \\
&\equiv\ a \in \mathsf{U} \wedge \bigvee_x(\mathcal{F}_a \wedge (a \stackrel{e}{=} x)) \\
&\equiv\ a \in \mathsf{U} \wedge \mathcal{F}_a \wedge \bigvee_x(a \stackrel{e}{=} x) \\
&\equiv\ a \in \mathsf{U} \wedge \mathcal{F}_a \wedge \mathsf{U} \\
&\equiv\ a \in \mathsf{U} \wedge \mathcal{F}_a.
\end{aligned}$$

In traditional first-order systems, such as **KM**, theorem 7.1 is usually presented as an axiom schema; in **EAC** it is a consequence of the (propositional) quantifiers and the polysynthetic axioms.

The following lemma gives both some of the familiar properties of the classifier as well as some that are specific to the polysynthetic nature of the system:

Lemma 7.5

1. $\vdash A \overset{e}{=} \mathsf{E}_x(x \in A)$
2. $\vdash \bigwedge_x(\mathcal{F}_x \rightarrow \mathcal{G}_x) \ \rightarrow \ \mathsf{E}_x\mathcal{F}_x \subseteq \mathsf{E}_x\mathcal{G}_x$
3. $\vdash \mathsf{E}_x((x \in \mathsf{U}) \wedge \mathcal{F}_x) \overset{e}{=} \mathsf{E}_x\mathcal{F}_x$
4. $\vdash \mathsf{sng}\, A \overset{e}{=} \mathsf{E}_x(x \overset{e}{=} A)$
5. $\vdash \mathsf{E}_x(\mathcal{F}_x \rightarrow \mathcal{G}_x) \overset{e}{=} (\mathsf{E}_x\mathcal{F}_x \rightarrow \mathsf{E}_x\mathcal{G}_x)$
6. $\vdash \mathsf{E}_x({\sim}\mathcal{F}_x) \overset{e}{=} {\sim}\mathsf{E}_x\mathcal{F}_x$
7. $\vdash \mathsf{E}_x(\bigwedge_y\mathcal{F}_{xy}) \overset{e}{=} \bigwedge_y \mathsf{E}_x\mathcal{F}_{xy}$
8. $\vdash {\sim}(\mathsf{E}_x[{\sim}(x \in x)] \in \mathsf{U})$
9. $\vdash \mathsf{E}_x[{\sim}(x \in x)] \subseteq \mathsf{U}$
10. $\vdash {\sim}(\mathsf{U} \in \mathsf{U})$

Already the above small sample of the theorems involving the classifier show that if **EAC** is viewed as an extended propositional calculus then E has the role of a definable propositional quantifier. Also it is clear that with the classifier one can *define* many classes; for example, the class of all ordinals[30]. However how many of them are in fact *different* classes can not be ascertained in **EAC**. All that can be proven is that there is at least **one** set, namely: **0**, and **one** proper class, namely: U. It is consistent with **EAC** that any other class be extensionally equal to one of those two.

[30] Using the definition of *ordinals* given in (Morse 1965).

7.2. A TWO ELEMENT MODEL FOR EAC

Let **2** be the two element Boolean algebra $\{0,1\}$. Then define the binary functions $\in$ and $\overset{\circ}{=}$ on **2** as follows:

$$\begin{array}{rclcrcl} \in(0,1) & = & 1 & \quad & \overset{\circ}{=}(0,1) & = & 0 \\ \in(0,0) & = & 0 & & \overset{\circ}{=}(0,0) & = & 1 \\ \in(1,1) & = & 0 & & \overset{\circ}{=}(1,1) & = & 1 \\ \in(1,0) & = & 0 & & \overset{\circ}{=}(1,0) & = & 0 \end{array}$$

It is then a routine matter to verify that in the algebra $\mathfrak{M} = \langle\{0,1\}, \in, \overset{\circ}{=}\rangle$ we have the following:

1. $(a \overset{e}{=} b) = 1$ iff $a = b$.
2. $(a \in b) = 1$ iff $a = 0$ and $b = 1$.
3. $\mathsf{sng}\, 0 = 1$ and $\mathsf{sng}\, 1 = 0$.
4. $\mathsf{E}_x[\sim(x \in x)] = 1$

Routine calculations show that all the axioms of **EAC** have the value 1.

7.2.1. THE THEORY EAC$^\omega$

The simplest additional axiom that would eliminate finite models is:

$$x \in \mathsf{U} \;\rightarrow\; (\mathsf{sng}\, x \in \mathsf{U});$$

however the natural numbers generated would correspond to the ones considered by Zermelo: $0, \{0\}, \{\{0\}\}, \ldots$. Hence I prefer to let **EAC**$^\omega$ be the extension of **EAC** by the addition of (the universal closure of):

$$x \in \mathsf{U} \;\rightarrow\; ((x \vee \mathsf{sng}\, y) \in \mathsf{U}).$$

Then in **EAC**$^\omega$ it can be shown that the Mirimanoff/von Neumann finite ordinals are distinct. If in addition one adds an axiom that guarantees that the union of a set is again a set, then one obtains a theory equivalent to Tarski's General Class Theory (presented in, for example, (Chuaqui 1981)).

8. THE AXIOM OF CHOICE IN MORSE'S SYSTEM

Morse tackles the axiom of Choice by introducing a new primitive symbol 'mel' together with the additional axiom:

$$\bigwedge_A \bigwedge_z (z \in A \;\; \rightarrow \;\; \mathsf{mel}A \in A).$$

It is well documented that this is a very strong form of the axiom of choice since it allows the universe U to be well-ordered by a (single) function of the system. Nevertheless it is not the strongest possible, Marek, in (Marek 1973), makes use of an even stronger form.

If one rewrites Morse's axiom as:

$$\bigwedge_A [\bigvee_z (z \in A) \;\equiv\; (\mathsf{mel}A \in A)],$$

then one cannot fail to see it as a special case of Hilbert's epsilon symbol ε:

$$\bigvee_z (z \in A) \;\equiv\; (\varepsilon_z (z \in A) \in A).$$

In other words, if one had Hilbert's epsilon symbol then Morse's mel could be defined as:

$$\mathsf{mel}A \quad \mathring{=} \quad \varepsilon_z (z \in A)$$

Thus instead of adding the primitive symbol 'mel' one could instead add the variable binding operator 'ε' together with its natural rules of inference—since we are dealing with a 'logical notion'—so as to lead to the theorem schema:

$$\bigvee_z \mathcal{F}^a_z \;\; \rightarrow \;\; \mathcal{F}^a_{\varepsilon_z \mathcal{F}_z}$$

Since in Morse's system:

$$A \overset{e}{=} B \;\; \rightarrow \;\; \mathsf{mel}A \overset{e}{=} \mathsf{mel}B,$$

the following axiom schema—since we are now dealing with a 'set theoretical notion'—would have to be to be added

$$\bigwedge_z(\mathcal{F}_z \equiv \mathcal{G}_z) \;\rightarrow\; (\varepsilon_z\mathcal{F}_z \overset{e}{\doteq} \varepsilon_z\mathcal{G}_z)$$

The resulting system is formally stronger than the one with 'mel' since $\mathsf{mel}A$ is always a set but there is no such restriction on $\varepsilon_z\mathcal{F}_z$.

9. THE PROPOSITIONAL PART OF EAC

It is clear that the formula '$\mathsf{sng}\,x$' is a formula of **EAC** which is *dependent* on the quantifier '$\bigwedge$' and thus should not be considered as a *propositional formula*. On the other hand the formula '**0**', whose definition is $[\mathbf{0} \overset{\circ}{\doteq} \bigwedge_p p]$, should be considered as a propositional formula!

Hence I propose that:

> *A* **basic propositional formula** *of* **EAC** *is any propositional formula generated by (i) the variables, (ii) the symbol* **0** *and (iii) the operational binary infix symbols:* $\rightarrow$ *, and* $\in$ *.*

The *propositional formulae* of **EAC** are those formulae which are equivalent, through 'unwinding' the definitional axioms, to a basic propositional formula.

By the '*Algebra of Classes*' and '*Algebra of Classes*$^\omega$' we understand the following sets of propositional formulae:

$$\begin{aligned} \mathbf{AoC} &= \{\mathcal{P} : \vdash_{\mathbf{EAC}} \mathcal{P} \text{ and } \mathcal{P} \text{ is a propositional formula of } \mathbf{EAC}\} \\ \mathbf{AoC}^\omega &= \{\mathcal{P} : \vdash_{\mathbf{EAC}^\omega} \mathcal{P} \text{ and } \mathcal{P} \text{ is a propositional formula of } \mathbf{EAC}\} \end{aligned}$$

The principal challenge before us is to determine the status of the above propositional calculi as well as their relation to their parent theories and to the *Algebra of Logic* of Schröder *et al.*

9.1. A POSSIBLE AXIOMATIZATION FOR AoC

For rules of inference of **AoC** let us take those of **EAC** concerning $\rightarrow$ and in addition let us add the following two rules about *falsum* and classical negation:

$$\frac{\mathbf{0}}{\mathcal{G}} \qquad\qquad \begin{array}{c}[(\mathcal{F} \rightarrow \mathbf{0})] \\ \Pi \\ \mathcal{F} \\ \hline \mathcal{F}\end{array}$$

Instead of using definitional axioms, let us use the more relaxed attitude of the traditional propositional calculus in which the 'definitions' are considered as temporary typographical abbreviations. The following are useful temporary abbreviations (where $\overset{\text{abbr}}{\longleftrightarrow}$ corresponds to '*abbreviates*'):

$$\begin{array}{llcl}
Ab1. & \sim\mathcal{P} & \overset{\text{abbr}}{\longleftrightarrow} & (\mathcal{P} \rightarrow \mathbf{0}) \\
Ab2. & \mathsf{U} & \overset{\text{abbr}}{\longleftrightarrow} & (\mathbf{0} \rightarrow \mathbf{0}) \\
Ab3. & (\mathcal{P} \wedge \mathcal{Q}) & \overset{\text{abbr}}{\longleftrightarrow} & \sim(\mathcal{P} \rightarrow\sim \mathcal{Q}) \\
Ab4. & (\mathcal{P} \vee \mathcal{Q}) & \overset{\text{abbr}}{\longleftrightarrow} & (\sim\mathcal{P} \rightarrow \mathcal{Q}) \\
Ab5. & (\mathcal{P} \equiv \mathcal{Q}) & \overset{\text{abbr}}{\longleftrightarrow} & ((\mathcal{P} \rightarrow \mathcal{Q}) \wedge (\mathcal{Q} \rightarrow \mathcal{P}))
\end{array}$$

As axiom schemata let us take the following:

AS1. $(\mathcal{P} \in \mathbf{0}) \rightarrow \mathbf{0}$

AS2. $(\mathsf{U} \in \mathsf{U}) \rightarrow \mathbf{0}$

AS3. $\mathcal{P} \equiv (\mathbf{0} \in \mathcal{P})$

AS4. $\mathcal{P} \in \mathcal{Q} \rightarrow [\mathcal{P} \in (\mathcal{A} \in \mathcal{B}) \equiv (\mathcal{A} \in \mathcal{B})]$

AS5. $\mathcal{P} \in \mathcal{Q} \rightarrow [\mathcal{P} \in (\mathcal{A} \rightarrow \mathcal{B}) \equiv ((\mathcal{P} \in \mathcal{A}) \rightarrow (\mathcal{P} \in \mathcal{B}))]$

Let us call the propositional calculus described above: 'MP'.

Lemma 9.1

1. $\vdash_{\mathsf{MP}} \sim(\mathcal{P} \in \mathbf{0})$
2. $\vdash_{\mathsf{MP}} \sim\mathbf{0}$
3. $\vdash_{\mathsf{MP}} \sim(\mathsf{U} \in \mathsf{U})$
4. $\vdash_{\mathsf{MP}} \mathsf{U}$
5. $\vdash_{\mathsf{MP}} (\mathcal{P} \in \mathbf{0}) \;\rightarrow\; (\mathcal{P} \in \mathcal{A})$
6. $\vdash_{\mathsf{MP}} (\mathcal{P} \in \mathcal{A}) \;\rightarrow\; (\mathcal{P} \in \mathsf{U})$

An informal way of reading the last two results is that:

$$\mathbf{0} \subseteq \mathcal{A} \subseteq \mathsf{U}.$$

In other words in **AoC** all the classes (*a.k.a.* propositions) are subclasses of the Universe U, furthermore the Universe is not a *member* of itself.

The following theorems give further credence to the claim that MP is indeed an algebra of classes:

Lemma 9.2

1. $\vdash_{\mathsf{MP}} \mathcal{P} \in (\mathcal{A} \wedge \mathcal{B}) \equiv (\mathcal{P} \in \mathcal{A} \wedge \mathcal{P} \in \mathcal{B})$
2. $\vdash_{\mathsf{MP}} \mathcal{P} \in (\mathcal{A} \vee \mathcal{B}) \equiv (\mathcal{P} \in \mathcal{A} \vee \mathcal{P} \in \mathcal{B})$
3. $\vdash_{\mathsf{MP}} \mathcal{P} \in \mathsf{U} \;\rightarrow\; [\sim(\mathcal{P} \in \mathcal{A}) \equiv (\mathcal{P} \in \sim\mathcal{A})]$
4. $\vdash_{\mathsf{MP}} \mathcal{P} \in \mathsf{U} \;\rightarrow\; [\mathcal{P} \in (\mathcal{A} \equiv \mathcal{B}) \;\equiv\; ((\mathcal{P} \in \mathcal{A}) \equiv (\mathcal{P} \in \mathcal{B})]$

It is now routine to verify that the propositional calculus MP is a subsystem of **AoC**.

10. FOR THE FUTURE

AoC and, if 'definitions' are replaced by 'abbreviations', **EAC**, are simply propositional and extended propositional logics, respectively. They are *invariant* logics in the sense that:

$$\text{If} \quad \vdash \mathcal{F}_{xyz\ldots} \quad \text{then} \quad \vdash \mathcal{F}_{\mathcal{ABC}\ldots}.$$

However, because of $\in$, they are not *logically invariant* since the following **is not** a theorem schema:

$$(\mathcal{A}{\equiv}\mathcal{B}) \wedge \mathcal{F}_{\mathcal{A}} \;\rightarrow\; \mathcal{F}_{\mathcal{B}}.$$

AoC and **EAC** can thus be considered as two (abstract) propositional logics obtained from an analysis of the *Paradise that Cantor created for us*. It is hoped that their exact place amongst the abstract logics will be determined in future papers.

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