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LOGIC, PARTIAL ORDERS AND TOPOLOGY

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Dedicated to Itala D'Ottaviano

Abstract: We give a version of Łós' ultraproduct result for forcing in Kripke structures in a first-order language with equality and discuss ultrafilters in a topology naturally associated to a partial order. The presentation also includes background material so as to make the exposition accessible to those whose main interest is Computer Science, Artificial Intelligence and/or Philosophy.

Key-words: Kripke structures. Partial orders. Topological ultrafilters. Generalized ultraproducts.

This paper originates in lectures delivered at the Logic Seminar of the Institute of Mathematics of the University of São Paulo in the academic year 2001-2002. The audience was composed of people with different backgrounds: Mathematics, Computer Science, Artificial Intelligence, Belief Revision and Philosophy. The aims of the lectures were to expound basic ideas from Logic, Topology and Partial Orders, to present new results but also – and very importantly –, to show that interdisplinarity, even if in this case restricted to Mathematics and Logic, has fruitful consequences. Moreover, the diversity of the audience required the development of common ground on which an appreciation of results and methods could be constructed. The trained eye will recognize sheaf-theoretic tactics, although the word "sheaf" is never mentioned in the text.

The original lecture notes were considerably more extensive, containing proofs of many basic results in the themes appearing in the title. To obtain a reasonable bound on the number of pages of this paper, a selection was inevitable. Nevertheless, we have tried to maintain the fundamental idea that inspired the seminar, separating a main body of development and including most of basic facts and definitions as Appendices. Hence, at the same time that it has an expository character, the paper also includes new results. The main ones are: Theorems 5.1 and 5.5 and its close relative, Theorem 6.4, characterizing the first-order theory of the inductive limit of a Kripke structure and giving a necessary and sufficient condition for the colimit of embeddings to be an elementary embedding; Theorem 7.7, generalizing to stalks of Kripke structures, Łós' well-known result on ultraproducts and Theorem 10.2, describing the first-order theory of stalks of Kripke structures at convergent ultrafilters.

The table that follows gives an idea of the contents:

Since Itala D'Ottaviano has always been an enthusiast and a catalyst of inter and cross disciplinary endeavors, we thought it appropriate to offer this contribution to a volume in her honor.

1 The $\mathfrak{U}$-topology

This section discusses a natural topology associated to any poset that will be fundamental in what follows. The definition and basic properties used forthwith are described in the Appendices.

1.1 Basic Notation. If X, Y are sets, $A, B \subseteq X$ and $f : X \longrightarrow Y$ is a map

(1) 2^X is the family of subsets of X;

(2) $card(X)$ is the *cardinality* of X;

(3) A^c and $X \setminus A$ stand for the complement of A in X;

(4) $A \subseteq_f B$ means that A is a *finite* subset of B. Note that $\emptyset \subseteq_f B$, for all sets B;

(5) $f_{|A} : A \longrightarrow Y$ is the *restriction* of f to A, that is, the map $a \in A \longmapsto f(a) \in Y$;

(6) Whenever possible, we omit parentheses in functional notation, writing fx for $f(x)$. □

If $\langle P, \leq \rangle$ is a poset and $x \in P$, recall (A.3.(a)) that

$$[x] = \{y \in P : x \leq y\} \quad \text{and} \quad x^{\leftarrow} = \{z \in P : z \leq x\}.$$

Define

$$\mathfrak{U}(P) = \{A \subseteq 2^P : \forall\, x \in P, \;\; x \in A \;\Rightarrow\; [x] \subseteq A\}.$$

The fundamental properties of $\mathfrak{U}(P)$ are described in

Proposition 1.2 *Let $\langle P, \leq \rangle$ be a poset, $T \subseteq P$ and $x \in P$.*

a) $\mathfrak{U}(P)$ is a topology on P, wherein the intersection of <u>any</u> family of opens is open and the union of <u>any</u> family of closed sets is closed.

b) For all $x \in P$, $[x]$ is a supercompact[1] *open and the smallest open neighborhood of x.*

c) $\overline{\{x\}} = x^{\leftarrow}$ and $\mathfrak{U}(P)$ is a T_0 topology on P.

d) $int\, T = \{t \in T : [t] \subseteq T\}$.

e) $\overline{T} = \bigcup_{t \in T} t^{\leftarrow}$.

[1] Every open covering has a one element subcovering.

f) int $(\overline{T}) = \{y \in P : T$ *is cofinal in* $[y]\}$.

g) An open set is dense in P *iff it is unbounded.*

Proof. a) It is easily established that

$*$ $P, \emptyset \in \mathfrak{U}(P)$;

$*$ $\forall\, x \in P$, $[x] \in \mathfrak{U}(P)$;

$*$ $\mathfrak{U}(P)$ is closed under arbitrary unions and intersections.

Hence, the closed sets are also closed under arbitrary unions and intersections.

b) To see that $[x]$ is supercompact, just notice that any open covering of $[x]$ has an open set containing x; this open set must then include all of $[x]$.

c) Assume that $y \in \overline{\{x\}}$; then the open neighborhood $[y]$ of y must have non-empty intersection with $\{x\}$ (A.18.(5)). But this means that $y \leq x$, that is, $y \in x^{\leftarrow}$. Hence, $\overline{\{x\}} \subseteq x^{\leftarrow}$. Conversely, it is clear that if $y \leq x$, then any open set containing y will have non-empty intersection with $\{x\}$, establishing the desired equality. To see that $\mathfrak{U}(P)$ is T_0 (A.19.(a)) just observe that because $\leq$ is a partial order, [*po* 2], the first part of (c) and (i) entail

$$x = y \quad \text{iff} \quad x^{\leftarrow} = y^{\leftarrow} \quad \text{iff} \quad \overline{\{x\}} = \overline{\{y\}}.$$

Item (d) is straightforward, while (e) is a straightforward application of A.18.(5).

f) Write $A = \{y \in P : T$ is cofinal in $[y]\}$. By (e), $\overline{T} = \bigcup_{t \in T} t^{\leftarrow}$; it is thus clear that $A \subseteq \overline{T}$. We shall verify that A is the largest open contained in $\overline{T}$. If $y \in A$ and $y \leq x$, then T is also cofinal in $[x]$, showing that $[y] \subseteq A$. Hence, A is open. If V is an open set contained in $\overline{T}$, for each $z \in V$, item (e) provides $t \in T$, such that $z \leq t$. This applies, in particular to the elements of $[x]$, where $x \in V$. Consequently, T is cofinal in $[x]$, for all $x \in V$, establishing that $V \subseteq A$, as desired. Item (g) follows immediately from (f). □

Corollary 1.3 *Let* $\langle P, \leq \rangle$ *be a poset and* A *be an open set in* P.

a) A *is clopen (A.35) iff for all* $x \in A$, $[x] \cup x^{\leftarrow} \subseteq A$.

b) A *is a regular (A.35) iff for all* $x \in P$, A *is cofinal in* $[x] \Rightarrow x \in A$.[2]

[2] The converse is trivial because A is open!

c) If $z \in P$, $[z]$ is regular iff for all $x \in P$, $[z]$ cofinal in $[x] \Rightarrow z \leq x$.

Proof. For (a), A being both open and closed, the conclusion follows from (d) and (e) in 1.2. Since *regular* means $A = \text{int}\,\overline{A}$, (b) follows from 1.2.(f), while (c) is immediate from (b). ☐

Proposition 1.4 *Let $\langle P, \leq \rangle$ be a poset and $A \subseteq P$.*

a) $A \subseteq P$ is compact iff there is a finite $S \subseteq A$ such that $A \subseteq \bigcup_{s \in S}[s]$. An open subset of P is compact iff it is a finite union of opens of the form $[x]$.

b) If P is ω-rd[3], then the compact opens in P are closed under finite intersections.

Proof. a) If $A \subseteq P$ is compact, consider the open covering of A given by $\{[x] : x \in A\}$; it must have a finite subcovering, and so there is a finite $S \subseteq A$ such that $A \subseteq \bigcup_{s \in S} [s]$. Conversely, if A satisfies the condition in the statement and $\{U_i : i \in I\}$ is an open covering of A, for each $s \in S$ select U_{i_s} such that $s \in U_{i_s}$. Then $[s] \subseteq U_{i_s}$ and so $A \subseteq \bigcup_{s \in S} U_{i_s}$, establishing the compactness of A. The remaining statement is an immediate consequence of what has been proven.

b) If P is ω-rd, then for all $x, y \in P$, there is a *finite* $S \subseteq P$, such that $[x] \cap [y] = \bigcup_{s \in S} [s]$. If A and B are compact opens in P, item (a) entails that

$$A = \bigcup_{i=1}^{n} [x_i] \quad \text{and} \quad B = \bigcup_{j=1}^{m} [y_j].$$

But then, $A \cap B = \bigcup_{i,j} [x_i] \cap [y_j]$, which can be written as a *finite* union of sets of the form $[z]$ because each intersection $[x_i] \cap [y_j]$ satisfies the same property. ☐

The topology $\mathfrak{U}(P)$ is called the **topology of upward order** on the poset $\langle P, \leq \rangle$. Whenever $\langle P, \leq \rangle$ is clear from context its mention will be omitted from the notation.The reader can certainly imagine the definitions of the topologies of order or downward order on P.

Example 1.5 Let I be a set, considered as a poset with the partial order of identity, i.e, $[i] = \{i\}$, for all $i \in I$. Thus, a set without any

[3] ω is the cardinal of the natural numbers, $\omega = \{0, 1, \ldots, n, \ldots\}$; ω-rd is defined in A.3.(c).

structure is a special case of posets. To identify the $\mathfrak{U}$-topology on I, note that all points are open ($[i]$ is open) and so $\mathfrak{U} = 2^I$, the discrete topology on I.

Now let $P = \mathfrak{U}^{op}$, the opposite of the inclusion on 2^I. For $A \in P$, *in the order of* P we have $[A] = 2^A$, the set of subsets of A; but in the original partial order of 2^I, $[A] = \{B \subseteq I : A \subseteq B\}$. This example shows that care must be exercised in using the notation. One could hang indices or exponents on the symbols (e.g., $[A]_P$ and $[A]_{2^I}$), but the best solution is attention to context. □

All topological notions hereafter refer to the upward order topology, $\mathfrak{U}$

Lemma 1.6 *Let $\langle P, \leq \rangle$ and $\langle L, \leq \rangle$ be posets and $f : P \longrightarrow L$ be a map. The following conditions are equivalent:*

(1) *f is continuous;* (2) *f is increasing, i.e., $x \leq y \;\Rightarrow\; fx \leq fy$.*

Proof. (1) ⇒ (2): First note that f is increasing iff for all x, $y \in P$

$$y \in [x] \quad \Rightarrow \quad fy \in [fx].$$

For $x \in P$, $[fx]$ is an open set in L; since f is continuous, we get that $f^{-1}([fx])$ is open in P. But $x \in f^{-1}([fx])$, and so $[x] \subseteq f^{-1}([fx])$. Hence, $y \in [x]$ entails $fy \in [fx]$, as needed.

(2) ⇒ (1): If C is an open set in L, then $f^{-1}(C) = \{x \in P : fx \in C\}$; hence, f being increasing, if $x \in f^{-1}(C)$ and $y \in [x]$, we get $fx \leq fy$, i.e., $fy \in [fx] \subseteq C$, because C is open in L. Therefore, $[x] \subseteq f^{-1}(C)$, verifying that it is open in P and f is continuous. □

The next result describes the closed irreducible subsets of $\mathfrak{U}(P)$ and shows that P is the union of its irreducible components (defined in A.22.(c)).

Proposition 1.7 *a) A closed set in P is irreducible iff it is right-directed.*

b) Every irreducible closet in P is contained in an irreducible component of P.

c) P is the union of its irreducible components.

Proof. a) Assume that F is irreducible (A.22.(b)) and let x, $y \in F$. Since the open sets $[x]$, $[y]$ have non-empty intersection with F, item (4) of the equivalence in A.24 implies that

$$[x] \cap [y] \cap F \neq \emptyset,$$

which, by A.4.(2) is equivalent to F being right-directed. Conversely, suppose that F is rd and let F_1, F_2 be non-empty closed sets such that $F = F_1 \cup F_2$. If $F_1 \neq F_2$, we may assume, without loss of generality that there is $x \in F_2 \setminus F_1$. For $y \in F_1$, since F is rd, there is $z \in F$ such that x, $y \leq z$. By 1.2.(c), we have $\overline{\{z\}} = z^{\leftarrow}$, and so x, $y \in \overline{\{z\}}$. If $z \in F_1$, then $\overline{\{z\}} \subseteq F_1$, which implies $x \in F_1$, contrary to assumption. Thus, $z \in F_2$ and so $z^{\leftarrow} = \overline{\{z\}} \subseteq F_2$, which in turn yields $y \in F_2$. We have shown that $F_1 \subseteq F_2$, establishing the irreducibility of F.

b) Fix an irreducible closed set G in P. Let

$$\mathcal{V} = \{F \subseteq P : F \text{ is an irreducible closed set containing } G\},$$

partially ordered by inclusion, $\mathcal{V} \neq \emptyset$ since $G \in \mathcal{V}$. We contend that $\mathcal{V}$ verifies the hypotheses of Zorn's Lemma (A.5). Indeed, if F_i, $i \in I$, is a chain of elements of $\mathcal{V}$, then $T = \bigcup_{i \in I} F_i$ is also in $\mathcal{V}$. To see this, note that

$*$ T is closed because the any union of closed sets in P is closed (1.2.(a));

$*$ By (a) each F_i is rd; since the union of a **chain** of rd subsets of P is again rd, we conclude that T is rd. Hence, T is irreducible, as claimed.

By Zorn's Lemma, $\mathcal{V}$ has maximal elements; any such is an irreducible component of P containing G.

c) For $x \in P$, item (a), 1.2.(c) and A.4.(2), imply that $x^{\leftarrow}$ is an irreducible closed set containing x. Now apply (b) to get obtain an irreducible component of P containing x, ending the proof. □

Definition 1.8 *For $x \in P$, define*

$$\Im(x) = \{F : F \textit{ is an irreducible component of } P \textit{ containing } x\},$$

called the ***irreducible hull of x***.

Remark A.28.(d), Lemma A.29.(b) and Proposition 1.2.(a) yield

Corollary 1.9 *If P is a poset and $x \in P$, then $\bigcup \Im(x)$ is a closed connected subset of P.*

Our next theme is the characterization of the connected components of the $\mathfrak{U}$ -topology.

Remark 1.10 For all $x \in P$, $[x]$ is a connected open set. This follows from supercompactness (1.2.(b)), for it is impossible to find disjoint opens satisfying the conditions in Definition A.27 (x must be in one of them!). □

Define a binary relation R on P by

$$x \; R \; y \quad \text{iff} \quad [x] \cap [y] \neq \emptyset.$$

Clearly, R is reflexive and symmetric. Let $\mathfrak{c}$ be the transitive closure of R. By Lemma A.2, $\mathfrak{c}$ is the equivalence relation generated by R in P. For $x \in P$, write $x/\mathfrak{c} = \{y \in P : x \; \mathfrak{c} \; y\}$ for the equivalence class of x with respect to $\mathfrak{c}$ and $P/\mathfrak{c} = \{x/\mathfrak{c} : x \in P\}$ for the quotient of P by $\mathfrak{c}$.

Proposition 1.11 *a) The equivalence classes of* $\mathfrak{c}$ *are clopen and the connected components of* P *in the* $\mathfrak{U}$*-topology.*

b) $\mathfrak{B}(P)$ (*the Boolean algebra of clopens in P, as in A.*37) *is naturally isomorphic to the complete Boolean algebra* $2^{P/\mathfrak{c}}$*, of subsets of the quotient* $P/\mathfrak{c}$*.*

Proof. First note that for all $t \in P$, $[t] \subseteq t/\mathfrak{c}$, because if $y \in [t]$, then $[y] \subseteq [t]$ and so $y \; R \; t$, which in turn entails $y \; \mathfrak{c} \; t$.

a) Assume, to get a contradiction, that $z/\mathfrak{c}$ is disconnected. Then, there are open sets U, V such that $z/\mathfrak{c} \subseteq U \cup V$, $U \cap z/\mathfrak{c} \neq \emptyset$, $V \cap z/\mathfrak{c} \neq \emptyset$ and $U \cap V \cap z/\mathfrak{c} = \emptyset$. Fix $x \in U \cap z/\mathfrak{c}$ and $y \in V \cap z/\mathfrak{c}$. Then, $x \; \mathfrak{c} \; z \; \mathfrak{c} \; y$, and so $x \; \mathfrak{c} \; y$, whence, by Lemma A.2,

$\exists \; n \geq 2$ and $t_1, \ldots, t_n$ in P such that

$$\begin{cases} (i) \; t_1 = x, \, t_n = y; \\ (ii) \; [t_i] \cap [t_{i+1}] \neq \emptyset, \, 1 \leq i \leq (n-1). \end{cases} \qquad (\sharp)$$

Note that for all $1 \leq k \leq n$, $t_k \in z/\mathfrak{c}$; moreover, since $z/\mathfrak{c}$ is contained in $U \cup V$, a point in the class of z is either in U or V. We proceed by induction on $2 \leq k \leq n$, to show that $t_k \in U$. If $t_2 \in V$, then, since $[x] \subseteq U$ and $[t_2] \subseteq V$, (1) and the condition in (ii) in $(\sharp)$ entail

$$\emptyset \;\neq\; [x] \cap [t_2] \;\subseteq\; U \cap V \cap z/\mathfrak{c},$$

which is impossible. Assume that for $k \leq n-1$, x, t_2, $\dots$, $t_k \in U$, and that $t_{k+1} \in V$. The argument used above, with t_k in place of x and t_{k+1} in place of t_2, shows that $U \cap V \cap z/\mathfrak{c}$ is non-empty, completing the induction step. Thus, $t_n = y \in U$, contrary to assumption, establishing the fact that the equivalence classes of $\mathfrak{c}$ are connected.

Since P is the *disjoint* union of the equivalence classes of $\mathfrak{c}$, to show that these classes are clopen it is enough to prove them open. Because if this is accomplished, then the complement of any class is the union of the classes distinct from it, being therefore also open in P. Moreover, since each class is connected, they must be the connected components of P. To show that $x/\mathfrak{c}$ is open it suffices to check that $y\ \mathfrak{c}\ x$ implies $[y] \subseteq x/\mathfrak{c}$. Fix $z \in [y]$ and suppose that $t_1, \dots, t_n$ is a sequence witnessing $y\ \mathfrak{c}\ x$. But then, the sequence z, $t_1, \dots, t_n$ witnesses that $z\ \mathfrak{c}\ x$, since $[z] \cap [t_1] = [z] \cap [y] = [z]$, completing the proof of (a).

b) We start with the following general

Fact 1 *Let X be a topological space, C a connected subset of X and U a clopen in X. Then,*

$$C \cap U \neq \emptyset \quad \Rightarrow \quad C \subseteq U.$$

Proof. If the conclusion is false, then $C \cap U^c \neq \emptyset$. But then U and U^c constitute a pair of opens in X satisfying the conditions required to guarantee that C is disconnected (A.27).

By Fact 1, if U is clopen in P, then it is the union of the equivalence classes of its elements: $U = \bigcup_{x \in U} x/\mathfrak{c}$. Now it straightforward to check that the map

$$U \in \mathfrak{B}(P) \ \longmapsto \ \{x/\mathfrak{c} \in P/\mathfrak{c} : x \in U\} \in 2^{P/\mathfrak{c}}$$

is natural Boolean algebra isomorphism between $\mathfrak{B}(P)$ and $2^{P/\mathfrak{c}}$, ending the proof. ☐

Corollary 1.9 and Proposition 1.11 yield

Corollary 1.12 *If $\langle P, \leq \rangle$ is a poset and $x \in P$, then $\bigcup \mathfrak{I}(x) \subseteq x/\mathfrak{c}$.*

Definition 1.13 *Two elements x, y of a poset $\langle P, \leq \rangle$ are* **compatible** *if $[x] \cap [y] \neq \emptyset$. Otherwise, x and y are* **incompatible**, *written $x \perp y$. Compatibility is reflexive and symmetric, while incompatibility is symmetric. P is* **ccc (countable chain condition)** *if every family of pairwise incompatible elements is at most countable.*

Remark 1.14 a) A poset is rd iff all pairs of elements are compatible.
b) There are (at least) two possibilities of defining compatibility and incompatibility. An alternative would be

$$x \text{ and } y \text{ are } \textit{(down) compatible} \text{ if } x^{\leftarrow} \cap y^{\leftarrow} \neq \emptyset.$$

In this case, x is incompatible with y iff $x^{\leftarrow} \cap y^{\leftarrow} = \emptyset$. We have chosen the notion in 1.13 because it corresponds to the concept of **compatible partial maps** as presented in Definition 1.15 and whose fundamental property is described in Lemma 1.16.

However, there is a canonical way to connect the two notions: just consider the opposite order. As an example, with the notion of compatibility in 1.13, the classical notion of **ccc topological space**[4] corresponds to $\mathcal{O}_*^{op}$ being ccc. For we have

Fact 1.14.A *If $\langle X, \mathcal{O} \rangle$ is a topological space and $U, V \in \mathcal{O}$, then with $\mathcal{O}_* = \mathcal{O} \setminus \{\emptyset\}$ (as in A.3.(d))*

$$U \perp V \text{ in } \mathcal{O}_*^{op} \quad \text{iff} \quad U \cap V = \emptyset.$$

Proof. Since $[U]_{op} = \{W \in \mathcal{O}_* : W \subseteq U\}$, we have $[U]_{op} \cap [V]_{op} = \emptyset$ iff $U \cap V = \emptyset$. □

Definition 1.15 *Let A, B be sets.*

a) A **partial map** *from A to B is a function whose domain is a subset of A, taking values in B. Write $pF(A, B)$ for the set of partial maps from A to B and $f : dom\, f \longrightarrow B$, for a typical element of $pF(A, B)$ (with $dom\, f \subseteq A$).*

b) $f, g \in pF(A, B)$ are **compatible** *if they coincide in the intersection of their domains.*

Lemma 1.16 (Gluing of compatible families) *Let A, B be sets and let $\{f_i : i \in I\} \subseteq pF(A, B)$ be a set of pairwise compatible partial maps from A to B. Then, there is a* unique *$f \in pF(A, B)$, written $\bigvee_{i \in I} f_i$ and called the* **gluing** *of the f_i, satisfying the following conditions:*

$$dom\, f \;=\; \bigcup_{i \in I} dom\, f_i \quad \text{and} \quad \forall\, i \in I, \quad f_i = f_{|dom\, f_i}.$$

Proof. If $D = \bigcup_{i \in I} dom\, f_i$ and $x \in D$, select $i \in I$ such that $x \in dom\, f_i$ and define $f(x) = f_i(x)$; the compatibility of the f_i entail that $f(x)$

[4]Every family of pairwise disjoint non-empty opens is countable.

is independent of the choice of $i \in I$. This method defines a *unique* partial map f satisfying the required properties. $\square$

Remark 1.17 Let P be a poset. If V_i, $i \in I$, is a collection of opens in the $\mathfrak{U}$-topology then 1.2.(a) and the definitions in Appendix V entail that in the frame $\mathfrak{U}(P)$ we have $\bigwedge_{i \in I} V_i = \bigcap_{i \in I} V_i$, while joins are unions *in any topology.* Next, items (d) and (f) in Proposition 1.2 yield, for U, V in $\mathfrak{U}(P)$

(1) $U \to V = \{q \in P : [q] \subseteq (U^c \cup V)\}$;

(2) $\neg U = \{q \in P : [q] \cap U = \emptyset\}$;

(3) $\neg\neg U = \{q \in P : U \text{ is cofinal in } [q]\}$. $\square$

We now present a natural way to embed a poset in a complete lattice. If $\langle P, \leq \rangle$ is a poset, the $\mathfrak{U}$-topology on P is a frame, as discussed in appendix A.V. Hence, $\mathfrak{U}(P)^{op}$ is also a complete distributive lattice (see A.8). We shall write $\leq$ for the partial order on $\mathfrak{U}(P)^{op}$, that is, for all opens U, V

$$U \leq V \quad \text{iff} \quad V \subseteq U.$$

Theorem 1.18 *Let $\langle P, \leq \rangle$ be a poset and $\mathfrak{U} = \mathfrak{U}(P)$. The map*

$$\gamma : P \longrightarrow \mathfrak{U}^{op}, \text{ given by } \gamma p = [p]$$

is a **join-preserving and meet-dense embedding** *of $\langle P, \leq \rangle$ into $\langle \mathfrak{U}^{op}, \leq \rangle$, that is,*

a) (Embedding) *For all $p, q \in P$, $p \leq q$ iff $\gamma p \leq \gamma q$.*

b) (Join-preserving) *If $\bigvee S$ exists in P, then $\gamma(\bigvee S) = \bigvee_{s \in S} \gamma s$.*[5]

c) (Meet-dense) *For all $U \in \mathfrak{U}^{op}$, there is $S \subseteq P$ such that $U = \bigwedge_{s \in S} \gamma s$.*

Moreover, γ preserves incompatibility, that is, for all $p, q \in P$

$$p \perp q \quad \text{iff} \quad \gamma p \perp \gamma q \quad \text{iff} \quad \gamma p \cap \gamma q = \emptyset.$$

Proof. For $p, q \in P$ we have

$$p \leq q \quad \text{iff} \quad [q] \subseteq [p] \quad \text{iff} \quad \gamma p \leq \gamma q$$

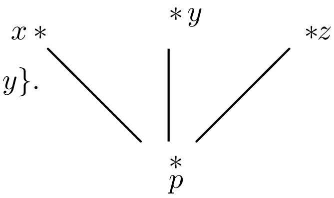
proving (a). If $S \subseteq P$ is such that $p = \bigvee S$, it must be shown that $\gamma p = \bigvee_{s \in S} \gamma s$. Unraveling notation and taking Remark 1.17 into account, this amounts to $[p] = \bigcap_{s \in S} [s]$. Since $p \geq s$, we get $[p] \subseteq [s]$,

[5] Notation as in A.6.

for all $s \in S$. If $q \in \bigcap_{s\in S} [s]$, then $q \geq s$, for all $s \in S$, and the definition of join in A.6 entails $p \leq q$, as needed to prove (b). For $U \in \mathfrak{U}$, we know that $U = \bigcup_{s\in U} [s]$, that is, $U = \bigwedge_{s\in U} \gamma s$, verifying (c). The assertion about incompatibility follows easily from the definition and Fact 1.14.A. □

Remark 1.19 a) To see that the embedding γ of 1.18 might not preserve meets just consider a poset with four distinct points, $P = \{x, y, z, p\}$, where p is its least element, while the other three are unrelated. Then, $x \wedge y = p$, but

$$[p] = P \quad \neq \quad [x] \cup [y] = \{x, y\}.$$



b) If P is a *chain* ($\forall x, y \in P$, $x \leq y$ or $y \leq x$), then the embedding γ is **regular**, that is, it preserves all meets and joins that exist in P.

c) By Theorem 1.18, every poset can be join-embedded into a complete *distributive* lattice. One may enquire whether it is possible to *regularly* embed a poset into a complete distributive lattice. The answer is **no**, even for *distributive lattices*. For a discussion of this, see [BD]. □

2 Kripke Structures and Colimits

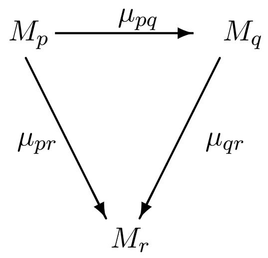
In this section we describe the models of the Intuitionistic Predicate Calculus presented in A.53, which will be our main concern here. We also give a presentation of the colimit (or inductive limit) associated to Kripke structures defined over a rd poset.

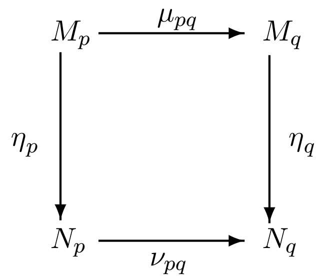
Definition 2.1 *Let* $\langle P, \leq \rangle$ *be a poset. A* **Kripke L-structure over P**, $\mathcal{M} = \langle M_p; \mu_{pq} \rangle$, $p \leq q$ *in* P, *consists of:*

∗ *A family of* L*-structures,* M_p, $p \in P$*;*

$*$ *Whenever* $p \leq q$*, a* L*-morphism* $\mu_{pq} : M_p \longrightarrow M_q$*, such that* $\mu_{pp} = Id_{M_p}$*;*

$*$ *If* $p \leq q \leq r$*, then* $\mu_{pr} = \mu_{qr} \circ \mu_{pq}$*, i.e., the diagram below left is commutative.*





If $\mathcal{M} = \langle M_p, \mu_{pq} \rangle$*,* $\mathcal{N} = \langle N_p, \nu_{pq} \rangle$ *are Kripke* L*-structures over* P*, a* **morphism***,* $\eta : \mathcal{M} \longrightarrow \mathcal{N}$*, is a family of* L*-morphisms,* $\eta = \{\eta_p : p \in P\}$*, where* $\eta_p : M_p \longrightarrow N_p$*, such that for all* $p \leq q$ *in* P*, the diagram above right is commutative.*

Kripke L*-structures over* P *and their morphisms are a category, written* $\mathfrak{K}(P, L)$*. When* L *is clear from context, its mention will be omitted from the notation.*

If $\mathcal{M} = \langle M_p; \mu_{pq} \rangle$ *is a Kripke* L*-structure over* P *and* $Q \subseteq P$*, write* $\mathcal{M}_{|Q} = \langle M_r; \mu_{rs} \rangle$*,* $r \leq s$ *in* Q*, for the restriction of* $\mathcal{M}$ *to* Q*.*

Remark 2.2 A poset $\langle P, \leq \rangle$ may be considered a *category*, whose objects are its elements and with arrows given, for p, $q \in P$

$$\mathrm{Mor}(p, q) = \begin{cases} \{\langle p, q \rangle\} & \text{if } p \leq q; \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus, there is a unique arrow from p to q iff $p \leq q$. Hence, in the language of Category Theory, a Kripke L-structure is a covariant functor from P to $\boldsymbol{L\,mod}$ and a *morphism* of Kripke L-structures over P is simply a natural transformation of covariant functors. Category theorists would call a Kripke L-structure a $\boldsymbol{P}$**-diagram in** $\boldsymbol{L\,mod}$. Logicians prefer to name it after Saul Kripke. $\square$

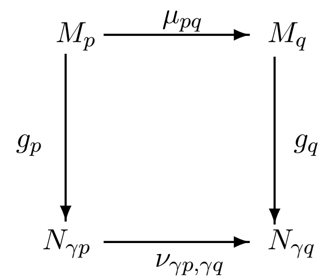
It is also useful to have a notion of morphism between Kripke structures over different bases.

Definition 2.3 *Let P, R be posets. If $\mathcal{M} = \langle\, M_p; \mu_{pq} \,\rangle$ is a Kripke structure over P and $\mathcal{N} = \langle\, N_r; \nu_{rs} \,\rangle$ is a Kripke structure over R, a* **morphism**, *$G : \mathcal{M} \longrightarrow \mathcal{N}$, is a pair, $G = \langle\, \gamma;\ (g_p)_{p \in P} \,\rangle$, such that*

$*$ $\gamma : P \longrightarrow R$ is an increasing map [6];

$*$ For each $p \in P$, g_p is a L-morphism from M_p to $N_{\gamma p}$;

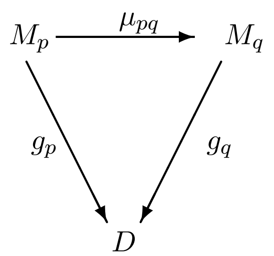
$*$ If $p \leq q$ in P, the following diagram is commutative:

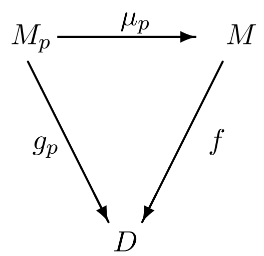


Clearly, this notion coincides with that discussed in 2.1 *and* 2.2 *in case $g = Id_P$.*

We now present the notions of *dual cone* and *colimit or inductive limit* in the category of Kripke structures.

Definition 2.4 *Let $\mathcal{M} = \langle\, M_p; \mu_{pq} \,\rangle$ be a Kripke structure over the poset P. A* **dual cone** *over $\mathcal{M}$ is a L-structure D, together with L-morphisms, $g_p : M_p \longrightarrow D$, $p \in P$, such that if $p \leq q$, the diagram below left is commutative:*





[6] That is, continuous in the $\mathfrak{U}$-topology, by 1.6.

A dual cone $\langle M; \mu_p \rangle$, $p \in P$, *is a* **colimit** *for* $\mathcal{M}$ *in* ***L mod*** *if for all dual cones* $\langle D; g_d \rangle$ *over* $\mathcal{M}$, *there is a* unique *L-morphism,* $f : M \longrightarrow D$, *such that for all* $p \in P$ *the diagram above right is commutative. In this case, write*

$$\langle M; \mu_p \rangle \;=\; \varinjlim \mathcal{M} \quad \text{or} \quad M = \varinjlim \mathcal{M},$$

to indicate that $\langle M; \mu_p \rangle$ *is the colimit of* $\mathcal{M}$ *in* ***L mod****.*

Remark 2.5 a) If $\mathcal{M}$ is a Kripke structure over P, such that $M = \varinjlim \mathcal{M}$ exists in ***L mod***, the universal property that defines M entails that it is **unique, up to *L*-isomorphism**.

b) It is clear from the preceding discussion that a Kripke structure $\mathcal{M}$ over a poset P *with a largest element*, $\top$, has a colimit, namely the dual cone $\varinjlim \mathcal{M} = \langle M_\top; \mu_{p\top} \rangle$. Hence, the interesting case is when P is "never ending". □

Our next result is part of the folklore of Model Theory, although a proof is not easy to find in the literature. It is included in [Mi2], to which the reader is referred.

Theorem 2.6 *Let* $\mathcal{M} : \langle P, \leq \rangle \longrightarrow L$-mod *be a Kripke L-structure over the rd poset* P*. Then,*

a) $\varinjlim \mathcal{M}$ *exists in* L-mod *and is unique up to isomorphism. Moreover, if* $Q \subseteq P$ *is cofinal in* I, *then* $\varinjlim \mathcal{M}_{|Q}$ *is naturally isomorphic to* $\varinjlim \mathcal{M}$.

b) A dual cone over $\mathcal{M}$, $\langle M, f_p \rangle$, $p \in P$, *is* (isomorphic to) $\varinjlim \mathcal{M}$ *iff it verifies:*

[colim 1]: *For all* p, $q \in P$, $x \in M_p$ *and* $y \in M_q$,

$$f_p(x) = f_q(y) \quad \text{iff} \quad \exists\, r \geq p, q \text{ such that } \mu_{pr}(x) = \mu_{qr}(y).$$

[colim 2]: *If* $\phi(v_1, \ldots, v_n)$ *is an atomic formula in* L, *and* $\overline{\xi} \in M^n$, *then*[7]

$$M \models \phi[\overline{\xi}] \quad \Leftrightarrow \quad \exists\, p \in P \text{ and } \overline{x} \in M_p^n \text{ such that } \overline{\xi} = f_p(\overline{x}) \text{ and } M_p \models \phi[\overline{x}].$$

[7]Since the maps f_p are L-morphisms, the significant implication here is ($\Rightarrow$).

Remark 2.7 a) Since P is rd, it is straightforward that [colim 2] in 2.6.(b) holds for any conjunction of atomic formulas, that is, if $\phi(v_1, \ldots, v_n)$ is a conjunction of atomic formulas and $\overline{\xi} \in M^n$ is such that $M \models \phi[\overline{\xi}]$, then there is $p \in P$ and $\overline{x} \in M_p^n$ such that $f_p(\overline{x}) = \overline{\xi}$ and $M_p \models \phi[\overline{x}]$. For a fuller discussion of the validity of [colim 2], see Lemma 2.8.(b).

b) It follows from (a) that [colim 2] implies that for all $\overline{\xi}$ in M^n, there is $p \in P$ and $\overline{x} \in M_p^n$ such that $f_p(\overline{x}) = \overline{\xi}$. To see this, just consider the conjunction of atomic formulas

$$\phi(v_1, \ldots, v_n) \equiv (v_1 = v_1) \wedge \ldots \wedge (v_n = v_n),$$

clearly verified by M at $\overline{\xi}$. In particular, $M = \bigcup_{p \in P} f_p(M_p)$. □

We now take a closer look at what happens when the transition morphisms are embeddings.

Lemma 2.8 *Let $\mathcal{M} = \langle M_p; \mu_{pq} \rangle$ be a Kripke structure over the rd poset P and let $\langle M; \mu_p \rangle = \varinjlim \mathcal{M}$.*

a) If for all $p \leq q$ in P, μ_{pq} is an embedding, then μ_p is an embedding for all $p \in P$.

b) Let $\phi(v_1, \ldots, v_n)$ be a positive quantifier-free formula (as in A.51) in L. For $\overline{\xi} \in M^n$, let $p \in P$ and $\overline{x} \in M_p^n$ be such that $\mu_p(\overline{x}) = \overline{\xi}$. Then

$$(*) \quad M \models \phi[\overline{\xi}] \quad \textit{iff} \quad \exists\, q \geq p \text{ such that } \forall\, r \geq q,\ \ M_r \models \phi[\mu_{pr}(\overline{x})].$$

If for all $p \leq q$ in P, μ_{pq} is an embedding, then $(*)$ holds for all quantifier-free formulas in L.

Proof. a) Fix $p \in P$ and $\overline{x} \in M_p^n$. By A.59.(b), it must be verified that if $\phi(v_1, \ldots, v_n)$ is an atomic formula in L, then

$$M_p \models \phi[\overline{x}] \quad \Leftrightarrow \quad M \models \phi[\mu_p(\overline{x})].$$

Since μ_p is a L-morphism, it suffices to check ($\Leftarrow$). By [colimit 2] in 2.6.(b), $M \models \phi[\mu_p(\overline{x})]$ is equivalent to the existence of $q \in P$ and $\overline{y} \in M_q^n$, such that $\mu_q(\overline{y}) = \mu_p(\overline{x})$ and $M_q \models \phi[\overline{y}]$. By [colimit 1] in 2.6.(b), there is $r \geq p,\ q$ such that $\mu_{pr}(\overline{x}) = \mu_{qr}(\overline{y})$. Since $M_r \models \phi[\mu_{qr}(\overline{y})]$, we get $M_r \models \phi[\mu_{pr}(\overline{x})]$ and the fact that μ_{pr} is an embedding entails $M_p \models \phi[\overline{x}]$, as needed.

b) We prove the result for positive quantifier-free formulas. The modifications needed for the not necessarily positive case will be mentioned latter. We proceed by induction on complexity. If ϕ is atomic, (*) reduces to [colimit 2] in 2.6.(b). If $\phi \equiv \phi_1 \wedge \phi_2$, then

$$M \models \phi[\overline{\xi}] \quad \text{iff} \quad M \models \phi_1[\overline{\xi}] \quad \text{and} \quad M \models \phi_2[\overline{\xi}].$$

By induction, there are q_1 and q_2 satisfying (*) with respect to ϕ_1 and ϕ_2, respectively. Take $q \geq q_1,\ q_2$ and recall that positive quantifier-free formulas are preserved by L-morphisms, to obtain an element of P satisfying (*) with respect to ϕ. The case of the connective $\vee$ is similar (in fact, simpler).

If each μ_{pq} is an embedding, we discuss the induction step through negation. If $M \models \neg\,\phi[\overline{\xi}]$, then, M does not satisfy $\phi[\overline{\xi}]$ and induction yields $r \geq p$ such that $M_r \models \neg\,\phi[\overline{\xi}]$. But embeddings preserve quantifier-free formulas; hence, the right-hand side of (*) is satisfied for $r \in P$. The converse is immediate from (a), ending the proof. □

A typical preservation result for colimits is

Theorem 2.9 *Let P is a rd poset, $\mathcal{M}$ a Kripke structure over P and $M = \varinjlim \mathcal{M}$. Let $\phi(v_1, \ldots, v_n)$ be a formula in L which is* **either**

(1) *A disjunction of negated atomic formulas;* **or**

(2) *A disjunction of formulas of the type $\forall\overline{w}(\psi_1 \rightarrow \exists\overline{u}\,\psi_2)$, where ψ_1, ψ_2 are positive and quantifier-free.*

Then, for $p \in P$ and $\overline{x} \in M_p^n$,

If $\{q \in P : q \geq p$ and $M_q \models \phi[\mu_{pq}(\overline{x})]\}$ is cofinal in P, then $M \models \phi[\mu_p(\overline{x})]$.

Proof. We have $\phi \equiv \phi_1 \vee \cdots \vee \phi_m$, where each ϕ_k is either of type (1) or (2). Fix $p \in P$, $\overline{x} \in M_p^n$ and consider, with $1 \leq k \leq m$,

$$(\sharp) \qquad \begin{cases} S = \{q \in P : q \geq p \text{ and } M_q \models \phi[\mu_{pq}(\overline{x})]\}; \\ S_k = \{q \in P : q \geq p \text{ and } M_q \models \phi_k[\mu_{pq}(\overline{x})]\}. \end{cases}$$

Since a L-structure satisfies ϕ iff it satisfies one of the ϕ_k, we get $S = \bigcup_{k=1}^{m} S_k$; thus, since S is cofinal in P, some S_k must also be cofinal in P. Hence, the proof is reduced to showing that the conclusion holds if ϕ is *a* formula of type (1) or type (2) in the statement. Let $M = \langle\, M; \mu_p \,\rangle$, $p \in P$.

$*$ Assume that $\phi \equiv \neg\,\theta$, where θ is an atomic formula and suppose that $M \models \theta[\mu_p(\overline{x})]$. By [colimit 2] in 2.6.(b) (or 2.8.(b)), there is $q \geq$ p such that $M_q \models \theta[\mu_{pq}(\overline{x})]$. With notation as in ($\sharp$), since S is cofinal in P, there is $r \in S$, with $r \geq q$. But then, because $\mu_{pr} = \mu_{qr} \circ \mu_{pq}$, we obtain $M_r \models \theta[\mu_{pr}(\overline{x})]$, contradicting the fact that $r \in S$.

$*$ Assume that $\phi \equiv \forall\overline{w}\ (\psi_1 \rightarrow \exists\overline{u}\ \psi_2)$, where ψ_i are positive and quantifier-free. To ease exposition, we shall suppose that $\psi \equiv \forall w(\psi_1 \rightarrow \exists u\, \psi_2)$. The reasoning is the same to deal with sequences of variables (but the overload in notation is not!).

Fix $\xi \in M$ and select $r \geq p$ together with $y \in M_r$ such that $\mu_r(y) = \xi$. Without loss of generality, we may suppose that $r = p$ and $y \in M_p$[8]. It must be shown that

$$M \models \psi_1 \rightarrow \exists u\ \psi_2\ \ [\xi;\, \mu_p(\overline{x})].$$

Suppose that $M \models \psi_1[\xi;\, \mu_p(\overline{x})]$; by Lemma 2.8.(b), there is $q \geq p$, such that

$$\text{For all } r \in [q], \quad M_r \models \psi_1[\mu_{pr}(y);\, \mu_{pr}(\overline{x})]. \qquad (\sharp\sharp)$$

Because P is rd, $[p] \cap S$ is also cofinal in P. Hence, there is $r \geq q$ such that $M_r \models \phi[\mu_{pr}(\overline{x})]$, and so ($\sharp\sharp$) implies $M_r \models \exists u\ \psi_2[\mu_{pr}(y);\, \mu_{pr}(\overline{x})]$. Choose $z \in M_r$, with $M_r \models \psi_2[z,\, \mu_{pr}(y);\, \mu_{pr}(\overline{x})]$. Since μ_r is a L-morphism and ψ_2 is positive quantifier-free, we have

$$M \models \psi_2[\mu_r(z),\, \mu_r(\mu_{pr}(y);\, \mu_r(\mu_{pr}(\overline{x}))],$$

and so $M \models \exists u\ \psi_2[\xi;\, \mu_p(\overline{x})]$, ending the proof. □

Remark 2.10 a) Exactly as in the case of Lemma 2.8.(b) if the connecting morphisms μ_{pq} of $\mathcal{M}$ are embeddings, Theorem 2.9 is valid whenever ψ_1, ψ_2 are *any* quantifier-free formulas.

b) It follows from 2.9 that colimits preserve many algebraic constructions. This is the case for groups, rings, local rings and fields. For the latter, recall that any ring homomorphism from a field into a ring must be injective, since fields have no proper ideals distinct from (0) and itself. □

[8] Otherwise, just reason with r in place of p and $\mu_{pr}(\overline{x})$ in that of $\overline{x}$.

An extremely useful and influential result, obtained by induction on the complexity of formulas is

Theorem 2.11 (Tarski) *Let* $\mathcal{M} = \langle M_p; \mu_{pq} \rangle$ *be a Kripke structure over the rd poset* P *and* $\langle M, \mu_p \rangle = \varinjlim \mathcal{M}$*. If for all* $p \leq q$, μ_{pq} *is an elementary embedding, then so are the* μ_p, $p \in P$. □

Lemma 2.12 (Colimit of morphisms) *Let* $\mathcal{M} = \langle M_p; \mu_{pq} \rangle$ *and* $\mathcal{N} = \langle N_p; \nu_{pq} \rangle$ *be Kripke structures over a rd poset* P*. Let* $\eta = (\eta_p)_{p \in P}$ *be a morphism from* $\mathcal{M}$ *to* $\mathcal{N}$*. Then, there is a* unique *L-morphism,* $\varinjlim \eta : \varinjlim \mathcal{M} \longrightarrow \varinjlim \mathcal{N}$*, such that the following diagram is commutative for all* $p \in P$,

$$\begin{array}{ccc} M_p & \xrightarrow{\ \mu_p\ } & \varinjlim \mathcal{M} \\ {\scriptstyle \eta_p}\big\downarrow & & \big\downarrow{\scriptstyle \varinjlim \eta} \\ N_p & \xrightarrow[\ \nu_p\ ]{} & \varinjlim \mathcal{N} \end{array}$$

Moreover, if each η_p *is an embedding, the same is true of* $\varinjlim \eta$.

Proof. Straightforward, although one must take care with notation. □

Example 2.13 We show that the colimit of elementary embeddings **might not be** an elementary embedding. Let $\mathbb{N}$ be the set of natural numbers (a linear order and so a rd poset), $\mathbb{Z}$ be the ring of integers and $\mathbb{Q}$ be the field of rational numbers. Let $\{p_n : n \geq 1\}$ be an enumeration of the positive primes, in increasing order. Define, by induction on n, a sequence of commutative rings with 1, Z_n, and ring homomorphisms, $\iota_n : Z_n \longrightarrow Z_{n+1}$, $n \geq 0$, as follows:

$*$ $Z_0 = Z$;

$*$ For $n \geq 0$, $Z_{n+1} = Z_n[\frac{1}{p_n}]$, the ring generated by Z_n and the inverse of the n^{th} prime p_n, inside the field $\mathbb{Q}$; we let ι_n be the canonical inclusion of Z_n into Z_{n+1}.

It is straightforward that $Z_n = \{k/m \in \mathbb{Q}$: The prime divisors of m are among the $p_1, \ldots, p_n\}$, while ι_n is the natural inclusion of Z_n into Z_{n+1}. Let $\mathcal{Z} = \langle\, Z_n;\, \iota_{nm} \,\rangle$ be the Kripke structure over $\mathbb{N}$ where for $\iota_{nn} = Id_{Z_n}$ and for $m \geq n+1$, $\iota_{nm} = \iota_{m-1} \circ \cdots \circ \iota_n$. Clearly, the colimit of $\mathcal{Z}$ is $\mathbb{Q}$, that is,

$$\varinjlim \mathcal{Z} \;=\; \mathbb{Q} \;=\; \bigcup_{n \geq 0} Z_n. \qquad (\sharp)$$

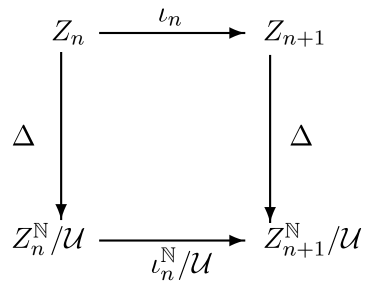
Now let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. We consider the Kripke structure obtained by applying the ultrapower functor determined by the pair $\langle\, \mathbb{N}, \mathcal{U} \,\rangle$ to $\mathcal{Z}$ (see A.75), that is,

$$\mathcal{Z}^{\mathbb{N}}/\mathcal{U} = \langle\, Z_n^{\mathbb{N}}/\mathcal{U};\;\; \iota_{nm}^{\mathbb{N}}/\mathcal{U} \,\rangle.$$

Being ultrapowers of embeddings, the connecting morphisms in $\mathcal{Z}^{\mathbb{N}}/\mathcal{U}$ are also ring embeddings. Hence,

$$\varinjlim \mathcal{Z}^{\mathbb{N}}/\mathcal{U} \;=\; \bigcup_{n \geq 0} Z_n^{\mathbb{N}}/\mathcal{U}. \qquad (\sharp\sharp)$$

By Remark A.75, we have a natural morphism of Kripke structures, $\mathcal{D} : \mathcal{Z} \longrightarrow \mathcal{Z}^{\mathbb{N}}/\mathcal{U}$, induced by the diagonal embeddings Δ of Z_n into $Z_n^{\mathbb{N}}/\mathcal{U}$, since for each n the following diagram is commutative:



By Corollary A.74.(a), each component of the morphism $\mathcal{D}$ is an elementary embedding. By ($\sharp$), the colimit of $\mathcal{Z}$ is a **field**, that is, every non-zero element in it has a multiplicative inverse. On the other hand, $\varinjlim \mathcal{Z}^{\mathbb{N}}/\mathcal{U}$ **is not a field**. Indeed, consider the sequence $\xi = \langle 1, 2, 3, \ldots, n, \ldots \rangle$, which belongs to $\mathbb{Z}^{\mathbb{N}}/\mathcal{U}$, the 0^{th} component of $\mathcal{Z}^{\mathbb{N}}/\mathcal{U}$; by ($\sharp\sharp$), if ξ had an inverse in $\varinjlim \mathcal{Z}^{\mathbb{N}}/\mathcal{U}$, then it would be in $Z_n^{\mathbb{N}}/\mathcal{U}$, for some $n \geq 1$. But this is impossible, because $\mathcal{U}$ is non-principal (A.45.(b)) and ξ contains arbitrarily large primes, with no inverse in Z_n. Hence, $\varinjlim \mathcal{D}$ is not an elementary embedding, as desired. $\square$

2.14 Problem. *Determine conditions on a morphism of Kripke structures over a rd poset, entailing its direct limit to be an elementary embedding.* □

3 Completion of Kripke Structures. Stalks

If P is a poset, Theorem 1.18 yields a join-preserving embedding γ of P into $\mathfrak{U}^{op}$. If we are given a Kripke structure over P, it is natural to enquire whether it can be *naturally extended*, along γ, to a Kripke structure over $\mathfrak{U}^{op}$. This section is devoted to showing that there is an affirmative answer to this question and to reaping the consequences thereof.

Let $\mathcal{M} = \langle M_p; \mu_{pq} \rangle$ be a Kripke L-structure over a poset P. For each $U \in \mathfrak{U}(P) = \mathfrak{U}$, define

$$\mathfrak{g}\mathcal{M}(U) = \{x \in \textstyle\prod_{r \in U} M_r : \forall\, p,\, q \in U, \;\; p \leq q \Rightarrow \mu_{pq}(x(p)) = x(q)\}.$$

Note that the definition makes sense because $x(p) \in M_p$ and $x(q) \in M_q$. Moreover, $\mathfrak{g}\mathcal{M}(U)$ contains the interpretation of constants and is closed under all operations in L. Therefore, we

Endow $\mathfrak{g}\mathcal{M}(U)$ with the L-structure induced by the product $\prod_{r\in U} M_r$, presented in appendix X, that is, for each $U \in \mathfrak{U}$, $\mathfrak{g}\mathcal{M}(U)$ is the L-structure wherein:

$*$ If c is a constant in L, its interpretation is the map $c^{\mathfrak{g}U}(r) \;=\; c^{M_r}$, clearly in $\mathfrak{g}\mathcal{M}(U)$;

$*$ If ω is a n-ary operation in L, its interpretation is given by

$$\omega^{\mathfrak{g}U}(x_1, \ldots, x_n)(r) \;=\; \omega^{M_r}(x_1(r), \ldots, x_n(r)),$$

again, clearly in $\mathfrak{g}\mathcal{M}(U)$;

$*$ If R is a n-ary relation symbol in L and $(x_1, \ldots, x_n) \in \mathfrak{g}\mathcal{M}(U)^n$, then

$$\mathfrak{g}\mathcal{M}(U) \models R[x_1, \ldots, x_n] \quad \text{iff} \quad \forall\, r \in U, \;\; M_r \models R[x_1(r), \ldots, x_n(r)].$$

It is straightforward that if $\tau(v_1, \ldots, v_n)$ is a term in L, then its interpretation in $\mathfrak{g}\mathcal{M}(U)$ is the map

[T] $$\begin{array}{l} \tau : \mathfrak{g}\mathcal{M}(U)^n \longrightarrow \mathfrak{g}\mathcal{M}(U), \text{ given by} \\ \tau(x_1, \ldots, x_n)(r) \;=\; \tau^{M_r}(x_1(r), \ldots, x_n(r)). \end{array}$$

Furthermore, if $\phi(v_1, \ldots, v_n)$ is an atomic formula in L, then for all $\overline{x} = \langle\, x_1, \ldots, x_n \,\rangle \in \mathcal{M}(U)^n$,

[atom] $\qquad \mathfrak{g}\mathcal{M}(U) \models \phi[\overline{x}] \quad$ iff $\quad \forall\, r \in U, \;\; M_r \models \phi[\overline{x}(r)]$.

If $V \subseteq U$, i.e., $U \leq V$ in $\mathfrak{U}^{op}$, there is a natural map

$\rho_{UV} : \mathfrak{g}\mathcal{M}(U) \longrightarrow \mathfrak{g}\mathcal{M}(V)$, given by $\rho_{UV}(x) = x_{|V}$.[9]

Note that ρ_{UV} is the restriction to $\mathfrak{g}\mathcal{M}(U)$ of the canonical projection, $\prod_{r \in U} M_r \longrightarrow \prod_{s \in V} M_s$ that forgets the coordinates outside V. Since this map is a L-morphism, we conclude that **$\boldsymbol{\rho_{UV}}$ is a $\boldsymbol{L}$-morphism**, a fact that can also be verified directly. We have just constructed a Kripke structure, $\mathfrak{g}\mathcal{M}$, over $\mathfrak{U}^{op}$, leading to

Definition 3.1 *The Kripke structure* $\mathfrak{g}\mathcal{M} = \langle\, \mathfrak{g}\mathcal{M}(U);\; \rho_{UV} \,\rangle$ *is the* **completion of $\boldsymbol{\mathcal{M}}$ over $\boldsymbol{\mathfrak{U}^{op}}$**.

We now show that $\mathfrak{g}\mathcal{M}$ deserves its name, by constructing a morphism of Kripke structures as in 2.3, that for all $p \in P$ is a L-isomorphism of M_p onto $\mathfrak{g}\mathcal{M}([p]) = \mathfrak{g}\mathcal{M}(\gamma p)$.

Theorem 3.2 *a) For each $p \in P$, the map*

$g_p : M_p \longrightarrow \mathfrak{g}\mathcal{M}([p])$, *defined by* $g_p(z) = \langle\, \mu_{pq}(z) \,\rangle_{q \geq p}$

is a L-isomorphism from M_p onto $\mathfrak{g}\mathcal{M}([p])$, such that for all $q \geq p$, the following diagram commutes:

$$\begin{array}{ccc} M_p & \xrightarrow{\;\mu_{pq}\;} & M_q \\ {\scriptstyle g_p}\downarrow & & \downarrow{\scriptstyle g_q} \\ \mathfrak{g}\mathcal{M}(\gamma p) & \xrightarrow[\rho_{\gamma p, \gamma q}]{} & \mathfrak{g}\mathcal{M}(\gamma q) \end{array}$$

where $\gamma : P \longrightarrow \mathfrak{U}^{op}$ is the embedding $p \mapsto [p]$ of Theorem 1.18.

b) The Kripke structure $\mathfrak{g}\mathcal{M}$ over $\mathfrak{U}^{op}$ verifies the following conditions:

[9] By definition, x is a map from U to $\bigcup_{r \in U} M_r$, and so has a restriction to V.

$*$ **extensionality**: *For all* $U \in \mathfrak{U}$, *all atomic formulas* $\phi(v_1, \ldots, v_n)$ *in* L *and* $\overline{x} \in \mathfrak{g}\mathcal{M}(U)^n$,

$$[ext] \begin{cases} \textit{If } S \textit{ is a collection of open subsets of } U, \textit{ such that} \\ (i) \bigcup S = U \ (S \textit{ is a covering of } U); \quad \Rightarrow \qquad \mathfrak{g}\mathcal{M}(U) \models \phi[\overline{x}]. \\ (ii) \textit{ For all } V \in S, \quad \mathfrak{g}\mathcal{M}(V) \models \phi[\rho_{UV}(\overline{x})], \end{cases}$$

$*$ **completeness**: *For all* $U \in \mathfrak{U}$, *if* $\langle V_i, x_i \rangle$, $i \in I$, *satisfies, for all* i, $j \in I$

$$[comp] \begin{cases} (i) \ x_i \in \mathfrak{g}\mathcal{M}(V_i); \\ (ii) \bigcup_{i \in I} V_i = U; \quad \Rightarrow \quad \textit{There is a } \text{unique } x \in \mathfrak{g}\mathcal{M}(U) \textit{ such that } \rho_{UV_i}(x) = x_i, \ \forall\, i \in I. \\ (iii) \ \rho_{V_i, V_i \cap V_j}(x_i) = \rho_{V_j, V_i \cap V_j}(x_j). \end{cases}$$

Proof. a) Note that g_p is well-defined; for if $r \geq q \geq p$, then, $\mu_{qr}(\mu_{pq}(z)) = \mu_{pr}(z)$, showing that $\langle \mu_{pq}(z) \rangle_{q \geq p} \in \mathfrak{g}\mathcal{M}([p])$. Furthermore, since $\mu_{pp} = Id_{M_p}$, $g_p(z) = \langle z, \ldots \rangle$, and so g_p is injective. To show it surjective, observe that if $x \in \mathfrak{g}\mathcal{M}([p])$, then $g_p(x(p)) = x$. Indeed, $x(p) \in M_p$ (by definition) and for $q \geq p$ we have $\mu_{pq}(x(p)) = x(q)$, as needed. Clearly, g_p preserves constants and operations. If $R \in rel(n)$ and $\overline{z} \in M_p^n$ is such that $M_p \models R[\overline{z}]$, then the μ_{pq} being L-morphisms, we conclude that for all $q \geq p$, $M_q \models R[\mu_{pq}(\overline{z})]$. This means that

$$\textstyle\prod_{q \geq p} M_q \quad \models \quad R[g_p(z_1), \ldots, g_p(z_n)]$$

and so, since $\mathfrak{g}\mathcal{M}([p])$ is a substructure of this product, we get that $\mathfrak{g}\mathcal{M}([p]) \models R[g_p(\overline{z})]$. Hence, g_p is a L-morphism. Now, suppose that $\mathfrak{g}\mathcal{M}([p]) \models R[g_p(\overline{z})]$. Since $p \in [p]$, condition [atom] in page 470 yields $M_p \models R[g_p(\overline{z})(p)]$, that is, $M_p \models R[\overline{z}]$, because $g_p(\overline{z})(p) = \overline{z}$. We have shown that g_p is a surjective L-embedding, being, therefore, a L-isomorphism. The commutativity of the displayed diagram is straightforward, ending the proof of (a). Observe that

$$\mathfrak{g} = \langle \gamma; (g_p)_{p \in P} \rangle$$

is a morphism of Kripke structures, $\mathfrak{g} : \mathcal{M} \longrightarrow \mathfrak{g}\mathcal{M}$, as defined in 2.3.

b) Condition [*ext*] is a consequence of the fact that the L-morphisms ρ_{UV} are induced by the projections that forget coordinates and satisfaction of atomic formulas in a product is determined coordinatewise (see the first paragraph of appendix X). Hence, if an atomic formula is true at the restrictions of elements of a product in a covering of their domain, the atomic formula must also be satisfied at these elements. Details are left to the reader.

As for [*comp*], note that the family $S = \{x_i : i \in I\}$ is a family of compatible partial maps from U to $\bigcup_{p \in U} M_p$. Indeed, each x_i is a function

$$x_i : V_i \longrightarrow \bigcup_{q \in V_i} M_q, \text{ such that } \forall\, q \leq r \in V_i,\ \mu_{qr}(x_i(q)) = x_i(r).$$

Now condition (*ii*) guarantees that the union of the domain of the x_i's is U, while (*iii*) entails their compatibility. Hence, Lemma 1.16 yields a unique $x : U \longrightarrow \bigcup_{p \in U} M_p$, whose restriction to each V_i equals x_i. For $p \leq q$ in U, there is $i \in I$ such that $p \in V_i$. Then, $q \in V_i$ and so, since x is an extension of x_i, we get $\mu_{pq}(x(p)) = x(q)$. Hence, $x \in \mathfrak{g}\mathcal{M}(U)$, ending the proof. □

Remark 3.3 a) The converse of [*ext*] is 3.2.(b) is trivial because the maps ρ_{UV} are L-morphisms.

b) For the atomic formula $v_1 = v_2$, [*ext*] entails that if $x, y \in \mathfrak{g}\mathcal{M}(U)$ coincide locally in a covering of U, then $x = y$. This is reminiscent of extentionality in Set Theory: two sets are equal iff they have the same elements. That is the origin of the terminology. In the literature one will also find the term *separated* used for the same concept. □

Example 3.4 Let I be a set, considered as a poset with the identity partial order. Let M_i, $i \in I$, be a family of L-structures. This family can be considered as a Kripke structure, $\mathcal{M}$, where the only connecting morphisms are the identities. We shall describe the completion of $\mathcal{M}$ over $\mathfrak{U}^{op}$.

In Example 1.5 it was shown that $\mathfrak{U} = 2^I$, that is, $\mathfrak{U}$ is the discrete topology on I (all subsets are open). For each $A \subseteq I$, we have

$$\mathfrak{g}\mathcal{M}(A) \;=\; \{x \in \textstyle\prod_{i \in A} M_i : \forall\, i \in A,\ x(i) = x(i)\} \;=\; \textstyle\prod_{i \in A} M_i.$$

Hence, the completion of $\mathcal{M}$ over $\mathfrak{U}^{op}$ associates to each $A \subseteq I$, the product of the L-structures whose indices are in A. For $A \subseteq B \subseteq I$, the map ρ_{BA} is just the canonical projection

$$\prod_{i\in B} M_i \longrightarrow \prod_{j\in A} M_j$$

that forgets the coordinates outside A. Note that for $A = \{i\}$, $i \in I$, $*\ \mathfrak{g}\mathcal{M}([i]) = \prod_{j\in\{i\}} M_j = M_i$; $*$ The maps $g_i : M_i \longrightarrow \mathfrak{g}\mathcal{M}([i])$ are simply the identity.

Even this simple example shows that the completion process, while preserving the L-structures originally given at the nodes of the poset P, provides enlargement via the "gluing of compatibles" according to the transition maps μ_{pq}. □

Lemma 3.5 *Let M_i, $i \in I$, be a family of L-structures and let $\mathcal{M}$ be the associated Kripke structure over I, partially ordered by identity. Let $\mathfrak{g}\mathcal{M}$ be the completion of $\mathcal{M}$ over $(2^I)^{op}$, as in Example* 3.4 *and let $\mathcal{F}$ be a filter on I. Then,*

a) $\mathcal{F}$ is a right-directed subset of $(2^I)^{op}$.

b) $\varinjlim (\mathfrak{g}\mathcal{M})_{|\mathcal{F}}$ is naturally L-isomorphic to the reduced product $\prod_{i\in I} M_i/\mathcal{F}$.

Proof. Item (a) is clear because $\mathcal{F}$ is closed under finite meets.

b) Note that $\mathfrak{g}\mathcal{M}_{|\mathcal{F}}$ is the Kripke structure $\mathcal{N}$ over $\mathcal{F} \subseteq (2^I)^{op}$, such that for $A \subseteq B$, both in $\mathcal{F}$,

$*$ $\mathcal{N}_A = \prod_{i\in A} M_i$;

$*$ $\nu_{BA} : \mathcal{N}_B \longrightarrow \mathcal{N}_A$ is the map forgetting coordinates outside A.

Write M for $\prod_{i\in I} M_i/\mathcal{F}$; for $A \in \mathcal{F}$, define

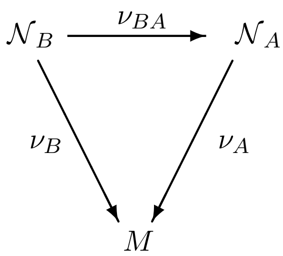
$$\nu_A : \mathcal{N}_A \longrightarrow M, \text{ given by } \nu_A(s) = x/\mathcal{F}, \qquad (\sharp)$$

where $x \in \mathcal{N}_I = \prod_{i\in I} M_i$ is *any* extension of s (which has domain A) to I. To see that ($\sharp$) is independent of the choice of extensions, suppose x, $y \in \mathcal{N}_I$ satisfy $x_{|A} = y_{|A} = s$. Then, since $A \in \mathcal{F}$, we obtain

$$A \subseteq \{i \in I : x(i) = y(i)\} \in \mathcal{F},$$

and so $x/\mathcal{F} = y/\mathcal{F}$. It is easily checked that ν_A is a L-morphism. Let $A \subseteq B$, both in $\mathcal{F}$, let $y \in \mathcal{N}_B$ and let $x = \nu_{BA}(y)$; let $z \in \mathcal{N}_I$ be an extension of y to I. Since ν_{BA} is the projection that forgets coordinates outside A, z is also an extension of x to I. Consequently,

$$\nu_B(y) \;=\; z/\mathcal{F} \;=\; \nu_A(x) \;=\; \nu_A(\nu_{AB}(y)),$$



i.e., the triangle above right is commutative. Thus, $\langle M, \nu_A \rangle$, $A \in \mathcal{F}$, is a dual cone over $\mathcal{N} = \mathfrak{g}\mathcal{M}_{|\mathcal{F}}$. To verify the stated L-isomorphism it suffices to check [colimit 1] and [colimit 2] in 2.6.(b); we shall prove [colimit 2], leaving the former to the reader. Since ν_A, $A \in \mathcal{F}$, are L-morphism, it is enough to verify the implication ($\Rightarrow$) in [colimit 2]. Let $\phi(v_1, \ldots, v_n)$ be an atomic formula in L and let $\overline{\xi} = \langle x_1/\mathcal{F}, \ldots, x_n/\mathcal{F} \rangle \in M^n$. Then, by Corollary A.70

$$\begin{array}{c} M \models \phi[x_1/\mathcal{F}, \ldots, x_n/\mathcal{F}] \quad \text{iff} \\ \mathfrak{v}\phi(\overline{x}) \;=\; \{i \in I : M_i \models \phi[x_1(i), \ldots, x_n(i)]\} \in \mathcal{F}. \end{array} \tag{$\sharp\sharp$}$$

Let $A = \mathfrak{v}\phi(x_1, \ldots, x_n)$ and, for $1 \leq k \leq n$, set $s_k = \nu_{IA}(x_k) = x_{k_{|A}}$; clearly, $\nu_A(s_k) = x_k/\mathcal{F}$ and so $\nu_A(\overline{s}) = \overline{\xi}$. Moreover, it follows immediately from ($\sharp\sharp$) and the definition of product L-structure (see first § of appendix X) that $\mathcal{N}_A = \prod_{i \in A} M_i \models \phi[\overline{s}]$, establishing [colimit 2], as desired. □

The completion constructed above will allow a generalization of reduced products to Kripke structures. Lemma 3.5 indicates the path to thread; its item (a) is true in general, with the same proof:

Lemma 3.6 *If $\mathcal{F}$ is a filter in $\mathfrak{U}(P)$, P a poset, then $\mathcal{F}$ is a rd subset of $\mathfrak{U}^{op}$.* □

This observation, together with Theorem 2.6, leads to

Definition 3.7 *Let $\mathcal{M}$ be a Kripke structure over a poset P and let $\mathfrak{g}\mathcal{M}$ be its completion over $\mathfrak{U}(P)^{op}$. If $\mathcal{F}$ is a filter in $\mathfrak{U}$, we define* **the stalk of $\mathcal{M}$ at $\mathcal{F}$** *as* $\mathcal{M}_{\mathcal{F}} \;=\; \varinjlim \,(\mathfrak{g}\mathcal{M})_{|\mathcal{F}}$.

By Lemma 3.5, the stalk of the completion of a discrete family of structures at a filter $\mathcal{F}$ is precisely the reduced product by $\mathcal{F}$. Thus, the notion of *stalk* generalizes reduced products (and ultraproducts).

Definition 3.8 *If $\mathcal{F}$ is a filter in $\mathfrak{U}(P)$, the* **trace of $\mathcal{F}$ in P** *is*

$$\gamma^{-1}(\mathcal{F}) = \{p \in P : [p] \in \mathcal{F}\},$$

where γ is the embedding of 1.18. *If we identify P with its image by γ inside $\mathfrak{U}^{op}$, then the trace of $\mathcal{F}$ in P is simply $\mathcal{F} \cap P$.*

Note that the trace of filter may be very small, even though the filter is large. For instance, if $\mathcal{U}$ is a non-principal ultrafilter on a set I (necessarily infinite by A.45.(b)), the trace of $\mathcal{U}$ in I is empty. Nevertheless, there are interesting situations in which the opposite occurs:

Proposition 3.9 *Let P be a poset and let $\mathcal{F}$ be a filter in $\mathfrak{U}(P) = \mathfrak{U}$. Assume that $\mathcal{F}$ satisfies*

$[E]$ *For all $U \in \mathcal{F}$, there is $p \in \gamma^{-1}(\mathcal{F})$, such that $[p] \subseteq U$.*

Then,

a) $\gamma^{-1}(\mathcal{F})$ is right-directed in P.

b) For all Kripke structures $\mathcal{M}$ over P, $\mathcal{M}_{\mathcal{F}} = \varinjlim \mathcal{M}_{|\gamma^{-1}(\mathcal{F})}$.

Proof. a) For $p, q \in \gamma^{-1}(\mathcal{F})$, we have $[p] \cap [q] \in \mathcal{F}$ and so conditions $[E]$ yields $r \in \gamma^{-1}(\mathcal{F})$ such that $[r] \subseteq [p] \cap [q]$. Hence, $p, q \leq r$ and $\gamma^{-1}(\mathcal{F})$ is rd.

b) Recall that for $U, V \in \mathfrak{U}$, $V \subseteq U$ in $\mathfrak{U}$ iff $U \leq V$ in $\mathfrak{U}^{op}$. Hence, condition $[E]$ guarantees that $\{[p] : p \in \gamma^{-1}(\mathcal{F})\}$ is *cofinal* in $\mathcal{F}$ in $\mathfrak{U}^{op}$, that is,

For all $U \in \mathcal{F}$, there is $p \in \gamma^{-1}(\mathcal{F})$ such that $U \leq [p]$.

Since $\mathfrak{g}\mathcal{M}_{|\gamma(P)}$ is isomorphic to $\mathcal{M}$ (3.2.(a)), the conclusion follows from Theorem 2.6.(a). □

Here are some applications of 3.9.

Corollary 3.10 *Let P be a right-directed poset. With notation as above,*

a) The collection $\mathfrak{U} \setminus \{\emptyset\}$ is a filter on P, in fact, the filter $\mathfrak{D}(\mathfrak{U})$, of dense elements in the topology $\mathfrak{U}$. Moreover, $\mathfrak{D}(\mathfrak{U})$ is the **unique** *ultrafilter in $\mathfrak{U}$.*

b) $\gamma^{-1}(\mathfrak{D}(\mathfrak{U})) = P$.

c) If $\mathcal{M}$ *is a Kripke structure over* P*, then* $\varinjlim \mathcal{M} = \mathcal{M}_{\mathfrak{D}(\mathfrak{U})}$*, the stalk of* $\mathcal{M}$ *at* $\mathfrak{D}(\mathfrak{U})$.

Proof. Item (b) follows immediately from (a), while (c) is a consequence of (b) and 3.9. For (a), recall that P is irreducible (1.7.(a)) and so every non-empty open is dense (A.24.(3)). The remaining assertion in (a) follows from the equivalence in A.43.(e). □

Corollary 3.10 is our first example of a *generalized ultraproduct*. One may ask if there is an analogue of the Łós ultraproduct Theorem (A.73). The answer is **yes**: see Theorems 5.1, 6.4 and 7.7.

Definition 3.11 *A proper filter* $\mathcal{P}$ *in a topology* $\mathcal{O}$ *is* **completely prime**[10] *if*

$[CP]$ *For all* $S \subseteq \mathcal{O}$, $\bigcup S \in \mathcal{P} \;\Rightarrow\; \exists\, u \in S$ *such that* $u \in \mathcal{P}$.

Example 3.12 The filter ν_p of open neighborhoods of a point p in any space is completely prime. □

Corollary 3.13 *Let* $\mathcal{M}$ *be a Kripke structure over a poset* P *and let* $\mathcal{P}$ *be a completely prime filter in* $\mathfrak{U}(P)$*. Then*

a) $\gamma^{-1}(\mathcal{P})$ *verifies condition* $[E]$ *in* 3.9. *b)* $\mathcal{M}_{\mathcal{P}} = \varinjlim \mathcal{M}_{|\gamma^{-1}(\mathcal{P})}$.

c) For all $p \in P$, $\mathcal{M}_{\nu_p} = M_p$.

Proof. Item (b) follows from (a) and 3.9. To check (a), let $U \in \mathcal{P}$; then $U = \bigcup_{p \in U} [p]$ and the fact that $\mathcal{P}$ is completely prime entails that there is $p \in U$ such that $[p] \in \mathcal{P}$, as needed.

c) Since ν_p is completely prime, (b) yields $\mathcal{M}_{\nu_p} = \varinjlim \mathcal{M}_{|\gamma^{-1}(\nu_p)}$. Now observe that

$$\text{For all } q \in P, \quad [q] \in \nu_p \quad \text{iff} \quad q \leq p,$$

and so $\gamma^{-1}(\nu_p) = p^{\leftarrow}$. Since $p^{\leftarrow}$ has a largest element p, the conclusion follows from 2.5.(b). □

[10] Also called **point** or **pure state**.

4 Forcing in Kripke Structures

In this section we develop a sound interpretation of Intuitionism in Kripke structures, called **forcing**. There are several types of forcing, closely related to each other. We start with the concept which most fundamental. As before, L is a first-order language with equality.

Definition 4.1 *Let P be a poset and $\mathcal{M} = \langle\, M_p;\, \mu_{pq}\,\rangle$ be a Kripke L-structure over P. Let $\phi(v_1, \ldots, v_n)$ be a L-formula, $p \in P$ and $\overline{x} \in M_p^n$. Define a relation*

$$M_p \Vdash \phi[\overline{x}],$$

read "M_p forces ϕ at $\overline{x}$", by induction on complexity, as follows[11]*:*

(1) *If ϕ is atomic, then* $M_p \Vdash \phi[\overline{x}]$ *iff* $M_p \models \phi[\overline{x}]$;

(2) $M_p \Vdash \phi \wedge \psi\,[\overline{x}]$ *iff* $M_p \Vdash \phi[\overline{x}]$ *and* $M_p \Vdash \psi[\overline{x}]$;

(3) $M_p \Vdash \phi \vee \psi\,[\overline{x}]$ *iff* $M_p \Vdash \phi[\overline{x}]$ *or* $M_p \Vdash \psi[\overline{x}]$;

(4) $M_p \Vdash \neg\,\phi[\overline{x}]$ *iff* $\forall\, q \geq p$, *it is not true that* $M_q \Vdash \phi[\mu_{pq}(\overline{x})]$;

(5) $M_p \Vdash \phi \rightarrow \psi\,[\overline{x}]$ *iff* $\forall\, q \geq p$, $M_q \Vdash \phi[\mu_{pq}(\overline{x})] \;\Rightarrow\; M_q \Vdash \psi[\mu_{pq}(\overline{x})]$;

(6) $M_p \Vdash \exists v\, \phi[v; \overline{x}]$ *iff* $\exists\, y \in M_p$ *such that* $M_p \Vdash \phi[y; \overline{x}]$;

(7) $M_p \Vdash \forall v\, \phi[v; \overline{x}]$ *iff* $\forall\, q \geq p$ *and* $\forall\, y \in M_q$, $M_q \Vdash \phi[y;\, \mu_{pq}(\overline{x})]$.

If $\Gamma(v_1, \ldots, v_n)$ is a set of formulas in L, $p \in P$ and $\overline{x} \in M_p^n$, $M_p \Vdash \Gamma[\overline{x}]$ *means that for all $\phi \in \Gamma$,* $M_p \Vdash \phi[\overline{x}]$.

Lemma 4.2 *If $\mathcal{M} = \langle\, M_p;\, \mu_{pq}\,\rangle$ is a Kripke structure, $\phi(\overline{v})$ is a formula in L, $p \in P$ and $\overline{x} \in M_p^n$,*

a) (Extension) $M_p \Vdash \phi[\overline{x}]$ *and* $q \geq p \;\Rightarrow\; M_q \Vdash \phi[\mu_{pq}(\overline{x})]$.

b) (Consistency) *It cannot happen that* $M_p \Vdash \phi[\overline{x}]$ *and* $M_p \Vdash \neg\,\phi[\overline{x}]$.

c) (Double Negation) $M_p \Vdash \neg\neg\,\phi[\overline{x}]$ *iff* $\forall\, q \geq p$, $\exists\, r \geq q$ *such that* $M_r \Vdash \phi[\mu_{pr}(\overline{x})]$.

Proof. Item (b) is immediate from clause (4) in 4.1. For (a), proceed by induction on complexity. For atomic formulas the result is true

[11] All the connectives on the right-hand side of the definitions below are the *classical connectives* in the metalanguage.

because the L-morphisms μ_{pq} preserves atomic formulas. We treat the existential quantifier and implication, leaving the the other logical symbols to the reader.

If $M_p \Vdash \exists v\ \phi[\overline{x}]$ and $q \geq p$, then there is $y \in M_p$ such that $M_p \Vdash \phi[y; \overline{x}]$. By the induction hypothesis, $M_q \Vdash \phi[\mu_{pq}(y); \mu_{pq}(\overline{x})]$ and so $M_q \Vdash \exists v\ \phi[v; \mu_{pq}(\overline{x})]$, as needed.

Assume that $M_p \Vdash \phi \to \psi\ [\overline{x}]$ and $q \geq p$. If $r \geq q$ and $M_r \Vdash \phi[\mu_{qr}(\mu_{pq}(\overline{x}))]$, since $\mu_{qr}(\mu_{pq}(\overline{x})) = \mu_{pr}(\overline{x})$, clause (5) in 4.1 entails $M_r \Vdash \psi[\mu_{pr}(\overline{x})]$. Hence, $M_r \Vdash \psi[\mu_{qr}(\mu_{pq}(\overline{x}))]$ and $M_q \Vdash \phi \to \psi\ [\mu_{pq}(\overline{x})]$, as desired. Item (c) is just an unraveling of definitions. □

Corollary 4.3 *Let $\mathcal{M} = \langle M_p; \mu_{pq} \rangle$ be a Kripke structure over the poset P. If p is a maximal point in P, i.e., $[p] = \{p\}$, and $\phi(\overline{v})$ is a formula in L, then for all $\overline{x} \in M_p^n$ $M_p \Vdash \phi[\overline{x}]$ iff $M_p \models \phi[\overline{x}]$.*

Proof. An easy induction on complexity. □

It follows from Corollary 4.3 that *forcing generalizes satisfaction.* Moreover, it is interesting when there is an actual "never ending" process going on.

Theorem 4.4 (Soundness) *Let $\Gamma(v_1, \ldots, v_n) \cup \{\phi(v_1, \ldots, v_n)\}$ be a set of formulas in L. For all $p \in P$ and $\overline{x} \in M_p^n$,*

$$M_p \Vdash \Gamma[\overline{x}] \quad \textit{and} \quad \Gamma(v_1, \ldots, v_n) \vdash_{\mathcal{H}} \phi(v_1, \ldots, v_n) \quad \Rightarrow \quad M_p \Vdash \phi[\overline{x}].$$

Proof. Straightforward, but patience and perseverance are required. In fact, this is an example of a result that, in the words of Serge Lang, "one should prove once, and only once, in a lifetime". □

We now connect forcing in a Kripke structure over a poset P with the $\mathfrak{U}$-topology on P. As expected, one must deal with finite sequences and so we extend the notational conventions in A.57 as follows:

4.5 Notation. If $\mathcal{M} = \langle M_p; \mu_{pq} \rangle$ is a Kripke L-structure over a poset P

(1) Let $M_* = \coprod_{p \in P} M_p = \bigcup_{p \in P} M_p \times \{p\}$.

(2) If $n \geq 1$ is an integer, write $\langle \overline{x}, \overline{p} \rangle \in M_*^n$ for $\langle \overline{x}, \overline{p} \rangle = \langle \langle x_1, p_1 \rangle, \ldots, \langle x_n, p_n \rangle \rangle$.

(3) Write $\underline{p}$ for the constant n-sequence $\underbrace{\langle p, \ldots, p \rangle}_{ntimes}$. Hence,

$$\langle \overline{x}, \underline{p} \rangle \ = \ \langle \langle x_1, p \rangle, \ldots, \langle x_n, p \rangle \rangle.$$

(4) If $\overline{p} \in P^n$ and $q \in P$, $q \geq \overline{p}$ means that $q \geq p_1, \ldots, p_n$.

(5) If $\langle \overline{x}, \overline{p} \rangle \in M_*^n$ and $q \geq \overline{p}$, then

$$\mu_{\overline{p}\,q}(\overline{x}) = \langle \mu_{p_1 q}(x_1), \ldots, \mu_{p_n q}(x_n) \rangle \in M_q^n.$$

(6) If $\langle \overline{x}, \overline{p} \rangle \in M_*^n$, the **extent of $\langle \overline{x}, \overline{p} \rangle$** is $E\langle \overline{x}, \overline{p} \rangle = \bigcap_{i=1}^n [p_i]$. Note that $E\langle x, p \rangle = [p]$. □

Definition 4.6 *If $\mathcal{M} = \langle M_p; \mu_{pq} \rangle$ is a Kripke L-structure over a poset P and $\phi(v_1, \ldots, v_n)$ is a formula in L, we define a map*

$$[\![\phi(\cdot)]\!]_{\mathcal{M}} : M_*^n \longrightarrow 2^P,$$

the $\mathfrak{U}$-value of ϕ, *given, for $\langle \overline{x}, \overline{p} \rangle \in M_*^n$ by*

$$\begin{aligned} [\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!]_{\mathcal{M}} &= \{q \in E\langle \overline{x}, \overline{p} \rangle : M_q \Vdash \phi[\mu_{\overline{p}\,q}(\overline{x})]\} \\ &= \{q \geq \overline{p} : M_q \Vdash \phi[\mu_{\overline{p}\,q}(\overline{x})]\}. \end{aligned}$$

When $\mathcal{M}$ is clear from context its mention will be omitted from the notation.

Lemma 4.7 *If $\mathcal{M} = \langle M_p; \mu_{pq} \rangle$ is a Kripke structure over a poset P and $\phi(v_1, \ldots, v_n)$ is a formula in L, then*

a) For all $\langle \overline{x}, \overline{p} \rangle \in M_*^n$, $[\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!]$ is an open set in the $\mathfrak{U}$-topology.

b) For all $p \in P$ and $\overline{x} \in M_p^n$, $M_p \Vdash \phi[\overline{x}]$ iff $p \in [\![\phi(\langle \overline{x}, \underline{p} \rangle)]\!]$.

Proof. Straightforward from the definitions and Lemma 4.2.(a). □

Remark 4.8 If σ is a *sentence* in L (i.e., a formula with no free variables) then

$$[\![\sigma]\!] = \{p \in P : M_p \Vdash \sigma\}$$

is an open set in P, **the $\mathfrak{U}$-value of σ** in the Kripke structure $\mathcal{M}$. □

By 4.7, $[\![\phi]\!]$ is actually a map from M_*^n into $\mathfrak{U}(P)$. Theorem 4.4 and Lemma 4.7 yield

Corollary 4.9 *Let $\mathcal{M} = \langle M_p; \mu_{pq} \rangle$ be a Kripke structure and $\Gamma \cup \{\phi\}$ be a set of formulas in L in the free variables $v_1, \ldots, v_n$. Then,*
$\Gamma \vdash_{\mathcal{H}} \phi \quad \Rightarrow \quad \forall \langle \overline{x}, \overline{p} \rangle \in M_*^n, \ \bigcap_{\psi \in \Gamma} [\![\psi(\langle \overline{x}, \overline{p} \rangle)]\!] \ \subseteq \ [\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!]$.

Example 4.10 Let $\mathcal{M} = \langle M_p; \mu_{pq} \rangle$ be a Kripke structure over the poset P. Consider the formula $\phi \equiv (v = u)$, where v, u are distinct variables in L. If $\langle \langle x, p \rangle, \langle y, q \rangle \rangle \in M_*^2$, then

$$[\![\phi(\langle \langle x, p \rangle, \langle y, q \rangle \rangle)]\!] \ = \ \{r \geq p, q : \mu_{pr}(x) = \mu_{qr}(y)\}.$$

We shall simply write this using *infix* notation as

$$[\![\langle x, p \rangle = \langle y, q \rangle]\!] \ = \ \{r \geq p, q : \mu_{pr}(x) = \mu_{qr}(y)\}.$$

It is easily established that for $\langle x, p \rangle$, $\langle y, q \rangle$, $\langle z, r \rangle \in M_*$
[equ 1]: $[\![\langle x, p \rangle = \langle y, q \rangle]\!] = [\![\langle y, q \rangle = \langle x, p \rangle]\!]$;
[equ 2]: $[\![\langle x, p \rangle = \langle y, q \rangle]\!] \cap [\![\langle y, q \rangle = \langle z, r \rangle]\!] \subseteq [\![\langle x, p \rangle = \langle z, r \rangle]\!]$.
Moreover, $[\![\langle x, p \rangle = \langle x, p \rangle]\!] = [p] = E\langle x, p \rangle$, the extent of $\langle x, p \rangle$ as defined in 4.5.

This may be generalized to sequences $\langle \overline{x}, \overline{p} \rangle$, $\langle \overline{y}, \overline{q} \rangle \in M_*^n$, as follows:

$$[\![\langle \overline{x}, \overline{p} \rangle = \langle \overline{y}, \overline{q} \rangle]\!] \ = \ \bigcap_{i=1}^n [\![\langle x_i, p_i \rangle = \langle y_i, q_i \rangle]\!].$$

Properties [equ 1] and [equ 2] still hold for the equality of finite sequences in M_*. Moreover,

$$[\![\langle \overline{x}, \overline{p} \rangle = \langle \overline{x}, \overline{p} \rangle]\!] \ = \ E\langle \overline{x}, \overline{p} \rangle,$$

as defined in 4.5. This notation *is compatible* with that used for products (A.66): just consider the completion of $\mathcal{M}$ over $\mathfrak{U}^{op}$, discussed in Theorem 3.2, as well as Example 3.4. □

The next Lemma shows that Kripke structures are extensional and that the $\mathfrak{U}$-values of formulas satisfy the Leibniz substitution rule ([L] in A.53).

Lemma 4.11 *With notation as in* 4.10*, let* $\mathcal{M}$ *be a Kripke structure over a poset* P*.*

a) For all $\langle x, p\rangle$*,* $\langle y, q\rangle \in M_*$*,*[12]

$$[ext] \quad E\langle x,p\rangle = E\langle y,q\rangle = [\![\langle x,p\rangle = \langle y,q\rangle]\!] \ \textit{ implies } \ \langle x,p\rangle = \langle y,q\rangle.$$

b) If $\phi(v_1,\ldots,v_n)$ *is a formula in* L *and* $\langle \overline{x},\overline{p}\rangle$*,* $\langle \overline{y},\overline{q}\rangle \in M_*^n$*, then*

$$[L] \qquad [\![\phi(\langle \overline{x},\overline{p}\rangle)]\!] \ \cap \ [\![\langle \overline{x},\overline{p}\rangle = \langle \overline{y},\overline{q}\rangle]\!] \ \subseteq \ [\![\phi(\langle \overline{y},\overline{q}\rangle)]\!].$$

Proof. Let $\mathcal{M} = \langle M_p; \mu_{pq}\rangle$; the hypothesis in (a) means

$$[p] = [q] = [\![\langle x,p\rangle = \langle y,q\rangle]\!] = \{r \geq p,\, q : \mu_{pr}(x) = \mu_{qr}(y)\}.$$

Hence, $p = q$ and $x = \mu_{pp}(x) = \mu_{qq}(y) = y$.

b) If r is the intersection on the left-hand side, then $\mu_{\overline{p}r}(\overline{x}) = \mu_{\overline{q}r}(\overline{y})$ and $M_r \Vdash \phi[\mu_{\overline{p}r}(\overline{x})]$. Thus, $M_r \Vdash \phi[\mu_{\overline{q}r}(\overline{y})]$, and so $r \in [\![\phi(\langle \overline{y},\overline{q}\rangle)]\!]$, as desired. ☐

Since the largest $\mathfrak{U}$-value that a formula can have at $\langle \overline{x},\overline{p}\rangle \in M_*^n$ is $E\langle \overline{x},\overline{p}\rangle$, it is natural to set down the following

Definition 4.12 *If* $\mathcal{M}$ *is a Kripke structure over a poset* P*,* $\phi(v_1,\ldots,v_n)$ *is a formula in* L *and* $\langle \overline{x},\overline{p}\rangle \in M_*^n$*, define*

$$\mathcal{M} \Vdash \phi[\langle \overline{x},\overline{p}\rangle] \quad \textit{iff} \quad [\![\phi(\langle \overline{x},\overline{p}\rangle)]\!] = E\langle \overline{x},\overline{p}\rangle,$$

read $\boldsymbol{\mathcal{M}}$ **forces** $\boldsymbol{\phi}$ **at** $\boldsymbol{\langle \overline{x},\overline{p}\rangle}$*, corresponding to* classical satisfaction*. Note that if* σ *is a* sentence *in* L*, then* $\mathcal{M} \Vdash \sigma$ *iff* $[\![\sigma]\!] = P$*.*

Example 4.13 Let $\mathcal{Z}$ be the Kripke structure of commutative rings with identity of Example 2.13, that is,

$$\mathbb{Z} \xrightarrow{\iota_1} Z_1 \ldots Z_n \xrightarrow{\iota_n} Z_{n+1} \ldots$$

where $Z_n = \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{p_n}]$ is the ring generated, inside $\mathbb{Q}$, by $\mathbb{Z}$ and the inverse of the first n primes. The reader can check that

$$\begin{aligned} &\mathcal{Z} \Vdash \forall v\ (v \neq 0 \to \neg\neg\, \exists u(uv = 1)) \quad \text{and} \\ &\mathcal{Z} \Vdash \forall v\ (\neg\, \exists u(uv = 1) \to v = 0), \end{aligned} \qquad (\sharp)$$

but that **we do not have**

$$\mathcal{Z} \Vdash \forall v\ (v \neq 0 \ \to \ \exists u(uv = 1)). \qquad (\sharp\sharp)$$

[12] Compare with $[ext]$ in the statement of Theorem 3.2.

Classically, all three sentences define a field. Intuitionistically, there are several distinct concepts of “field”. Thus, $\mathcal{Z}$ is a field in the sense of ($\sharp$), but not in the sense of ($\sharp\sharp$). This phenomenon is important in applications of Intuitionistic reasoning to Mathematics. □

Remark 1.17 yields an inductive description of the $\mathfrak{U}$-values of formulas, as follows:

Theorem 4.14 *Let $\mathcal{M} = \langle M_p; \mu_{pq} \rangle$ be a Kripke structure. Let $\phi(v_1, \ldots, v_n)$ be a formula in L and $\langle \overline{x}, \overline{p} \rangle \in M_*^n$. Then,*

a) If ϕ is atomic, $[\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!] = \{q \in E\langle \overline{x}, \overline{p} \rangle : M_q \models \phi[\mu_{\overline{p}q}(\overline{x})]\}$.

b) If $\phi \equiv \psi_1 \diamond \psi_2$, then $[\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!] = [\![\psi_1(\langle \overline{x}, \overline{p} \rangle)]\!] \diamond [\![\psi_2(\langle \overline{x}, \overline{p} \rangle)]\!]$, *where $\diamond \in \{\wedge, \vee\}$ and the $\diamond$ in the right-hand side of the equation refer to the corresponding operations in the frame $\mathfrak{U}$.*[13]

c) $[\![\neg \phi(\langle \overline{x}, \overline{p} \rangle)]\!] = E\langle \overline{x}, \overline{p} \rangle \cap \neg [\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!]$.

d) $[\![\psi_1 \to \psi_2(\langle \overline{x}, \overline{p} \rangle)]\!] = E\langle \overline{x}, \overline{p} \rangle \cap \Big([\![\psi_1(\langle \overline{x}, \overline{p} \rangle)]\!] \to [\![\psi_2(\langle \overline{x}, \overline{p} \rangle)]\!]\Big)$.

e) $[\![\exists v\, \psi(v; \langle \overline{x}, \overline{p} \rangle)]\!] = \bigcup_{\xi \in M_*} [\![\psi(\xi; \langle \overline{x}, \overline{p} \rangle)]\!]$.

f) $[\![\forall v\, \psi(v; \langle \overline{x}, \overline{p} \rangle)]\!] = E\langle \overline{x}, \overline{p} \rangle \cap \Big(\bigwedge_{\xi \in M_*} E\xi \to [\![\psi(\xi; \langle \overline{x}, \overline{p} \rangle)]\!]\Big)$.

Proof. a) Because forcing and satisfaction coincide for atomic formulas, it follows that

$$\begin{aligned} [\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!] &= \{q \geq \overline{p} : M_q \Vdash \phi[\mu_{\overline{p}q}(\overline{x})]\} \\ &= \{q \geq \overline{p} : M_q \models \phi[\mu_{\overline{p}q}(\overline{x})]\}. \end{aligned}$$

b) Let $\diamond \in \{\wedge, \vee\}$; the definition of forcing (4.1) yields

$$\begin{aligned} [\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!] &= \{q \geq \overline{p} : M_q \Vdash \phi[\mu_{\overline{p}q}(\overline{x})]\} = \\ &= \{q \geq \overline{p} : M_q \Vdash \psi_1 \diamond \psi_2[\mu_{\overline{p}q}(\overline{x})]\} \\ &= \{q \geq \overline{p} : M_q \Vdash \psi_1[\mu_{\overline{p}q}(\overline{x})]\} \diamond \{q \geq \overline{p} : M_q \Vdash \psi_1[\mu_{\overline{p}q}(\overline{x})]\} \\ &= [\![\psi_1(\langle \overline{x}, \overline{p} \rangle)]\!] \diamond [\![\psi_2(\langle \overline{x}, \overline{p} \rangle)]\!]. \end{aligned}$$

[13] The same convention holds in item (c), (d) and (f) of the present statement.

Since (c) is a special case of (d), we treat only the latter.

d) We shall use the description of implication in the $\mathfrak{U}$-topology appearing in 1.17.(1), as well as the fact that the interior operation distributes over finite meets (A.18.(4)). We have

$$\begin{aligned}
[\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!] &= \{q \geq \overline{p} : M_q \Vdash (\psi_1 \to \psi_2)[\mu_{\overline{p}\,q}(\overline{x})]\} \\
&= \{q \geq \overline{p} : \ \forall\, r \geq q, \ M_r \Vdash \psi_1[\mu_{\overline{p}\,r}(\overline{x})] \ \Rightarrow \ M_r \Vdash \psi_1[\mu_{\overline{p}\,r}(\overline{x})]\} \\
&= \{q \geq \overline{p} : \ \forall\, r \geq q, \ \neg \Big(M_r \Vdash \psi_1[\mu_{\overline{p}\,r}(\overline{x})] \Big) \ \ or \ \ M_r \Vdash \psi_1[\mu_{\overline{p}\,r}(\overline{x})]\} \\
&= \{q \geq \overline{p} : \ [q] \ \subseteq \ \Big([\![\psi_1(\langle \overline{x}, \overline{p} \rangle)]\!]^c \cup [\![\psi_2(\langle \overline{x}, \overline{p} \rangle)]\!] \Big)\} \\
&= \textstyle\bigcap_{i=1}^n \, [p_n] \ \cap \ \Big([\![\psi_1(\langle \overline{x}, \overline{p} \rangle)]\!] \to [\![\psi_2(\langle \overline{x}, \overline{p} \rangle)]\!] \Big) \\
&= E\langle \overline{x}, \overline{p} \rangle \cap \Big([\![\psi_1(\langle \overline{x}, \overline{p} \rangle)]\!] \to [\![\psi_2(\langle \overline{x}, \overline{p} \rangle)]\!] \Big),
\end{aligned}$$

as needed.

e) If $q \in [\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!]$, then $M_q \Vdash \exists v \ \psi[v;\, \mu_{\overline{p}\,q}(\overline{x})]$ and $\exists\, y \in M_q$ such that $M_q \Vdash \psi[y;\, \mu_{\overline{p}\,q}(\overline{x})]$. Hence, if $\zeta = \langle y, q \rangle$, we get $q \in [\![\psi(\zeta; \langle \overline{x}, \overline{p} \rangle)]\!]$, and so $[\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!] \ \subseteq \ \bigcup_{\xi \in M_*} [\![\psi(\xi; \langle \overline{x}, \overline{p} \rangle)]\!]$.

For the reverse containment, suppose $\xi = \langle y, r \rangle \in M_*$ is such that $q \in [\![\psi(\xi; \langle \overline{x}, \overline{p} \rangle)]\!]$; then,

$$q \geq r, \overline{p} \quad \text{and} \quad M_q \Vdash \psi[\mu_{rq}(y);\, \mu_{\overline{p}\,q}(\overline{x})],$$

and so $M_q \Vdash \exists v \ \psi[v;\, \mu_{\overline{p}\,q}(\overline{x})]$; thus, $q \in [\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!]$, completing the proof of (e).

f) The proof is divided in two parts:

(1) Since $[\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!] \subseteq E\langle \overline{x}, \overline{p} \rangle$ (by definition, see 4.6), to show that the left-hand side of the displayed equation is contained in its right-hand side , it is enough to check that

$$[\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!] \ \subseteq \ \textstyle\bigwedge_{\xi \in M_*} E\xi \to [\![\psi(\xi; \langle \overline{x}, \overline{p} \rangle)]\!].$$

For $\xi = \langle z, r \rangle \in M_*$, we must then verify that $[\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!] \subseteq E\xi \to [\![\psi(\xi; \langle \overline{x}, \overline{p} \rangle)]\!]$, which, by the adjointness relation [adj] in Lemma A.33.(a), is equivalent to

$$[\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!] \cap E\xi \;\subseteq\; [\![\psi(\xi; \langle \overline{x}, \overline{p} \rangle)]\!]. \qquad (\sharp)$$

If $q \in [\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!] \cap E\xi$, then $q \geq r, \overline{p}$ and $M_q \Vdash \forall v\ \psi[v;\ \mu_{\overline{p}\,q}(\overline{x})]$. Hence, for $s \geq q$ and $t \in M_s$, we have $M_s \Vdash \psi[t;\ \mu_{\overline{p}\,s}(\overline{x})]$. In particular, $M_q \Vdash \psi[\mu_{rq}(z);\ \mu_{\overline{p}\,q}(\overline{x})]$, and so $q \in [\![\psi(\xi; \langle \overline{x}, \overline{p} \rangle)]\!]$, verifying ($\sharp$).

(2) It must be checked that

$$E\langle \overline{x}, \overline{p} \rangle \;\cap\; \bigcap_{\xi \in M_*} [\![\psi(\xi; \langle \overline{x}, \overline{p} \rangle)]\!] \;\subseteq\; [\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!]. \qquad (\sharp\sharp)$$

We have written $\bigcap$ in place of $\bigvee$ because by 1.17, the meet in the frame $\mathfrak{U}(P)$ is set-theoretic intersection. To prove ($\sharp\sharp$), suppose that $q \in P$ satisfies

$$(i)\quad q \geq \overline{p} \quad \text{and} \quad (ii)\quad \forall\, \xi \in M_*,\;\; q \in (E\xi \to [\![\psi(\xi; \langle \overline{x}, \overline{p} \rangle)]\!]).$$

Since the implication in (ii) is open in P (4.7.(a)), condition (ii) is equivalent to

$$\forall\, \xi \in M_*,\;\; [q] \;\subseteq\; E\xi \to [\![\psi(\xi; \langle \overline{x}, \overline{p} \rangle)]\!],$$

which, by the adjointness relation in A.33.(a) is yet equivalent to

$$(ii)'\quad \forall\, \xi \in M_*,\;\; [q] \cap E\xi \;\subseteq\; [\![\psi(\xi; \langle \overline{x}, \overline{p} \rangle)]\!].$$

For $r \geq q$ and $z \in M_r$, consider $\xi = \langle z, r \rangle$; then, $r \in [q] \cap E\xi = [q] \cap [r]$, and $(ii)'$ entails $r \in [\![\psi(\xi; \langle \overline{x}, \overline{p} \rangle)]\!]$. Hence, $M_r \Vdash \psi[z;\ \mu_{\overline{p}\,r}(\overline{x})]$, that is $M_q \Vdash \forall v\ \psi[v;\ \mu_{\overline{p}\,q}(\overline{x})]$. This, in turn means that $q \in [\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!]$, establishing ($\sharp\sharp$) and ending the proof. □

Remark 4.15 The original idea for the open set values of formulas in Theorem 4.14 comes from Dana Scott ([FS]), although it was also envisaged by Danny Ellerman ([El]) and, in a different context, by G. Takeuti. This is not surprising, given Scott's and Takeuti's connection to Boolean valued models. However, since our metatheory is classical – in [FS] it is Intuitionism –, the author introduced modifications to simplify the treatment, which appeared in print for the first time in [Mi1]. □

5 Forcing and Truth in Colimits

In this section we relate forcing in a Kripke structure over a right-directed poset and truth in its colimit. Recall from A.51 that $L_\exists$ is the fragment of L consisting of the formulas constructed from the atomic formulas using only the logical symbols $\{\wedge, \vee, \neg, \rightarrow, \exists\}$. Recall that $\mathfrak{D}(\mathfrak{U})$ is the filter of dense opens in $\mathfrak{U}(P)$, the *only ultrafilter* in $\mathfrak{U}(P)$, by 3.10.(a). The next result gives a version of Łós' Theorem A.73 for forcing in a rd Kripke structure, as promised right after 3.10:

Theorem 5.1 *Let $M = \langle M; \mu_p \rangle = \varinjlim \mathcal{M}$, where $\mathcal{M} = \langle M_p; \mu_{pq} \rangle$ is a Kripke structure over a rd poset P. For a formula $\phi(v_1, \ldots, v_n)$ in $L_\exists$, $p \in P$ and $\overline{x} \in M_p^n$, the following are equivalent:*

(1) $M \models \phi[\mu_p(\overline{x})]$; (2) $[\![\phi(\langle \overline{x}, \underline{p} \rangle)]\!] \neq \emptyset$;

(3) $[\![\phi(\langle \overline{x}, \underline{p} \rangle)]\!]$ *is cofinal in* P; (4) $[\![\phi(\langle \overline{x}, \underline{p} \rangle)]\!] \in \mathfrak{D}(\mathfrak{U})$.

Proof. We shall show that (1) $\Leftrightarrow$ (2), noting that

$*$ (2) $\Leftrightarrow$ (3) because P being right-directed, an open set is cofinal iff it is non-empty;

$*$ (3) $\Leftrightarrow$ (4) follows from 1.2.(g).

The verification of (1) $\Leftrightarrow$ (2) is by induction on complexity. For atomic formulas it is an immediate consequence of condition [colim 2] in Theorem 2.6.(b).

<u>Conjunction:</u> If $M \models \phi \wedge \psi[\mu_p(\overline{x})]$, then $M \models \phi[\mu_p(\overline{x})]$ and $M \models \psi[\mu_p(\overline{x})]$. By the induction hypothesis, there are $q \in [\![\phi(\langle \overline{x}, \underline{p} \rangle)]\!]$ and $r \in [\![\psi(\langle \overline{x}, \underline{p} \rangle)]\!]$, such that $M_q \Vdash \phi[\mu_{pq}(\overline{x})]$ and $M_r \Vdash \psi[\mu_{pr}(\overline{x})]$. Select $s \geq q$, r; then Lemma 4.2.(a) and 4.1.(2) guarantee that $M_s \Vdash \phi \wedge \psi[\mu_{ps}(\overline{x})]$ and $[\![\phi \wedge \psi(\langle \overline{x}, \underline{p} \rangle)]\!] \neq \emptyset$. The converse is immediate (in fact, simpler). The induction step for disjunction can be treated similarly.

<u>Implication:</u> Assume $M \models \phi \rightarrow \psi[\mu_p(\overline{x})]$. If it is not true that $M \models \phi[\mu_p(\overline{x})]$, then $[\![\phi(\langle \overline{x}, \underline{p} \rangle)]\!] = \emptyset$ and 4.14.(d) entails $[\![\phi \rightarrow \psi(\langle \overline{x}, \underline{p} \rangle)]\!] = [p] \neq \emptyset$. If $M \models \phi[\mu_p(\overline{x})]$, then $M \models \psi[\mu_p(\overline{x})]$. Hence, induction and 4.14.(d) yield

$$\begin{aligned}\emptyset \neq [\![\psi(\langle \overline{x}, \underline{p} \rangle)]\!] &\subseteq [p] \cap \operatorname{int}\Big([\![\phi(\langle \overline{x}, \underline{p} \rangle)]\!]^c \cup [\![\psi(\langle \overline{x}, \underline{p} \rangle)]\!]\Big) \\ &= [\![\phi \to \psi(\langle \overline{x}, \underline{p} \rangle)]\!],\end{aligned}$$

showing that (1) $\Rightarrow$ (2). For the converse, assume that

$$q \in [\![\phi \to \psi(\langle \overline{x}, \underline{p} \rangle)]\!] \text{ and that } M \models \phi[\mu_p(\overline{x})].$$

The induction hypothesis, the fact that P is rd and Lemma 4.2.(a) yield $r \geq q$ such that

$$M_r \Vdash \phi \to \psi[\mu_{pr}(\overline{x})] \quad \text{and} \quad M_r \Vdash \phi[\mu_{pr}(\overline{x})],$$

and so we obtain $M_r \Vdash \psi[\mu_{pr}(\overline{x})]$, which in turn implies $M \models \psi[\mu_p(\overline{x})]$, as needed. The negation connective is a special case of implication (take any contradiction for the consequent).

Existential quantifier: If $M \models \exists v \phi[v;\, \mu_p(\overline{x})]$, there is $\xi \in M$, such that $M \models \phi[\xi;\, \mu_p(\overline{x})]$. By Remark 2.7.(b), there is $q \in P$ and $z \in M_q$, such that $\mu_q(z) = \xi$. Since P is rd, we may assume that $q \geq p$. Therefore, $M \models \phi[\mu_q(z);\, \mu_q(\mu_{pq}(\overline{x}))]$. By the induction hypothesis, there is $r \in [\![\phi(\langle z, q \rangle; \langle \overline{x}, \underline{p} \rangle)]\!]$. Hence, $M_r \Vdash \phi[\mu_{qr}(z);\, \mu_{pr}(\overline{x})]$, and 4.1.(6) yields $M_r \Vdash \exists v \phi[v;\, \mu_{pr}(\overline{x})]$, that is, $[\![\exists v\, \phi(v; \langle \overline{x}, \underline{p} \rangle)]\!] \neq \emptyset$. The converse is immediate from 4.14.(e) and the induction hypothesis, completing the proof. □

5.2 Problem. *Suppose $\mathcal{M}$ is a Kripke structure over a (not necessarily rd) poset P and assume that $\varinjlim \mathcal{M}$ exists in **L mod**. Can Theorem 5.1 be extended to this situation ?* □

Remark 5.3 One must be careful in using Theorem 5.1, because of the mixture between intuitionistic and classical values. Consider the sentences in the language of rings with identity

$$\begin{cases} \sigma_1 & \equiv & \forall v\ (v \neq 0 \to \exists u\ (vu = 1)); \\ \sigma_2 & \equiv & \forall v\ (\neg \exists u\ (vu = 1) \to v = 0); \\ \sigma_3 & \equiv & \neg \exists v\ (v \neq 0 \wedge \neg \exists u (vu = 1)). \end{cases}$$

Classically, these are all equivalent, *but not intuitionistically.* However, we do have

(*) $$\sigma_2 \vdash_{\mathcal{H}} \sigma_3.$$

In Example 4.13 it was observed that in the Kripke structure $\mathcal{Z}$,

(1) $[\![\sigma_1]\!] = \emptyset$, while $\varinjlim \mathcal{Z} = \mathbb{Q} \models \sigma_1$, showing that the statement of Theorem 5.1 is **false** for arbitrary formulas in L.[14]

(2) Note that $[\![\neg\neg\, \sigma_1]\!] = \neg\neg\, [\![\sigma_1]\!] = \emptyset$. Hence, **the double negation of a classically valid sentence is not necessarily intuitionistically valid** as already mentioned in A.64.

(3) $[\![\sigma_2]\!] = \mathbb{N}$ and so 4.9 and (*) imply $[\![\sigma_3]\!] = \mathbb{N}$. Hence, Theorem 5.1 applies to guarantee that $\varinjlim \mathcal{Z}$ is a field, because truth satisfies the rules of classical logic.

Moral: to check if $\varinjlim \mathcal{M}$ satisfies a sentence, choose a classical equivalent for it in $L_\exists$ [15], and then check if this equivalent is forced in $\mathcal{M}$.

There is a way to include the universal quantifier in the formulas to which the statement of 5.1 applies: use the Gödel transform, discussed in appendix IX, as will be done in Theorem A.63. □

To give an answer to to Problem 2.14, we introduce

Definition 5.4 *Let* $\mathcal{M} = \langle\, M_p;\, \mu_{pq}\,\rangle$, $\mathcal{N} = \langle\, N_p;\, \nu_{pq}\,\rangle$ *be* L*-Kripke structures over* P*. A a morphism of Kripke structures* (2.1, 2.2), $\eta = (\eta_p) : \mathcal{M} \longrightarrow \mathcal{N}$, *is* **stably elementary**[16] *iff for all formulas* $\phi(v_1, \ldots, v_n)$ *in* $L_\exists$, $p \in P$ *and* $\overline{x} \in M_p^n$,

$$M_p \Vdash \phi[\overline{x}] \quad \Rightarrow \quad \exists\, q \geq p \text{ such that } \; N_q \Vdash \phi[\nu_{pq}(\eta_p(\overline{x}))].$$

Theorem 5.5 *Let* $\eta : \mathcal{M} \longrightarrow \mathcal{N}$ *be a morphism of Kripke* L*-structures over a right-directed poset* P*. The following are equivalent:*

(1) η *is stably elementary;*

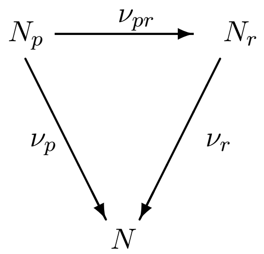
(2) $\varinjlim \eta : \varinjlim \mathcal{M} \longrightarrow \varinjlim \mathcal{N}$ *is an elementary embedding.*

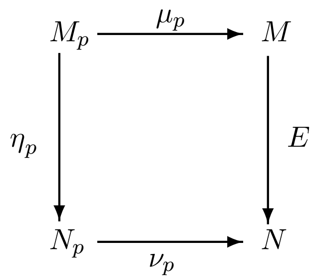
Proof. Write $M = \varinjlim \mathcal{M}$, $N = \varinjlim \mathcal{N}$ and $E = \varinjlim \eta$. Recall that the following diagrams are commutative, for $p \leq r$ in P:

[14] This reasons for this phenomenon deserve some thought.

[15] As shown by (2), it is not enough to take its double negation!

[16] The name originates in standard terminology in K-theory and Cohomology.





(1) $\Rightarrow$ (2): We show that for all formulas $\phi(v_1, \ldots, v_n)$ in $L_\exists$ and $\overline{\xi} = \langle \xi_1, \ldots, \xi_n \rangle \in M^n$,

$$M \models \phi[\overline{\xi}] \quad \Rightarrow \quad N \models \phi[E(\overline{\xi})]. \tag{$\sharp$}$$

Since in the classical Predicate Calculus every formula is equivalent to one in $L_\exists$, ($\sharp$) will be true of *all* formulas; by Remark A.60.(c), E is an elementary embedding of M into N. Assume that $M \models \phi[\overline{\xi}]$; by Remark 2.7.(b), there is $p \in P$ and $\overline{x} \in M_p^n$ such that $\overline{\xi} = \mu_p(\overline{x})$. Hence, $M \models \phi[\mu_p(\overline{x})]$. By Theorem 5.1, there is $q \geq p$ such that $M_q \Vdash \phi[\mu_{pq}(\overline{x})]$. Since η is stably elementary, there is $r \geq q$, such that $N_r \Vdash \phi[\nu_{pr}(\eta_p(\overline{x}))]$. Note that (see diagrams above)

$$\nu_r(\nu_{pr}(\eta_p(\overline{x}))) \;=\; \nu_p(\eta_p(\overline{x})) \;=\; E(\mu_p(\overline{x})) \;=\; E(\overline{\xi}). \tag{$\sharp\sharp$}$$

Therefore, another application of 5.1 yields $N \models \phi[E(\overline{\xi})]$, as desired.

(2) $\Rightarrow$ (1): Suppose that for $p \in P$ and $\overline{x} \in M_p^n$, $M_p \Vdash \phi[\overline{x}]$, that is, $[\![\phi(\langle \overline{x}, \underline{p} \rangle)]\!] \neq \emptyset$. By Theorem 5.1, $M \models \phi[\mu_p(\overline{x})]$. Since E is an elementary embedding, we get $N \models \phi[E(\mu_p(\overline{x}))]$, or equivalently, in view of ($\sharp\sharp$) above, $N \models \phi[\nu_p(\eta_p(\overline{x}))]$. Hence, there is $q \in [\![\phi(\langle \eta_p(\overline{x}), \underline{p} \rangle)]\!]$, that is, $q \geq p$ and $N_q \Vdash \phi[\nu_{pq}(\eta_p(\overline{x}))]$, and η is stably elementary, ending the proof. $\square$

5.6 Germs. Let X, Y be topological spaces and let $\mathbb{C}(X, Y)$ be the set of continuous maps from X to Y. For $p \in X$, recall (A.42) that $\nu_p = \{U \in \mathcal{O}(X) : p \in U\}$ is the filter of open neighborhoods of p in X. It was noted in A.42 that ν_p with the opposite order of inclusion is a right-directed poset. A Kripke structure (of sets) over ν_p is given by

$$\mathbb{C}_p(X, Y) = \langle\, \mathbb{C}(U, Y);\ \cdot_{|V} \,\rangle \qquad (V \subseteq U, \text{ both in } \nu_p)$$

where, for $V \subseteq U$, the **restriction map** $\cdot_{|V} : \mathbb{C}(U, Y) \longrightarrow \mathbb{C}(V, Y)$, is given by $f \mapsto f_{|V}$. If Y is a topological structure (group, ring, algebra, etc.) then $\mathbb{C}_p(X, Y)$ is a Kripke structure of the same kind. When Y is the real line or the complex numbers and X is a manifold, this applies just as well to differentiable, C^∞ or analytic maps.

The colimit of $\mathbb{C}_p(X, Y)$ is called the **stalk of $\mathbb{C}(X, Y)$ at p** or the **structure of germs** of maps from X to Y at p. It is a fundamental construction in many areas of Mathematics.

In this case, the equivalence relation that originates the colimit (see [colimit 1] in 2.6) is given by: if $f \in \mathbb{C}(U, Y)$ and $g \in \mathbb{C}(W, Y)$, where U, $W \in \nu_p$, then

$$f \; \theta \; g \quad \text{iff} \quad \exists\, V \in \nu_p,\, V \subseteq U \cap W, \text{ such that } f_{|V} = g_{|V}.$$

Thus, f and g have the same germ at p iff they coincide in an open neighborhood of p.

Theorem 2.9 guarantees that the stalk construction preserves the axioms for monoids, groups, rings and many other algebraic structures. However, to characterize the **classical first-order theory** of the stalk, one needs to use forcing, via Theorem 5.1. This might come as a surprise to a classical mathematician: that intuitionistic reasoning is helpful in understanding classical problems. □

6 Weak and ∗-forcing

Definition 6.1 *Let P be a poset and $\mathcal{M} = \langle M_p; \mu_{pq} \rangle$ be a Kripke L-structure over P. Let $\phi(v_1, \ldots, v_n)$ be a L-formula, $p \in P$ and $\overline{x} \in M_p^n$. Define relations,*

$$\boldsymbol{M_p} \textbf{ ∗-forces } \boldsymbol{\phi} \textbf{ at } \boldsymbol{\overline{x}} \quad \textit{by} \quad M_p \Vdash_* \phi[\overline{x}] \quad \textit{iff} \quad M_p \Vdash \phi^G[\overline{x}];$$

$$\boldsymbol{M_p} \textbf{ w-forces } \boldsymbol{\phi} \textbf{ at } \boldsymbol{\overline{x}} \quad \textit{by} \quad M_p \Vdash_w \phi[\overline{x}] \quad \textit{iff} \quad M_p \Vdash \neg\neg\, \phi[\overline{x}],$$

called ∗ and weak forcing, respectively.

By 4.4 and A.64, ∗-forcing and weak forcing are distinct notions.

Lemma 6.2 *Let $\mathcal{M} = \langle M_p; \mu_{pq} \rangle$ be a Kripke structure over the poset P. Let $\phi(v_1, \ldots, v_n)$ be a formula in L.*

a) If ϕ is in $L_\exists$, then for all $p \in P$ and $\overline{x} \in M_p^n$, $M_p \Vdash_* \phi[\overline{x}]$ iff $M_p \Vdash_w \phi[\overline{x}]$.

b) For $p \in P$ and $\overline{x} \in M_p^n$, the following are equivalent:

(1) $M_p \Vdash_w \phi[\overline{x}]$;

(2) *For all* $q \geq p$, $\exists\ r \geq q$ *such that* $M_r \Vdash \phi[\mu_{pr}(\overline{x})]$;

(3) $M_p \Vdash \neg\neg\,\phi[\overline{x}]$;

(4) $p \in [\![\neg\neg\,\phi(\langle\,\overline{x}, \underline{p}\,\rangle)]\!]$.

c) If P is right-directed, the following are equivalent:

(1) $M_p \Vdash_w \exists v\,\phi[v;\overline{x}]$

(2) $\exists\ s \geq p$ *and* $z \in M_s$, *such that* $M_s \Vdash \phi[z;\mu_{pq}(\overline{x})]$.

Proof. Item (a) follows from A.65. For (b), the equivalence of (1) $-$ (3) is clear. As for (3) $\Leftrightarrow$ (4), we apply 1.2.(f), the description of double negation in a topology (A.33.(e)) and 4.14.(c) to conclude that

$$p \in [p] \cap \neg\neg\,[\![\phi(\langle\,\overline{x}, \underline{p}\,\rangle)]\!] = E\langle\,\overline{x}, \underline{p}\,\rangle \cap \neg\neg\,[\![\phi(\langle\,\overline{x}, \underline{p}\,\rangle)]\!] = [\![\neg\neg\,\phi(\langle\,\overline{x}, \underline{p}\,\rangle)]\!],$$

as needed. For (c), in view of (b), it is enough to verify that (2) $\Rightarrow$ (1). Fix $q \geq p$; since P is right-directed, there is $r \geq q, s$. Hence, 4.2.(a) yields $M_r \Vdash \phi[\mu_{sr}(z);\, \mu_{qr}(\mu_{pq}(\overline{x}))]$. Because $\mu_{qr}(\mu_{pq}(\overline{x})) = \mu_{pr}(\overline{x})$, we get $M_r \Vdash \phi[\mu_{sr}(z);\ \mu_{pr}(\overline{x})]$. Thus, for all $q \geq p$, there is $r \geq q$ and $y \in M_r$ such that $M_r \Vdash \phi[y;\ \mu_{pr}(\overline{x})]$, and so the equivalence in (b) yields $M_p \Vdash_w \exists v \phi[v;\, \overline{x}]$, as needed. □

Lemma 6.3 *Let $\mathcal{M} = \langle M_p; \mu_{pq} \rangle$ be a Kripke structure over P. Let $\phi(v_1, \ldots, v_n)$ be a formula in L, $p \in P$, $\overline{x} \in M_p^n$ and $\langle\,\overline{x}, \overline{p}\,\rangle \in M_*^n$. Then,*

a) $M_p \Vdash_* \phi[\overline{x}] \;\Rightarrow\; \forall\ q \geq p,\ \ M_q \Vdash_* \phi[\mu_{pq}(\overline{x})]$.

b) $[\![\phi^G(\langle\,\overline{x}, \overline{p}\,\rangle)]\!]$ *is a regular open in* $E\langle\,\overline{x}, \overline{p}\,\rangle = \bigcap_{i=1}^n [p_n]$.[17]

[17]That is $[\![\phi^G(\langle\,\overline{x}, \overline{p}\,\rangle)]\!] \in Reg(E\langle\,\overline{x}, \overline{p}\,\rangle)$ in the induced topology from P.

Proof. Item (a) follows from the definition of $\Vdash_*$ and 4.2.(a). For (b), since $\vdash_{\mathcal{H}} \phi^G \leftrightarrow \neg\neg\,\phi^G$ (A.62.(b)), Corollary 4.9 entails $[\![\phi^G(\langle\,\overline{x},\overline{p}\,\rangle)]\!]$ $=$ $[\![\neg\neg\,\phi^G(\langle\,\overline{x},\overline{p}\,\rangle)]\!]$ $=$ $E\langle\,\overline{x},\overline{p}\,\rangle \cap \neg\neg\,[\![\phi^G(\langle\,\overline{x},\overline{p}\,\rangle)]\!]$, and $[\![\phi^G(\langle\,\overline{x},\overline{p}\,\rangle)]\!]$ is a regular open in the topology induced by $\mathfrak{U}(P)$ on $E\langle\,\overline{x},\overline{p}\,\rangle$. ☐

With the notion of $*$-forcing we can restate Theorem 5.1 as

Theorem 6.4 *Let $\mathcal{M} = \langle\, M_p;\ \mu_{pq}\,\rangle$ be a Kripke structure over a rd poset P and $M = \varinjlim \mathcal{M}$. If $\phi(v_1, \ldots, v_n)$ is a formula in L, $p \in P$ and $\overline{x} \in M_p^n$, then* $M \models \phi[\mu_p(\overline{x})] \quad \Leftrightarrow \quad M_p \Vdash_* \phi[\overline{x}]$.

Proof. $\underline{(\Rightarrow)}$: By A.60.(c), we may assume that $\phi \in L_\exists$. Since $\vdash_C \phi \leftrightarrow \phi^G$ (A.62.(a)), we get that $M \models \phi^G[\mu_p(\overline{x})]$. By Theorem 5.1, $[\![\phi^G(\langle\,\overline{x},\underline{p}\,\rangle)]\!]$ is cofinal in P. An application of 1.2.(f) yields $\neg\neg\,[\![\phi^G(\langle\,\overline{x},\underline{p}\,\rangle)]\!] = P$, and so

$$[\![\neg\neg\,\phi^G(\langle\,\overline{x},\underline{p}\,\rangle)]\!] \;=\; [p] \cap \neg\neg\,[\![\phi^G(\langle\,\overline{x},\underline{p}\,\rangle)]\!] \;=\; [p].$$

Thus, $M_p \Vdash \neg\neg\,\phi^G[\overline{x}]$; by A.62.(b) and 4.4 we obtain $M_p \Vdash \phi^G[\overline{x}]$, that is, $M_p \Vdash_* \phi[\overline{x}]$, as desired.

$\underline{(\Leftarrow)}$: By induction on complexity. If ϕ is atomic, then $M_p \Vdash \phi^G[\overline{x}]$ means that $M_p \Vdash \neg\neg\,\phi[\overline{x}]$ and Theorem 5.1 yields $M \models \phi[\mu_p\,(\overline{x})]$.

$*$ $M_p \Vdash (\phi \wedge \psi)^G[\overline{x}]$ amounts to $M_p \Vdash \phi^G \wedge \psi^G[\overline{x}]$, i.e., $M_p \Vdash \phi^G[\overline{x}]$ and $M_p \Vdash \psi^G[\overline{x}]$. Thus, induction gives $M \models \phi \wedge \psi[\mu_p(\overline{x})]$.

$*$ $M_p \Vdash (\phi \vee \psi)^G[\overline{x}]$ is equivalent to $M_p \Vdash \neg\neg\,(\phi^G \vee \psi^G)[\overline{x}]$. By 4.2.(c), there is $q \geq p$ such that $M_q \Vdash \phi^G \vee \psi^G[\mu_{pq}(\overline{x})]$, i.e., $M_q \Vdash \phi^G[\mu_{pq}(\overline{x})]$ or $M_q \Vdash \psi^G[\mu_{pq}(\overline{x})]$. Since,

$$\mu_p(\overline{x}) = \mu_q(\mu_{pq}(\overline{x})), \tag{$\sharp$}$$

the induction hypothesis yields $M \models \phi \vee \psi[\mu_p(\overline{x})]$.

$*$ Suppose that $M_p \Vdash (\phi^G \to \psi^G)[\overline{x}]$ and $M \models \phi[\mu_p(\overline{x})]$. By the first part of the proof, $M_p \Vdash \phi^G[\overline{x}]$, and so $M_p \Vdash \psi^G[\overline{x}]$. Consequently, $M \models (\phi\!\to \psi)[\mu_p(\overline{x})]$.

The induction step through the existential quantifier is similar to that of disjunction.

$*$ Suppose $M_p \Vdash \forall v\ \phi^G[v;\,\overline{x}]$ and let $\xi \in M$. By Remark 2.7.(b), there is $q \geq p$ and $z \in M_q$ such that $\mu_q(z) = \xi$. The definition of forcing entails that $M_q \Vdash \phi^G[z;\ \mu_{pq}(\overline{x})]$. Recalling (♯) above, induction yields $M \models \phi[\xi;\ \mu_p(\overline{x})]$; ξ being arbitrary in M, we get $M \models \forall v\ \phi[v;\ \mu_p(\overline{x})]$, as needed. □

7 A Łós Theorem for Kripke Structures

Having achieved a description of truth in colimits, a first example of generalized ultraproduct, the natural question is: is there a natural extension to arbitrary ultrafilters and Kripke structures ? In this section we provide an affirmative answer to this question. In fact, Theorems 5.1 and 6.4 are consequences of the results proven herein, but the authors thought it more profitable − against practice now current in Mathematics −, to present the special case before the general one.

In this section, fix a Kripke structure $\mathcal{M} = \langle\, M_p;\ \mu_{pq} \,\rangle$ over a poset P and write $\mathfrak{U}$ for $\mathfrak{U}(P)$. Let $\mathfrak{g}\mathcal{M} = \langle\, \mathfrak{g}\mathcal{M}(U);\ \rho_{UV} \,\rangle$ be the completion of $\mathcal{M}$ over $\mathfrak{U}^{op}$ (3.1).

To simplify notation, if $V \subseteq U$ in $\mathfrak{U}$, write the restriction map ρ_{UV} as $(\cdot)_{|V}$. Hence, for $t \in \mathfrak{g}\mathcal{M}(U)$,

$$t_{|V} \ = \ \rho_{UV}(t).$$

One should keep in mind that for $U \in \mathfrak{U}$,

$$\mathfrak{g}\mathcal{M}(U) = \{t \in \textstyle\prod_{p \in U} M_p : \forall\ r \geq q \text{ in } U,\ \ \mu_{qr}(t(q)) = t(r)\},$$

with the L-structure induced by the product. We shall henceforth treat elements of $\mathfrak{g}\mathcal{M}(U)$ as maps with domain U. If $V \subseteq U$, the restriction $(\cdot)_{|V}$ is exactly the usual restriction of maps. Moreover, the identification of M_p and $\mathfrak{g}\mathcal{M}([p])$ guarantees that if $[p] \subseteq U$, then the restriction map $(\cdot)_{|[p]}$ is *calculation at the point* p, that is, $t_{|[p]} = t(p)$.

In the results that follow it is important to avoid overload in notation. Moreover, to handle finite sequences in $\mathfrak{g}\mathcal{M}$, we introduce, yet again, notation generalizing that in 4.5.

7.1 Notation. (1) Define the **domain of $\mathfrak{g}\mathcal{M}$** by

$$|\mathfrak{g}\mathcal{M}| \ = \ \textstyle\coprod_{U \in \mathfrak{U}} \mathfrak{g}\mathcal{M}(U) \ = \ \bigcup_{U \in \mathfrak{U}} \mathfrak{g}\mathcal{M}(U) \times \{U\}.$$

Note that M_* (as in 4.5.(1)) is a subset of $|\mathfrak{g}\mathcal{M}|$.

(2) Define a map, $E : |\mathfrak{g}\mathcal{M}| \longrightarrow \mathfrak{U}$, called **extent**, by $E\langle t, U \rangle = U$. Although the elements of $|\mathfrak{g}\mathcal{M}|$ are pairs, $\langle t, U \rangle$, with $t \in \mathfrak{g}\mathcal{M}(U)$, we abuse notation and write $t \in |\mathfrak{g}\mathcal{M}|$, meaning of course $\langle t, Et \rangle$. This applies to sequences as well, that is, $\bar{t} \in |\mathfrak{g}\mathcal{M}|^n$ stands for

$$\bar{t} \;=\; \langle \langle t_1, Et_1 \rangle, \ldots, \langle t_n, Et_n \rangle \rangle.$$

We may extend the map E to $\bar{t} \in |\mathfrak{g}\mathcal{M}|^n$ by posing $E\bar{t} = \bigcap_{i=1}^n Et_i$. In particular, if $\bar{t} \in \mathfrak{g}\mathcal{M}([p])^n$, the notation of 4.5.(2) is replaced by our new conventions, recalling that p may be identified with $[p]$ (1.18) and M_p with $\mathfrak{g}\mathcal{M}([p])$ (3.2.(a)).

(4) Restriction is extended and simplified, as follows: for $\bar{t} \in |\mathfrak{g}\mathcal{M}|^n$ and $V \in \mathfrak{U}$, define

$$\bar{t}_{|V} \;=\; \bar{t}_{|V \cap E\bar{t}} \;=\; \langle \langle t_{1_{|V \cap E\bar{t}}}, V \cap E\bar{t} \rangle, \ldots, \langle t_{n_{|V \cap E\bar{t}}}, V \cap E\bar{t} \rangle \rangle.$$

(5) If $\bar{t} \in |\mathfrak{g}\mathcal{M}|^n$ and $p \in E\bar{t}$, then $\bar{t}(p) = \langle t_1(p), \ldots, t_n(p) \rangle \in M_p^n$. As observed earlier, this is a special case of the restriction notation introduced in (4). □

With an eye on the Feferman-Vaught value of a formula, defined in A.66, we state

Definition 7.2 *In the setting established above, let $\phi(v_1, \ldots, v_n)$ be a formula in L. Define a map*

$$[\![\phi(\cdot)]\!]_{\mathfrak{g}} : |\mathfrak{g}\mathcal{M}|^n \longrightarrow 2^P, \text{ given by } [\![\phi(\bar{t})]\!]_{\mathfrak{g}} \;=\; \{p \in E\bar{t} : M_p \Vdash \phi[\bar{t}(p)]\}.$$

The next result is clearly related to 4.7:

Lemma 7.3 *If $\phi(v_1, \ldots, v_n)$ is a formula in L, $\bar{t} \in |\mathfrak{g}\mathcal{M}|^n$ and $\langle \overline{x}, \overline{p} \rangle \in M_*^n$, then*

a) $[\![\phi(\bar{t})]\!]_{\mathfrak{g}}$ is open in the $\mathfrak{U}$-topology.

b) $[\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!]_{\mathfrak{g}} \;=\; \{q \geq \overline{p} : M_q \Vdash \phi[\mu_{pq}(\overline{x})]\} \;=\; [\![\phi(\langle \overline{x}, \overline{p} \rangle)]\!]$.

c) For all $p \in P$ and $\overline{x} \in M_p^n$, $M_p \Vdash \phi[\overline{x}]$ iff $p \in [\![\phi(\langle \overline{x}, \underline{p} \rangle)]\!]_{\mathfrak{g}}$.

Proof. a) If $p \in [\![\phi(\bar{t})]\!]_{\mathfrak{g}}$ and $q \geq p$, then 4.2.(a) together with the fact that $\mu_{pq}(\bar{t}(p)) = \bar{t}(q)$, entails $M_q \Vdash \phi[\bar{t}(q)]$. Thus, $[p] \subseteq [\![\phi(\bar{t})]\!]_{\mathfrak{g}}$, as needed; (b) and (c) are straightforward. □

Note that 4.8, 4.9, 4.10, 4.11 and 4.12 all apply to the present situation. The next step is the analogue of 4.14.

Theorem 7.4 *If $\phi(v_1, \ldots, v_n)$ is a formula in L and $\bar{t} \in |\mathfrak{g}\mathcal{M}|^n$, then*
a) If ϕ is atomic, $[\![\phi(\bar{t})]\!]_{\mathfrak{g}} = \{q \in E\bar{t} : M_q \models \phi[\bar{t}(q)]\}$.
b) If $\phi \equiv \psi_1 \diamond \psi_2$, then $[\![\phi(\bar{t})]\!]_{\mathfrak{g}} = [\![\psi_1(\bar{t})]\!]_{\mathfrak{g}} \diamond [\![\psi_2(\bar{t})]\!]_{\mathfrak{g}}$, *where $\diamond \in \{\wedge, \vee\}$ and the $\diamond$ in the right-hand side of the equation refer to the corresponding operations in the frame $\mathfrak{U}$.*[18]
c) $[\![\neg\, \phi(\bar{t})]\!]_{\mathfrak{g}} = E\bar{t} \cap \neg\, [\![\phi(\bar{t})]\!]_{\mathfrak{g}}$.
d) $[\![\psi_1 \rightarrow \psi_2(\bar{t})]\!]_{\mathfrak{g}} = E\bar{t} \cap \left([\![\psi_1(\bar{t})]\!]_{\mathfrak{g}} \rightarrow [\![\psi_2(\bar{t})]\!]_{\mathfrak{g}}\right)$.
e) $[\![\exists x\, \psi(x; \bar{v})(\bar{t})]\!]_{\mathfrak{g}} = \bigcup_{\xi \in |\mathfrak{g}\mathcal{M}|} [\![\psi(\xi; \bar{t})]\!]_{\mathfrak{g}}$.
f) $[\![\forall x\, \psi(x; \bar{v})(\bar{t})]\!]_{\mathfrak{g}} = E\bar{t} \cap \left(\bigwedge_{\xi \in |\mathfrak{g}\mathcal{M}|} E\xi \rightarrow [\![\psi(\xi; \bar{t})]\!]_{\mathfrak{g}}\right)$.

Proof. It is so *similar* to that of Theorem 4.14 that it can be safely left to the reader. □

Here is an agreeable property of the values we have defined:

Lemma 7.5 *Let $\phi(v_1, \ldots, v_n)$ be a formula in L and $\mathcal{M}$ a Kripke structure over a poset P. If $\bar{t} \in |\mathfrak{g}\mathcal{M}|^n$ and $V \in \mathfrak{U}$, then* $[\![\phi(\bar{t}_{|V})]\!]_{\mathfrak{g}} = V \cap [\![\phi(\bar{t})]\!]_{\mathfrak{g}}$.

Proof. Straightforward from the definitions. □

Let $\mathcal{F}$ be an **ultrafilter** in $\mathfrak{U}$ and $\mathcal{M}_{\mathcal{F}} = \varinjlim \mathfrak{g}\mathcal{M}_{|\mathcal{F}}$ be the stalk of $\mathcal{M}$ at $\mathcal{F}$, as in 3.7. For $U \in \mathcal{F}$, let $\rho_U : \mathfrak{g}\mathcal{M}(U) \longrightarrow \mathcal{M}_{\mathcal{F}}$ be the L-morphism that comes with the colimit construction. For $t \in \mathfrak{g}\mathcal{M}(U)$, write

$$t_{\mathcal{F}} = \rho_U(t),$$

for the class of t in $\mathcal{M}_{\mathcal{F}}$. This applies also to sequences, that is, if $\bar{t} \in |\mathfrak{g}\mathcal{M}|^n$ and $E\bar{t} \in \mathcal{F}$,

$$\bar{t}_{\mathcal{F}} = \langle\, t_{1\mathcal{F}}, \ldots, t_{n\mathcal{F}}\, \rangle \in \mathcal{M}^n_{\mathcal{F}}.$$

Before the generalization of Theorem A.73 to Kripke structures, we need a result from Topology.

[18] The same convention holds in items (c), (d) and (f).

Proposition 7.6 *Let $\langle X, \mathcal{O} \rangle$ be a topological space and $U \in \mathcal{O}$. If U_i, $i \in I$, is a covering of U, then there is a collection V_i, $i \in I$, of opens in X such that*

(1) $\forall\ i \in I,\ \ V_i \subseteq U_i$; *(2) $\forall\ i \neq j$ in I, $\ V_i \cap V_j = \emptyset$;*

(3) $\bigcup_{i \in I} V_i$ is dense in U.

Proof. There are several proofs, all dependent on the Axiom of Choice. We give one which is mildly "constructive". We assume that I is a cardinal λ (as in appendix III) and use induction on $\alpha \in \lambda$. Set $V_0 = U_0$; if the sequence has been constructed for all $\beta \in \alpha$, define $V_\alpha = U_\alpha \ \cap \ \neg \left(\bigcup_{\beta \in \alpha} V_\beta\right)$. Clearly, the V_α satisfy (1) and (2). To see that $V = \bigcup_{\alpha \in \lambda} V_\alpha$ is dense in U, it is enough to check that V is dense in U_α, for all $\alpha \in \lambda$. Note that

$$\begin{aligned} V \ &\supseteq \ V_\alpha \cup \bigcup_{\beta \in \alpha} V_\beta \ = \ \left(U_\alpha \cap \neg \left(\bigcup_{\beta \in \alpha} V_\beta\right)\right) \ \cup \ \bigcup_{\beta \in \alpha} V_\beta \\ &\supseteq \ U_\alpha \cap \left((\bigcup_{\beta \in \alpha} V_\beta) \ \cup \ \neg (\bigcup_{\beta \in \alpha} V_\beta)\right). \end{aligned}$$

By A.36.(i), $(\bigcup_{\beta \in \alpha} V_\beta) \ \cup \ \neg (\bigcup_{\beta \in \alpha} V_\beta)$ is dense in X. Hence, V is dense in U_α, as needed. □

We have all the ingredients for the generalization of Theorems A.73 and 5.1, namely

Theorem 7.7 *Let $\mathcal{M}$ be a Kripke structure over P and $\mathcal{F}$ an ultrafilter in $\mathfrak{U} = \mathfrak{U}(P)$. If $\phi(v_1, \ldots, v_n)$ is a formula in $L_\exists$ and $\bar{t} \in |\mathfrak{g}\mathcal{M}|^n$ is such that $E\bar{t} \in \mathcal{F}$, then $\mathcal{M}_\mathcal{F} \models \phi[\bar{t}_\mathcal{F}]$ iff $[\![\phi(\bar{t})]\!]_\mathfrak{g} \in \mathcal{F}$.*

Proof. By induction on complexity. The relations in 7.4 will be of current use. Fix $\bar{t} \in |\mathfrak{g}\mathcal{M}|^n$.

If ϕ is atomic and $\mathcal{M}_\mathcal{F} \models \phi[\bar{t}_\mathcal{F}]$, [colimit 2] in Theorem 2.6.(b) yields $V \in \mathcal{F}$ and $\bar{s} \in \mathfrak{g}\mathcal{M}(V)^n$ such that $\bar{s}_\mathcal{F} = \bar{t}_\mathcal{F}$ and $\mathfrak{g}\mathcal{M}(V) \models \phi[\bar{s}]$. Since $\mathcal{F}$ is closed under finite meets, we may assume that $V \subseteq E\bar{t}$. Because the L-structure in $\mathfrak{g}\mathcal{M}(V)$ is that induced by the product $\prod_{r \in V} M_r$, we obtain $M_r \models \phi[\bar{s}(r)]$, for all $r \in V$. But then $V \subseteq [\![\phi(\bar{t})]\!]_\mathfrak{g}$ and so $[\![\phi(\bar{t})]\!]_\mathfrak{g} \in \mathcal{F}$.

If $W = [\![\phi(\bar{t})]\!]_\mathfrak{g} \in \mathcal{F}$, then $\mathfrak{g}\mathcal{M}(W) \models \phi[\bar{t}_{|W}]$, because for all $q \in W$, $M_q \models \phi[\bar{t}(q)]$. Since the map $\rho_W : \mathfrak{g}\mathcal{M}(W) \longrightarrow \mathcal{M}_\mathcal{F}$ is a L-morphism, it follows that $\mathcal{M}_\mathcal{F} \models \phi[\bar{t}_\mathcal{F}]$.

For conjunction we have, recalling that $\mathcal{F}$ is closed under meets,

$$\begin{aligned}\mathcal{M}_{\mathcal{F}} \models (\phi \wedge \psi)[\bar{t}_{\mathcal{F}}] \quad &\text{iff} \quad \mathcal{M}_{\mathcal{F}} \models \phi[\bar{t}_{\mathcal{F}}] \quad \text{and} \quad \mathcal{M}_{\mathcal{F}} \models \psi[\bar{t}_{\mathcal{F}}] \\ &\text{iff} \quad [\![\phi(\bar{t})]\!]_{\mathfrak{g}} \in \mathcal{F} \quad \text{and} \quad [\![\psi(\bar{t})]\!]_{\mathfrak{g}} \in \mathcal{F} \\ &\text{iff} \quad [\![\phi \wedge \psi(\bar{t})]\!]_{\mathfrak{g}} = [\![\phi(\bar{t})]\!]_{\mathfrak{g}} \cap [\![\phi(\bar{t})]\!]_{\mathfrak{g}} \in \mathcal{F},\end{aligned}$$

as necessary. For disjunction, recall that an ultrafilter is prime (A.43.(e).(1)). Then,

$$\begin{aligned}\mathcal{M}_{\mathcal{F}} \models (\phi \vee \psi)[\bar{t}_{\mathcal{F}}] \quad &\text{iff} \quad \mathcal{M}_{\mathcal{F}} \models \phi[\bar{t}_{\mathcal{F}}] \quad \text{or} \quad \mathcal{M}_{\mathcal{F}} \models \psi[\bar{t}_{\mathcal{F}}] \\ &\text{iff} \quad [\![\phi(\bar{t})]\!]_{\mathfrak{g}} \in \mathcal{F} \quad \text{or} \quad [\![\psi(\bar{t})]\!]_{\mathfrak{g}} \in \mathcal{F} \\ &\text{iff} \quad [\![\phi \vee \psi(\bar{t})]\!]_{\mathfrak{g}} = [\![\phi(\bar{t})]\!]_{\mathfrak{g}} \cup [\![\phi(\bar{t})]\!]_{\mathfrak{g}} \in \mathcal{F},\end{aligned}$$

as desired. For negation, we obtain, because of A.43.(e).(3),

$$\begin{aligned}\mathcal{M}_{\mathcal{F}} \models \neg\,\phi[\bar{t}_{\mathcal{F}}] \quad &\text{iff it is false that } \mathcal{M}_{\mathcal{F}} \models \phi[\bar{t}_{\mathcal{F}}] \quad \text{iff} \quad [\![\phi(\bar{t})]\!]_{\mathfrak{g}} \notin \mathcal{F} \\ &\text{iff} \quad [\![\neg\,\phi(\bar{t})]\!]_{\mathfrak{g}} = \neg\,[\![\phi(\bar{t})]\!]_{\mathfrak{g}} \in \mathcal{F},\end{aligned}$$

establishing the induction step through negation. We leave the argument for implication to the reader and deal with existential quantification.

If $\mathcal{M}_{\mathcal{F}} \models \exists v\ \phi[v;\ \bar{t}_{\mathcal{F}}]$, there is $\xi \in \mathcal{M}_{\mathcal{F}}$ such that $\mathcal{M}_{\mathcal{F}} \models \phi[\xi;\ \bar{t}_{\mathcal{F}}]$. Select $V \subseteq E\bar{t}$ and $s \in \mathfrak{g}\mathcal{M}(V)$ such that $s_{\mathcal{F}} = \xi$. Then, $(\bar{t}_{|V})_{\mathcal{F}} = \bar{t}_{\mathcal{F}}$, and so $\mathcal{M}_{\mathcal{F}} \models \phi[\bar{z}_{\mathcal{F}}]$, where $\bar{z} = \langle\,\langle\, s, Es\,\rangle; \bar{t}\,\rangle \in |\mathfrak{g}\mathcal{M}|^{n+1}$. Induction and 7.4.(e) yield $[\![\phi(s;\bar{t})]\!]_{\mathfrak{g}} \in \mathcal{F}$ and $[\![\phi(s;\bar{t})]\!]_{\mathfrak{g}} \subseteq [\![\exists v\phi(v;\bar{t})]\!]_{\mathfrak{g}}$, whence, $[\![\exists v\phi(v;\bar{t})]\!]_{\mathfrak{g}} \in \mathcal{F}$, as needed. Conversely, suppose that $[\![\exists v\phi(v;\bar{t})]\!]_{\mathfrak{g}} \in \mathcal{F}$. For $\xi \in |\mathfrak{g}\mathcal{M}|$, let $U_\xi = [\![\phi(\xi;\ \bar{t})]\!]_{\mathfrak{g}}$. By 7.6, there is $V_\xi \subseteq U_\xi$, pairwise disjoint, and whose union is dense in V. Since the V_ξ are disjoint, the family

$$S = \{\xi_{|V_\xi} : \xi \in |\mathfrak{g}\mathcal{M}|\}$$

is compatible in $|\mathfrak{g}\mathcal{M}|$. Let $V = \bigcup_{\xi \in |\mathfrak{g}\mathcal{M}|} V_\xi$. Since $\mathfrak{g}\mathcal{M}$ satisfies [*comp*] in Theorem 3.2, there is a *unique* $z \in \mathfrak{g}\mathcal{M}(V)$, such that

$$\text{For all } \xi \in |\mathfrak{g}\mathcal{M}|, \quad z_{|V_\xi} = \xi_{|V_\xi}.$$

Next, we verify that $V = [\![\phi(z;\bar{t})]\!]_{\mathfrak{g}}$. By definition, $[\![\phi(z;\bar{t})]\!]_{\mathfrak{g}} \subseteq E\langle\, z, \bar{t}\,\rangle = V$, and so, it suffices to show that $V \subseteq [\![\phi(z;\bar{t})]\!]_{\mathfrak{g}}$, or equivalently, that for every $\xi \in |\mathfrak{g}\mathcal{M}|$, $V_\xi \subseteq [\![\phi(z;\bar{t})]\!]_{\mathfrak{g}}$. If $\xi \in |\mathfrak{g}\mathcal{M}|$, Lemma 7.5 yields

$$V_\xi \;\cap\; [\![\phi(z;\bar{t})]\!]_{\mathfrak{g}} \;=\; [\![\phi(z_{|V_\xi};\, \bar{t}_{|V\xi})]\!]_{\mathfrak{g}} \;=\; [\![\phi(\xi_{|V_\xi};\, \bar{t}_{|V_\xi})]\!]_{\mathfrak{g}}$$
$$= \; V_\xi \cap [\![\phi(\xi;\bar{t})]\!]_{\mathfrak{g}} \;=\; V_\xi \cap U_\xi \;=\; V_\xi,$$

and $V_\xi \;\subseteq\; [\![\phi(z;\bar{t})]\!]_{\mathfrak{g}}$, as needed. We now state

Fact 7.8 *If $\langle\, X, \mathcal{O}\,\rangle$ is a topological space, $\mathcal{F}$ is an ultrafilter in $\mathcal{O}$ and $U,\, V \in \mathcal{O}$, then*

$$V \text{ dense in } U \text{ and } U \in \mathcal{F} \quad\Rightarrow\quad V \in \mathcal{F}.$$

Proof. If V is dense in U, then $U \subseteq \neg\neg\, V$. Hence, $V \;\subseteq\; U \;\subseteq\; \neg\neg\, V$, items (e) and (f) in A.33, together with A.36.(a), yield $\neg\neg\, V = \neg\neg\, U$. Another application of A.36.(a) and we conclude that $\neg\, V = \neg\, U$, an impossibility because $\mathcal{F}$ is a proper filter. □

Since $\bigcup_{\xi\in|\mathfrak{g}\mathcal{M}|} U_\xi \;=\; [\![\exists v\phi(v;\bar{t})]\!]_{\mathfrak{g}} \;\in\; \mathcal{F}$, and V is dense in $[\![\exists v\phi(v;\bar{t})]\!]_{\mathfrak{g}}$, it follows from Fact 7.8 that $V = [\![\phi(z;\bar{t})]\!]_{\mathfrak{g}} \in \mathcal{F}$. Now, induction entails $\mathcal{M}_{\mathcal{F}} \models \phi[z_{\mathcal{F}};\, \bar{t}_{\mathcal{F}}]$, that is, $\mathcal{M}_{\mathcal{F}} \models \exists v\; \phi[v;\, \bar{t}_{\mathcal{F}}]$, ending the proof. □

Remark 7.9 The importance of "gluing compatibles" in the proof of 7.7 is clear. Without this, one would not be able to go through the induction step involving the existential quantifier. □

The Gödel transform can be used to give a version of 7.7 holding for **all** formulas in L. We state the pertinent results, omitting proofs, analogous to those presented above.

Lemma 7.10 *If $\phi(v_1,\ldots,v_n)$ is a formula in L and $\bar{t} \in |\mathfrak{g}\mathcal{M}|^n$, then $[\![\phi^G(\bar{t})]\!]_{\mathfrak{g}}$ is a regular open in $E\bar{t}$.*

Theorem 7.11 *Let $\mathcal{M}$ be a Kripke structure over P and $\mathcal{F}$ be an ultrafilter in $\mathfrak{U}(P)$. If $\phi(v_1,\ldots,v_n)$ is a formula in L and $\bar{t} \in |\mathfrak{g}\mathcal{M}|^n$ is such that $E\bar{t} \in \mathcal{F}$, then $\mathcal{M}_{\mathcal{F}} \models \phi[\bar{t}_{\mathcal{F}}]$ iff $[\![\phi^G(\bar{t})]\!]_{\mathfrak{g}} \in \mathcal{F}$.*

As always, one problem solved, another posed.

7.12 Problem. *Let $\mathcal{M}$ be a Kripke structure over a poset P and let $\mathcal{F}$ be a* filter *on $\mathfrak{U}(P)$. Is there a generalization of the Feferman-Vaught Theorem in* [FV] *to the intuitionistic situation ?* □

In the opinion of the authors, a nice solution to Problem 7.12 would constitute a very interesting contribution to Model Theory in general.

8 Free and Convergent Ultrafilters

In this section we discuss a basic topological classification of ultrafilters. It should be emphasized at the outset that, contrary to the standard (Bourbaki) practice, our ultrafilters are **in the topology**, **and not** in the underlying Boolean algebra of parts. Hence, all ultrafilters herein consist of *opens* in a certain space. It will be established that there is a close connection between ultrafilters in a topology and its irreducible closed sets.

Recall that if x is a point in a space X, ν_x is the filter of open neighborhoods of x, that is, $\nu_x = \{U \in \mathcal{O} : x \in U\}$.

Definition 8.1 *Let* $\langle X, \mathcal{O} \rangle$ *be a topological space,* $\mathcal{F}$ *be a proper filter in* $\mathcal{O}$ *and* x *a point in* X*.*

a) $\boldsymbol{x}$ **is isolated** *in* X *iff* $\{x\}$ *is open in* X*.*

b) $\boldsymbol{\mathcal{F}}$ **is principal** *iff there is* $x \in X$ *such that* $\mathcal{F} = \nu_x$*.*

c) Define $\mathbf{lim}\ \boldsymbol{\mathcal{F}} = \bigcap_{U \in \mathcal{F}} \overline{U}$*.*

d) $\boldsymbol{\mathcal{F}}$ **is convergent** *iff* $\lim \mathcal{F} \neq \emptyset$*. Otherwise,* $\mathcal{F}$ *is said to be* **free***.*

e) The expression $\mathcal{F} \longrightarrow K$ *is synonymous with* $\lim \mathcal{F} = K$*.*

A principal filter ν_x is convergent, for $x \in \bigcap_{U \in \nu_x} \overline{U}$. In general, $\lim \nu_x$ is much larger than $\overline{\{x\}}$[19].

Example 8.2 If P is a poset and $x \in P$, then

x is isolated in the $\mathfrak{U}$-topology iff $[x] = \{x\}$;

such points are called *maximal* or *isolated* in P. □

Proposition 8.3 *Let* $\langle X, \mathcal{O} \rangle$ *be a topological space and let* $\mathcal{F}$ *be an* ultrafilter *in* $\mathcal{O}$*.*

[19]For instance, in any linear order, with the $\mathfrak{U}$-topology.

a) For all $W \in \mathcal{O}$, $W \cap \lim \mathcal{F} \neq \emptyset \quad \Rightarrow \quad W \in \mathcal{F}$.

b) $\lim \mathcal{F} \; = \; \{x \in X : \nu_x \subseteq \mathcal{F}\}$.

c) $\mathcal{F}$ *is convergent iff there is* $x \in X$ *such that* $\nu_x \subseteq \mathcal{F}$.

d) $\lim \mathcal{F}$ *is an irreducible closed set in* X.

e) If $K \neq \emptyset$ *is an irreducible closed set in* X*, then there is an ultrafilter* $\mathcal{G}$ *in* $\mathcal{O}$ *such that* $K \subseteq \lim \mathcal{G}$.

f) Every non-empty irreducible component of X *is the limit of an ultrafilter in* $\mathcal{O}$.

Proof. a) Write $K = \lim \mathcal{F}$; if $W \cap K \neq \emptyset$, then $\mathcal{F} \cup \{W\}$ has the fip (A.43.(d)). Indeed, given $x \in W \cap K$, A.18.(5) entails that $W \cap U \neq \emptyset$, for all $U \in \mathcal{F}$. Since $\mathcal{F} \cup \{W\}$ generates a proper filter in $\mathcal{O}$, the maximality of $\mathcal{F}$ implies that it must be equal to $\mathcal{F}$ itself. But then, $W \in \mathcal{F}$, as desired.

b) If $x \in \lim \mathcal{F}$, then all open neighborhoods of x have non-empty intersection with $\lim \mathcal{F}$ and item (a) implies that $\nu_x \subseteq \mathcal{F}$. Conversely, if the latter containment holds, all open neighborhoods of x have non-empty intersection with each element of $\mathcal{F}$ and A.18.(5) yields $x \in \bigcap_{U \in \mathcal{F}} \overline{U} = \lim \mathcal{F}$. Item (c) is immediate from (b).

d) Clearly, $K = \lim \mathcal{F}$ is closed. To show it is irreducible it is enough to verify, by A.24.(3) that if $W \in \mathcal{O}$ is such that $W \cap K \neq \emptyset$, then this intersection is dense in K. By item (a), we have $W \in \mathcal{F}$. Since $\mathcal{F}$ is a proper filter, $\neg W \notin \mathcal{F}$, and so, $\neg W \cap K = \emptyset$ (item (a), again). Since $W \cup \neg W$ is dense in X (A.36.(i)), it follows that $W \cap K$ is dense in K, as needed.

e) Let $S = \bigcup_{x \in K} \nu_x$; by A.24, S has the fip and can, therefore, be extended to an ultrafilter $\mathcal{G}$ in $\mathcal{O}$ (A.43.(g)). Since for all $x \in K$ we have $\nu_x \subseteq \mathcal{G}$, (b) entails $K \subseteq \lim \mathcal{G}$, as desired. Item (f) follows immediately from (e). □

Remark 8.4 It will be shown in 9.9 that there are ultrafilters which converge to a *non-maximal* irreducible closed set. Hence, in general, the converse of 8.3.(f) is false. □

Lemma 8.5 *Let $\mathcal{F}$ be an* ultrafilter *in $\mathcal{O}$, where $\langle X, \mathcal{O} \rangle$ is a T_0 topological space.*

a) The following conditions are equivalent:

(1) $\bigcap \mathcal{F} \neq \emptyset$; (2) *$\mathcal{F}$ is a principal ultrafilter.*

If X is T_1 or if every point in X has a smallest open neighborhood[20], *the above conditions are equivalent to*

(3) *$\mathcal{F} = \nu_x$, where x is an isolated point in X.*

b) If X is Hausdorff space and $\mathcal{F}$ is a convergent ultrafilter in $\mathcal{O}$, then there is a unique *$x \in X$ such that* $\lim \mathcal{F} = \{x\}$.[21]

Proof. a) Clearly, (2) $\Rightarrow$ (1). For the converse, let $x \in \bigcap \mathcal{F}$. Then, the *proper* filter ν_x verifies $\mathcal{F} \subseteq \nu_x$. Since $\mathcal{F}$ is maximal, we get $\mathcal{F} = \nu_x$, as needed. To prove the remaining statements, note that:

$*$ Let $x \in \bigcap \mathcal{F}$ and let W be the smallest open containing x. Then, $W \subseteq U$, for all $U \in \mathcal{F}$ and so $W \subseteq \bigcap \mathcal{F}$. If $y \in W$, then the argument used above to show (2) $\Rightarrow$ (1) implies that $\nu_y = \mathcal{F} = \nu_x$, and the fact that X is T_0 (A.19.(a)) entails $x = y$[22]. Hence, $W = \{x\}$ and x is an isolated point in X, establishing (3).

$*$ Assume that X is T_1 (i.e., all points are closed) and that $\mathcal{F} = \nu_x$; let $O = \{x\}^c$. Clearly, $O \notin \mathcal{F}$. Hence, the set $\mathcal{F} \cup \{O\}$ cannot have the fip (A.43.(d)). Thus, there is $\emptyset \neq U \in \mathcal{F}$ with $U \cap O = \emptyset$. But then, $\{x\} = U$, and x is isolated in X.

b) In a Hausdorff space, the only irreducible closed sets are its points. Uniqueness follows from the fact that distinct points have disjoint neighborhoods and 8.3.(b). □

Example 8.6 a) In an infinite set I with the discrete topology (all points are open), the only convergent ultrafilters are the principal ones. In this case ultrafilters are either *principal* or *free.*

b) We shall shortly see (9.7) that there are important examples wherein all ultrafilters are convergent but none are principal.

[20] As is the case of the $\mathfrak{U}$-topology.

[21] In this case, x is the limit of $\mathcal{F}$ and we write $\mathcal{F} \longrightarrow x$.

[22] Clearly, $y \in \overline{\{x\}}$ iff $\nu_y \subseteq \nu_x$. Thus, X is T_0 iff $x \neq y \Rightarrow \nu_x \neq \nu_y$.

c) Let $X = \mathbb{N} \cup \{*\}$ be the set of natural numbers with a new point $*$, considered to be larger than all standard naturals. In X consider the following topologies:

(1) $\mathcal{O} = \{\emptyset\} \cup \{[n] : n \in \mathbb{N}\}$; this is a T_0 topology, in which every non-empty open is dense. Hence, $\mathfrak{D}(\mathcal{O})$ is the only ultrafilter in $\mathcal{O}$, with

$$\mathfrak{D}(\mathcal{O}) = \nu_* \quad \text{and} \quad \bigcap \mathfrak{D}(\mathcal{O}) = \{*\}.$$

However, $*$ **is not** isolated in X. Thus, 8.5.(a).(3) is false if X is not T_1 or if some point in X does not have a smallest open neighborhood. Note that $\mathcal{O}$ **is not** the $\mathfrak{U}$-topology on X, because $[*] = \{*\} \notin \mathcal{O}$.

(2) $\mathcal{O} = \{\emptyset\} \cup \{F^c : F$ is a finite subset of $X\}$; this topology is T_1 (all points are closed, since their complements are open, by definition) and, once more, all non-empty opens are dense. Hence, $\mathfrak{D}(\mathcal{O})$ is the only ultrafilter in $\mathcal{O}$ and we have $\bigcap \mathfrak{D}(\mathcal{O}) = \emptyset$ and $\mathfrak{D}(\mathcal{O}) \longrightarrow X$. Hence, $\mathfrak{D}(\mathcal{O})$ is a *non-principal* ultrafilter, convergent to X. This shows that 8.5.(b) is false if X is not Hausdorff. □

Lemma 8.5.(a) and Example 8.2 yield

Corollary 8.7 *If P is a poset and $\mathcal{F}$ is an ultrafilter in $\mathfrak{U}(P)$, then $\mathcal{F}$ is principal iff $\mathcal{F} = \nu_x$, for some isolated (or maximal) point x in P.*

Corollary 8.8 *If P is a* finite *poset, then all ultrafilters in $\mathfrak{U}(P)$ are principal.*

A topological condition for the convergence of all ultrafilters is *compactness* (A.25):

Proposition 8.9 *For a topological space $\langle X, \mathcal{O} \rangle$, consider the following conditions:*

(1) *X is compact;*

(2) *All ultrafilters in $\mathcal{O}$ are convergent.*

Then, (1) $\Rightarrow$ (2)*. If X is* <u>regular</u> *(A.19.(d)), these conditions are equivalent.*

Proof. (1) $\Rightarrow$ (2): If X is compact and $\mathcal{F}$ is an ultrafilter in $\mathcal{O}$, then $\overline{\mathcal{F}} = \{\overline{u} : u \in \mathcal{F}\}$ has the fip; by A.26.(a), $\bigcap \overline{\mathcal{F}} \neq \emptyset$, as needed.

Now assume that X is regular (A.19.(d)). We start with

Fact 8.10 *For a topological space $\langle X, \mathcal{O} \rangle$, the following are equivalent:*

(i) *X is regular;*

(ii) *For all $u \in \mathcal{O}$ and $x \in u$, there is $v \in \nu_x$ such that $\overline{v} \subseteq u$;*

(iii) *Every $u \in \mathcal{O}$ has an open covering, $\{v_i : i \in I\}$, such that $\overline{v}_i \subseteq u$, for all $i \in I$.*

Proof. Clearly, $(ii) \Leftrightarrow (iii)$. If X is regular and $x \in X$, let $u \in \nu_x$. Then, $x \notin F = u^c$, and regularity yields disjoint opens v, w, with $v \in \nu_x$ and $F \subseteq w$. But then $v \subseteq w^c \subseteq u$ and so $\overline{v} \subseteq w^c \subseteq u$, showing that $(i) \Rightarrow (ii)$. The converse is similar: if x is not in F, then $u = F^c$ is an open set containing x. By (ii), there is $v \in \nu_x$ such that $\overline{v} \subseteq u$; then v and $(\overline{v})^c$ are disjoint opens, with $F \subseteq (\overline{v})^c$.

Suppose, to get a contradiction, that X is not compact. Then, there is a open covering, $\mathcal{C}$, of X, with no finite subcovering. For $x \in X$, select $u_x \in \mathcal{C}$ such that $x \in u_x$ and then, using Fact 8.10.(ii), choose $v_x \in \nu_x$ such that $\overline{v}_x \subseteq u_x$. Consider $w_x = X \setminus \overline{v}_x$; since w_x is the complement of a closed set, it is open in X. The family

$$G = \{w_x : x \in X\} \subseteq \mathcal{O}$$

has the following properties:

($\sharp$) <u>G has the finite intersection property.</u> Suppose $w_{x_1} \cap \ldots \cap w_{x_n} = \emptyset$, for $x_1, \ldots, x_n \in X$. Then, since $\overline{v}_{x_k} \subseteq u_{x_k}$, we obtain

$$X = w^c_{x_1} \cup \cdots \cup w^c_{x_n} = \overline{v}_{x_1} \cup \cdots \cup \overline{v}_{x_n} \subseteq u_{x_1} \cup \cdots \cup u_{x_n},$$

and $\mathcal{C}$ would have a finite subcovering of X, contrary to assumption.

($\sharp\sharp$) <u>$\bigcap_{x \in X} \overline{w}_x = \emptyset$.</u> First note that for all $z \in X$, because v_z is open and $v_z \cap w_z = v_z \cap (\overline{v_z})^c = \emptyset$, A.18.(5) entails that $v_z \cap \overline{w_z} = \emptyset$. Now, if $y \in \bigcap_{x \in X} \overline{w}_x$, then we would have $y \in v_y$ (by construction) and $y \in \overline{w_y}$, which, as just noted, is impossible.

Since G has the fip, A.43.(g) guarantees that there is an ultrafilter F **in $\mathcal{O}$** with $G \subseteq F$. Then, ($\sharp\sharp$) entails $\bigcap_{U \in F} \overline{U} \subseteq \bigcap_{x \in X} \overline{w}_x = \emptyset$ and so F is not convergent, violating (2) and ending the proof. $\square$

Remark 8.11 a) If we endow an infinite linear order without first element with the $\mathfrak{U}$-topology, we obtain an example of a **non-compact** topological space in which all topological ultrafilters are convergent. The reasoning is the same as in 8.6.(c): there is only one ultrafilter, the filter of dense opens, and it converges to the whole space.

b) It is harder to find an example of a *Hausdorff* space in which all topological ultrafilters are convergent. By Proposition 8.9, such a space cannot be *regular*. In the original version of these notes we constructed such an example, which will be omitted here in order to save space.

c) Readers familiar with slogan "a space is compact iff all ultrafilters are convergent" may find 8.9 slightly odd; but it underlines the distinction between ultrafilters in the algebra of parts and topological ones. In fact, it is true – and simple to prove – the equivalence in quotes. For if X is not compact, then there is a family T of closed sets in X that has the fip, but empty intersection (A.26.(a)). Any ultrafilter **in $\mathbf{2}^{\boldsymbol{X}}$** extending T cannot be convergent. The example mentioned in (b) shows that, in general, T *cannot be extended to a topological ultrafilter.* □

Before the next result, we introduce

Definition 8.12 *A poset* $\langle P, \leq \rangle$ *is* **rooted** *if there is a set* R *of pairwise incompatible* (1.13) *elements in* P *such that* $P = \bigcup_{x \in R} [x]$. *We refer to* R *as the set of* **roots** *of* P.

Clearly, a root is a minimal element of P. Moreover, a rooted poset is the union of supercompact opens. Since compactness is preserved by finite unions(A.26.(c)), 8.9 yields

Corollary 8.13 *If* P *is a rooted poset with a finite number of roots, then all ultrafilters in* $\mathfrak{U}(P)$ *are convergent.* □

9 Trees and Tree-like Posets

Definition 9.1 *a) A poset* P *is* **tree-like (tl)** *if for all* $x \in P$, *with the order induced from* P, $x^{\leftarrow}$ *is linearly ordered (or a chain).*

b) If P is tree-like, a **branch** *of P is a maximal linearly ordered subset of P.*

c) A poset P is a **tree** *if for all $x \in P$, with the order induced by P, $x^{\leftarrow}$ is well-ordered* [23].

It is clear that any tree is a tree-like poset. Moreover, any discrete set, (i.e., with the identity partial order) is a tree.

Lemma 9.2 *Let P be a tree-like poset and $x, y \in P$. Then,*

a) $[x] \cap [y] \neq \emptyset \quad \Leftrightarrow \quad x \leq y \quad or \quad y \leq x$.

b) A subset of P is rd iff it is a chain in P.

c) For a subset $A \subseteq P$, the following are equivalent:

(1) A is a branch in P;

(2) A is an irreducible component of P $(A.22.(c))$.

d) If P is a tree, then any irreducible closed set in P is well-ordered.

Proof. a) It is enough to check ($\Rightarrow$). If $r \geq x, y$, then $x, y \in r^{\leftarrow}$, and the fact that the latter is chain immediately implies $x \leq y$ or $y \leq x$.

b) Clearly, any chain is rd. Conversely, if T is a rd subset of P, given $p, q \in T$, we have $[p] \cap [q] \neq \emptyset$, and it follows from (a) that $p \leq q$ or $q \leq p$, verifying that T is a chain in P.

c) It is straightforward from 1.7.(a) and item (b) that any irreducible component of P is a branch. Conversely, let A be a branch in P. We first verify that A is closed in the $\mathfrak{U}$-topology. For $x \in A$, we wish to show that $x^{\leftarrow} \subseteq A$. Consider $S = x^{\leftarrow} \cup A$; to show that S is linearly ordered, let $y, z \in S$. The only non-trivial alternative is $y \in x^{\leftarrow}$ and $z \in A$. If $z \geq x$, then $y \leq z$. If $z \in x^{\leftarrow} \cap A$, the fact that $x^{\leftarrow}$ is linearly ordered implies $y \leq z$ or $z \leq y$. Hence, S is a chain in P and the maximality of A entails $x^{\leftarrow} \subseteq A$, as desired. By 1.7.(a), A is irreducible, while its maximality and (b) guarantee that it is a maximal irreducible closed set in P, i.e., an irreducible component in P.

d) If $A \subseteq P$ is closed and irreducible, then A is rd and thus, by (b), A is a chain in P. Hence, $A = \bigcup_{x \in A} x^{\leftarrow}$. To see that A is well-ordered, let $\emptyset \neq S \subseteq A$. Then, for some $x \in A$, $S \cap x^{\leftarrow} \neq \emptyset$. Since $x^{\leftarrow}$ is

[23] As in A.14.

well-ordered, there is $a = \min\ (S \cap x^{\leftarrow})$. We shall verify that $a = \min$ S. For $b \in S$, select $y \in A$ such that $b \in y^{\leftarrow}$. If $y \leq x$, then

$$b \in (S \cap y^{\leftarrow}) \ \subseteq \ (S \cap x^{\leftarrow}),$$

and so $a \leq b$. The other possibility is that $x < y$. If $b \leq x$, the preceding argument entails $a \leq b$. But if $x \leq b$, then $a \leq x \leq b$. Hence, $a = \min$ S, as desired. □

Definition 9.3 *A* branch *B in a tree-like poset P is* **principal** *if $B = x^{\leftarrow}$. Otherwise, B is said to be* **non-principal**.

Note that a principal branch is an irreducible component of P of the form $x^{\leftarrow}$, where x is an *isolated point* in P (as in Example 8.2).

Before our next result, we recall some of the standard terminology for trees. Appendix III will be used without further reference.

Definition 9.4 *Let T be a tree, x an element of T and $S \subseteq T$.*

a) The **height of x in T**, *$h(x,T)$, is the ordinal type of $x^{\leftarrow}$. Hence, $h(x,T)$ is the* unique *ordinal isomorphic to the well-ordering $x^{\leftarrow}$.*

b) If $\alpha \in Ord$, the **α level of T** *is $T_\alpha = \{x \in T : h(x,T) = \alpha\}$.*

c) The **height of S** *is $h(S) = \sup\ \{h(x,T) : x \in S\}$.*

d) The **immediate successors of x** *is the set $S(x) = [x] \cap T_{\alpha+1}$, where $\alpha = h(x,T)$.*

e) Let $k \geq 0$ be an integer. T is **k-branching** *if $h(T)$ is a limit ordinal and for all $x \in T$, $S(x)$ is non-empty and has cardinality $\leq k$.*

Note that a k-branching tree has no isolated points (or principal branches) and all its elements have at least one immediate successor, but no more than k such points.

Lemma 9.5 *Let T be a tree and x, $y \in T$.*

a) $h(x,T) = h(y,T)$ and $x \neq y \ \Rightarrow \ x \perp y$[24].

b) $[x] = \{x\} \cup \bigcup_{s \in S(x)} [s]$.

c) $\bigcup_{s \in S(x)} [s]$ is dense in $[x]$.

[24] x is incompatible with y, see 1.13.

d) If $x^{\leftarrow}$ is properly contained in an irreducible closet K, then $K \cap S(x) \neq \emptyset$.

e) T is a rooted poset (8.12)*, where T_0 is the set of roots of T.*

Proof. a) If $[x] \cap [y] \neq \emptyset$, then 9.2.(a) implies $x \leq y$ or $y \leq x$. Since $x \neq y$, the above alternatives entail $h(x,T) \neq h(y,T)$, which is impossible. Hence $[x] \cap [y] = \emptyset$ and x and y are incompatible.

b) Let $\alpha = h(x,T)$; if $z > x$, the fact that $z^{\leftarrow}$ is well-ordered entails $h(z,T) \geq \alpha + 1$. Hence, $z^{\leftarrow}$ must intersect $T_{\alpha+1} \cap [x]$, that is, $z \in [s]$, for some $s \in S(x)$. Item (c) is straightforward.

d) Let $z \in K \setminus x^{\leftarrow}$. Then, 9.2.(d) implies that $z > x$ and the the desired conclusion follows from (b).

e) By item (a), T_0 consists of pairwise incompatible elements. If $x \in T$, then the first element of the well-ordered set $x^{\leftarrow}$ is in T_0 and below x. Thus, $T = \bigcup_{s \in T_0} [s]$, as desired. □

Corollary 9.6 *If T is a k-branching tree and $\mathcal{F}$ is a convergent ultrafilter in $\mathfrak{U}(T)$, then $h(\lim \mathcal{F})$ is a limit ordinal.*

Proof. Let $K = \lim \mathcal{F} \neq \emptyset$; by 8.3.(d) and 9.2.(d), K is well-ordered. If $h(K) = \alpha + 1$, K would have a largest element x and $K = x^{\leftarrow}$[25]. In particular, $[x] \in \mathcal{F}$. By 9.5.(c), $\bigcup_{s \in S(x)} [s]$ is dense in $[x]$, and so Fact 7.8 implies that $\left(\bigcup_{s \in S(x)} [s]\right) \in \mathcal{F}$. Since $\mathcal{F}$ is prime (A.43.(e)) and this union is finite, we get that for some $s \in S(x)$, $[s] \in \mathcal{F}$. But then, $\nu_s \subseteq \mathcal{F}$, that is, $s \in K$, a contradiction, since $s > x$. □

Proposition 9.7 *a) If T is a tree in which T_0 is finite, then all ultrafilters in $\mathfrak{U}(T)$ are convergent.*

b) Let $k \geq 1$ be an integer. If T is a k-branching tree of height ω in which T_0 is finite, then all ultrafilters in $\mathfrak{U}(T)$ converge to an irreducible component of T.[26]

Proof. Item (a) is immediate from Lemma 9.5.(e) and Corollary 8.13. For (b), fix an ultrafilter $\mathcal{F}$ in $\mathfrak{U} = \mathfrak{U}(T)$. By induction on $n \geq 0$, a sequence $x_n \in T$ shall be constructed, such that for all n

[25] Since $h(K) = \sup \{h(y,T) : y \in K\} = \alpha + 1$, this supremum has to be attained.

[26] Equivalently, by 9.2.(c), to a branch in T.

(i) $x_n \in T_n$; (ii) $x_{n+1} \in S(x_n)$; (iii) $\nu_{x_n} \subseteq \mathcal{F}$.

Since T_0 is finite and $T = \bigcup_{s \in T_0} [s] \in \mathcal{F}$, the fact that $\mathcal{F}$ is prime (A.43.(e)) yields the existence of $s \in T_0$, such that $[s] \in \mathcal{F}$. Set $x_0 = s$. Having chosen $x_1, \ldots, x_n$, verifying the conditions above, we may write, by 9.5.(b), $[x_n] = \{x_n\} \cup \bigcup_{y \in S(x)} [y]$. Since $[x_n] \in \mathcal{F}$ and $\bigcup_{y \in S(x)} [y]$ is dense in $[x_n]$ (9.5.(c)), the Fact 7.8 yields $\left(\bigcup_{y \in S(x)} [y]\right) \in \mathcal{F}$. Because $S(x)$ is finite and $\mathcal{F}$ is prime, there is $y \in S(x)$ such that $[y] \in \mathcal{F}$. Set $x_{n+1} = y$. Note that $[x_{n+1}] \in \mathcal{F}$ guarantees that $\nu_{x_{n+1}} \in \mathcal{F}$, as needed. Let $K = \bigcup_{n \geq 0} x_n^{\leftarrow}$; then K is a branch in T (it is well-ordered and has height ω) and thus an irreducible component of T (9.2.(c)). Since $K \subseteq \{x \in T : \nu_x \subseteq \mathcal{F}\}$, items (b) and (d) in 8.3, together with the maximality of K, imply $K = \lim \mathcal{F}$, ending the proof. □

Remark 9.8 If T is a tree in which T_0 is *infinite*, then there are free ultrafilters in $\mathfrak{U}(T)$. To see this, note that the collection of opens $S = \{\bigcup_{s \in A} [s] : A \text{ is cofinite in } T_0\}$ has the fip (A.43.(e)) and so can be extended to an ultrafilter $\mathcal{F}$ in $\mathfrak{U}(T)$. It is easily established that $\mathcal{F}$ is not convergent. □

Example 9.9 Let ω_1 be the first uncountable cardinal and set

$$T = \{s \in 2^{\alpha} : \alpha \in \omega_1\}.$$

An element of T is a map, $s : \alpha \longrightarrow 2 = \{0, 1\}$, where α is a countable ordinal (an element of ω_1). Moreover, T is poset: if $s, t \in T$, then

$$s \leq t \quad \text{iff} \quad dom\, s \subseteq dom\, t \text{ and } t_{|doms} = s.$$

T is the **complete binary tree on $\boldsymbol{\omega_1}$**. Clearly, it is a 2-branching tree. The branches in T correspond to 2^{ω_1}, that is, the set of maps from ω_1 to $2 = \{0, 1\}$. For $s \in T$, the **immediate successors** of s are written $s\hat{\ }0$ and $s\hat{\ }1$, defined as follows, where $dom\, s = \alpha \in \omega_1$:

$$\begin{cases} dom\ s\hat{\ }0 = dom\ s\hat{\ }1 = \alpha + 1 = \alpha \cup \{\alpha\}; \\ (s\hat{\ }0)_{|\alpha} = (s\hat{\ }1)_{|\alpha} = s; \\ s\hat{\ }0(\alpha) = 0 \text{ and } s\hat{\ }1(\alpha) = 1. \end{cases}$$

We shall construct an ultrafilter in $\mathfrak{U}(T)$ that converges to a *non-maximal* irreducible closet set in T. Write $\widehat{0}$ for the map constantly equal to 0 on ω_1; $\widehat{0}$ corresponds to a branch in T, namely

$$B = \{\widehat{0}_{|\alpha} : \alpha \in \omega_1\}.$$

Define, for $n \in \omega$, $z_n = \widehat{0}_{|n}$, $z = \widehat{0}_{|\omega}$ and $Z = \bigcup_{n \geq 0} z_n^{\leftarrow}$. Note that Z is a closed irreducible subset of T, clearly non-maximal, for it is properly contained in $\widehat{0}$ (and in uncountably many other branches of T). Moreover, Z **is not** a closed set of the type $s^{\leftarrow}$, since it does not possess a maximum.

Fact. *The set of opens in* $\mathfrak{U}$, $S = \{[z_n] : n \in \omega\} \cup \{\neg [z]\}$ *has the fip.*

Proof. It suffices to show that for $n \geq 0$, $[z_n] \cap \neg [z] \neq \emptyset$. Let $V = [z_n\hat{\ }1] \subseteq [z_n]$. Since, $z(n+1) = 0$, it is clear that no extension of z can be an extension of $z_n\hat{\ }1$. Thus, $[z_n\hat{\ }1] \cap [z] = \emptyset$, and so A.33.(c) implies $[z_n\hat{\ }1] \subseteq \neg [z]$, establishing the Fact.

Let $\mathcal{F}$ be an ultrafilter in $\mathfrak{U}$, with $S \subseteq \mathcal{F}$. It is claimed that lim $\mathcal{F}$ $= Z$. First, since $\nu_{z_n} \subseteq \mathcal{F}$, we have $Z \subseteq$ lim $\mathcal{F}$ (8.3.(b)). Next, suppose that there is $x \in$ lim $\mathcal{F} \setminus Z$. Since lim $\mathcal{F}$ is an irreducible closed set (8.3.(d)), it is well-ordered (9.2.(d)). Because Z consists of a sequence of immediate successors, we conclude that $x > z_n$, $n \geq 0$. Hence, $\omega \subseteq dom\, x$ and $x_{|n} = z_n$, for all $n \geq 0$. Then, $z = x_{|\omega} \in$ lim $\mathcal{F}$; but this is impossible, because, by construction, $\neg [z] \in \mathcal{F}$, which implies $[z] \notin \mathcal{F}$and so ν_z cannot be contained in $\mathcal{F}$. Therefore, lim $\mathcal{F} \subseteq Z$, establishing the claimed equality.

The same argument applies to show that:

If K is an irreducible closed set in T whose height is a limit ordinal in ω_1, then K is the limit of an ultrafilter in $\mathfrak{U}(T)$.

By 9.6, the limits of ultrafilters in $\mathfrak{U}(T)$ are precisely the irreducible closed sets whose height are limit ordinals in ω_1. □

10 Stalks at Convergent Ultrafilters

In 3.9 we introduced condition $[E]$ and applied it to the description, **by isomorphism**, of stalks of the completion of a Kripke structure at ultrafilters, in terms of the original Kripke structure. In Model Theory there are other ways to classify L-structures. One of them is to describe the *elementary theory* of the structure in terms of simpler components. The Feferman-Vaught result in [FV] is a seminal example of this. We shall here do the same for stalks over convergent ultrafilters. Since there are significant examples of posets in which every ultrafilter is convergent, the results below describe, in these cases, the elementary theory of all generalized ultraproducts. For principal ultrafilters, the old technique still works:

Corollary 10.1 *Let P be a poset and $\mathcal{F} = \nu_p$ be a principal ultrafilter in $\mathfrak{U}(P)$. For all Kripke structures $\mathcal{M}$ over P, $\mathcal{M}_{\nu_p} \approx M_p$.*

Proof. By 8.5.(a), if $\mathcal{F} = \nu_p$ is a principal ultrafilter in $\mathfrak{U}(P)$, then p is isolated in P. Hence, for all $U \in \mathcal{F}$, $[p] \subseteq U$ and $\mathcal{F}$ verifies condition $[E]$ in 3.9. Thus, by 3.13.(c), $\mathcal{M}_{\nu_p}$ is isomorphic to M_p. □

Fix a poset P and a *convergent* ultrafilter $\mathcal{F}$ in $\mathfrak{U} = \mathfrak{U}(P)$. Set $K = \lim \mathcal{F}$. By 8.3.(d), K is an irreducible closed set in $\mathfrak{U}$ and consequently, a rd subset of P (1.7.(a)). Let

$$\mathcal{M} = \langle\, M_p;\, \mu_{pq}\,\rangle \quad \text{and} \quad \mathfrak{g}\mathcal{M} = \langle\, \mathfrak{g}\mathcal{M}(U);\, (\cdot)_{|V}\,\rangle$$

be a Kripke structure over P and its completion over $\mathfrak{U}^{op}$, respectively. By Theorem 3.2.(a), we may identify $\mathfrak{g}\mathcal{M}([p])$ with M_p and, whenever $p \leq q$, the restriction $(\cdot)_{|[q]}$ with μ_{pq}. Let

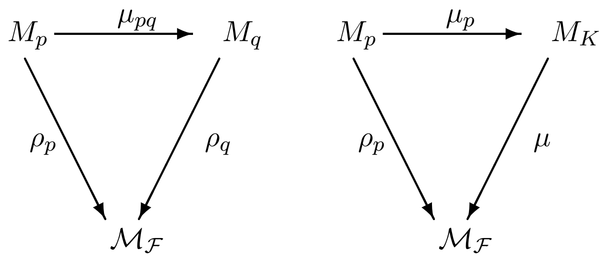
$$M_K = \langle\, M_K; \mu_p\,\rangle_{p \in K} \quad \text{and} \quad \mathcal{M}_{\mathcal{F}} = \langle\, \mathcal{M}_{\mathcal{F}};\, \rho_U\,\rangle_{U \in \mathcal{F}}$$

be the colimit of $\mathcal{M}_{|K}$ and the stalk of $\mathfrak{g}\mathcal{M}$ at $\mathcal{F}$, respectively. Here, for $p \in K$ and $U \in \mathcal{F}$,

$$\mu_p : M_p \longrightarrow M_K \quad \text{and} \quad \rho_U : \mathfrak{g}\mathcal{M}(U) \longrightarrow \mathcal{M}_{\mathcal{F}}$$

are the L-morphisms that come with the colimit construction. For each $p \in K$, by 8.3.(b), $\nu_p \subseteq \mathcal{F}$; in particular, $[p] \in \mathcal{F}$ and so there is a L-morphism, $\rho_p : M_p \longrightarrow \mathcal{M}_{\mathcal{F}}$. Hence, for $p \leq q$ in K, the diagram (D1) below left is commutative:

(D1)



(D2)

Because $M_K = \varinjlim \mathcal{M}_{|K}$, the universal property of colimits yields a *unique* L-morphism $\mu : M_K \longrightarrow \mathcal{M}_{\mathcal{F}}$, such that for all $p \in K$, the diagram (D2) above right is commutative.

If $\overline{\xi} \in M_K^n$, Remark 2.7.(b) yields $p \in K$ and $\overline{x} \in M_p^n$ such that $\mu_p(\overline{x}) = \overline{\xi}$. The commutativity of diagram (D2) entails $\mu(\overline{\xi}) = \rho_p(\overline{x})$.

Notation as above, Theorems 5.1 and 7.7 yield

Theorem 10.2 *Let P be a poset and $\mathcal{F}$ a convergent ultrafilter in $\mathfrak{U}(P)$, with* $\lim \mathcal{F} = K$. *Then, the canonical L-morphism,*

$$\mu : M_K \longrightarrow \mathcal{M}_{\mathcal{F}},$$

is an elementary embedding.

Proof. Let $\overline{\eta} \in M_K^n$; fix $p \in K$ and $\overline{x} \in M_p^n$ such that $\mu_p(\overline{x}) = \overline{\eta}$. We shall verify that for all formulas $\phi(v_1, \ldots, v_n)$ in $L_\exists$,

$$M_K \models \phi[\overline{\eta}] \quad \Rightarrow \quad \mathcal{M}_{\mathcal{F}} \models \phi[\mu(\overline{\eta})],$$

and conclude by Remark A.60.(c). Theorem 5.1 entails

$$\begin{aligned} M_K \models \phi[\overline{\eta}] \quad &\text{iff} \quad [\![\phi(\langle \overline{x}, \underline{p} \rangle)]\!]_K \neq \emptyset \\ &\text{iff} \quad \exists\, q \in K, \text{ with } q \geq p \text{ and } M_q \Vdash \phi[\mu_{pq}(\overline{x})]. \end{aligned}$$

Since $q \in K$, we have $[q] \in \mathcal{F}$; moreover, the last statement in the equivalence above yields $q \in [\![\phi(\mu_{pq}(\overline{x}))]\!]_{\mathfrak{g}}$. Since $[\![\phi(\mu_{pq}(\overline{x}))]\!]_{\mathfrak{g}}$ is open (7.3.(a)), we get $[q] \subseteq [\![\phi(\mu_{pq}(\overline{x}))]\!]_{\mathfrak{g}}$, and so $[\![\phi(\mu_{pq}(\overline{x}))]\!]_{\mathfrak{g}} \in \mathcal{F}$. Theorem 7.7 then yields $\mathcal{M}_{\mathcal{F}} \models \phi[\mu_{pq}(\overline{x})_{\mathcal{F}}]$. To finish the proof, recall that $\mu_{pq}(\overline{x})_{\mathcal{F}} = \rho_q(\mu_{pq}(\overline{x}))$ and note that diagrams (D1) and (D2) above furnish $\mu(\overline{\eta}) = \rho_p(\overline{x}) = \rho_q(\mu_{pq}(\overline{x}))$, and so $\mathcal{M}_{\mathcal{F}} \models \phi[\mu(\overline{\eta})]$, as needed. □

From Theorem 10.2 and Proposition 9.7 we get

Corollary 10.3 *If T is a finitely rooted tree and $\mathcal{M}$ is a Kripke structure over T, then the elementary theory of the stalks of $\mathcal{M}$ at the ultrafilters in $\mathfrak{U}(T)$ is the same as the elementary theory of the colimits of $\mathcal{M}$ over the irreducible components of T.*

A Appendices

Our notational conventions in 1.1 remain in force in these Appendices.

I Equivalence Relations

A.1 Equivalence Relations. An **equivalence relation** on a set S is a binary relation $E \subseteq S^2$ such that for $x, y, z \in S$

[equ 1]: $x\ E\ x$; [equ 2]: $x\ E\ y \;\Rightarrow\; y\ E\ x$;

[equ 3]: $x\ E\ y$ and $y\ E\ z \;\Rightarrow\; x\ E\ z$.

The set of equivalence relations on S is closed under arbitrary intersections. Hence, if $T \subseteq S^2$, the **equivalence relation generated by *T*** is defined as

$$E_T = \bigcap \{E \subseteq S^2 : E \text{ is an equivalence relation and } T \subseteq E\},$$

the *smallest* equivalence relation on S containing T. Let R be a reflexive and symmetric binary relation on S, i.e., R satisfies [equ 1] and [equ 2] above. The **transitive closure** of R is defined as

$$E_R = \{\langle a, b\rangle \in S^2 : \begin{array}{c} \exists\ n \geq 2 \text{ and } a_1, \ldots, a_n \text{ in } S \text{ such that} \\ a_1 = a,\ a_n = b \text{ and } a_i\ R\ a_{i+1},\ 1 \leq i \leq (n-1). \end{array}\Bigg\}$$

Lemma A.2 *If R is a reflexive and symmetric binary relation on a set S, then its transitive closure is the equivalence relation generated by R in S.*

Proof. Clearly, E_R is reflexive and any equivalence relation containing R must contain E_R. Thus, it is enough to check that E_R is symmetric and transitive. Let $x_1, \ldots, x_n$ be a sequence witnessing the fact that $a\ E_R\ b$; the inversion of order in $x_1, \ldots, x_n$ (i.e., the sequence $t_i = x_{n-i+1}$), shows that $b\ E_R\ a$. If $x_1, \ldots, x_n$ witnesses $a\ E_R\ B$ and $y_1, \ldots, y_m$ witnesses $b\ E_R\ c$, the concatenation of the x_i's with the y_j's shows that $a\ E_R\ c$ and E_R is transitive, as needed. □

II Partial Orders

This is an introduction to partial orders, semilattices, lattices and their complete counterparts.

Definition A.3 *A* **partially ordered set (poset)**, $\langle P, \leq \rangle$, *is a set* P, *and a binary relation* $\leq$ *on* P, *satisfying, for all* $x, y, z \in P$
[*po* 1]: $x \leq x$;
[*po* 2]: $x \leq y$ *and* $y \leq x \;\Rightarrow\; x = y$;
[*po* 3]: $x \leq y$ *and* $y \leq z \;\Rightarrow\; x \leq z$.
a) For $x \in P$, *write*

$$[x] = \{y \in P : x \leq y\} \qquad \text{and} \qquad x^{\leftarrow} = \{y \in P : y \leq x\}.$$

b) If S, T *are subsets of* P, $\boldsymbol{S}$ **is cofinal in** $\boldsymbol{T}$ *if for all* $t \in T$, $[t] \cap S \neq \emptyset$. *A subset of* P *is* **unbounded** *if it is cofinal in* P[27].
c) A subset D *of* P *is*

(1) **right-directed (rd)** [28] *if*

For all $x, y \in D$, *there is* $z \in D$ *such that* $x, y \leq z$.

(2) $\boldsymbol{\omega}$ **right-directed (**$\boldsymbol{\omega}$**-rd)** *if*

$\forall\ x, y \in D$ *there is a* *finite*[29] $S \subseteq D$ *such that* $[x] \cap [y] = \bigcup_{s \in S} [s]$.
d) Write $\perp$ (bottom) *and* $\top$ (top) *for the least and greatest elements of a poset* P (if they exist), *respectively. If* $\perp \in P$, *set* $P_* = P \setminus \{\perp\}$.
e) Let D *be a subset of* P *and let* $x \in P$.

(1) x *is* **maximal** *in* D *if* $x \in D$ *and* $\forall\ d \in D$, $d \geq x \;\Rightarrow\; d = x$;

(2) x *is* **minimal** *in* D *if* $x \in D$ *and* $\forall\ d \in D$, $d \leq x \;\Rightarrow\; d = x$.

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[27] Although this might not be strictly appropriate.
[28] Some authors use *up-directed* or *right-filtered*.
[29] Possibly empty; keep in mind that $\bigcup \emptyset = \emptyset$.

(3) *x is an* **upper bound** *for D if for all $d \in D$, $x \geq d$. If x is an upper bound for D and $x \in D$, then x is called the* **maximum** *of D, written* $\max D$*;*

(4) *x is a* **lower bound** *for D if for all $d \in D$, $x \leq d$. If x is a lower bound for D and $x \in D$, then x is called the* **minimum** *of D, written* $\min D$.

f) A poset $\langle P, \leq \rangle$ is a **linear order or a chain** *if for all x, $y \in P$, we have $x \leq y$ or $y \leq x$.*

Remark A.4 a) If D is a subset of a poset $\langle P, \leq \rangle$, then

(1) D, with the partial order induced by P, is a poset;

(2) D is rd iff $\forall\, x, y \in D$, $[x] \cap [y] \cap D \neq \emptyset$.

b) Any poset with a largest element is rd. For instance, $x^{\leftarrow}$ is a rd subset of P.

c) The concepts of rd and ω-rd are incomparable: neither one implies the other. □

A.5 The following statement has many application and, in spite of its name, has the status of an axiom of Set Theory, being equivalent, among others, to the Axiom of Choice:

Zorn's Lemma. *If $\mathcal{V}$ is a non-empty poset such that all chains in $\mathcal{V}$ have an upper bound, then $\mathcal{V}$ has a maximal element.* □

Definition A.6 *Let $\langle P, \leq \rangle$ be a poset, $S \subseteq P$ and $x \in P$.*

a) (1) ***x* is the join of *S*** *if it is an upper bound for S such that for all $y \in P$, if y is an upper bound for S, then $x \leq y$. Write $x = \bigvee S$, if x is the join of S in P;*

(2) ***x* is the meet of *S*** *if it is a lower bound for S such that for all $y \in P$, if y is a lower bound for S, then $x \geq y$. Write $x = \bigwedge S$, if x is the meet of S in P.*

Whenever they exist in P, $\bigvee S$ and $\bigwedge S$ are also called least upper bound *and* greatest lower bound *of S, respectively.*

b) ***P* is a join-semilattice** *if for all x, $y \in P$, $\bigvee \{x, y\}$ exists in P, being written as $x \vee y$. It is clear that $x, y \leq x \vee y$.*

c) ***P* is a meet-semilattice** *if for all* $x, y \in P$, $\bigwedge \{x, y\}$ *exists in* P, *being written as* $x \wedge y$. *Clearly,* $x \wedge y \le x, y$.

d) ***P* is a lattice** *if it is both a join and meet semilattice.*

e) ***P* is complete (or a complete lattice)** *if for all* $S \subseteq P$, $\bigvee S$ *and* $\bigwedge S$ *exist in* P [30]. *In particular, complete lattices have* $\bot$ *and* $\top$ (*cf.* A.3.(*d*)).

Example A.7 If $\langle P, \le \rangle$ is a poset, write $P^{op} = \langle P, \le^{op} \rangle$, for the poset whose domain is P, but whose relation is the *opposite* of that in P, that is,

$$x \le^{op} y \quad \text{iff} \quad y \le x.$$

P^{op} is called the **opposite** of P. It will be used frequently latter on. Note that

[op] Joins and meets in P become meets and joins in P^{op}, respectively.

Hence, the opposite of join-semilattice is a meet-semilattice and vice-versa. Moreover, the properties of being a lattice or a complete lattice are preserved by the passage from P to P^{op}. □

The lattices that will be of interest here are the *distributive* ones:

Definition A.8 *A lattice* L *is* **distributive** *if for all* $x, y, z \in L$

[D 1]: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

[D 2]: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Remark A.9 It is well-known that a lattice L verifies [D 1] in A.8 *iff* it verifies [D 2]. Moreover, note that L is a distributive lattice iff the same is true of its opposite, L^{op}. □

Definition A.10 *Let* L *be a distributive lattice with* $\bot$ *and* $\top$ *and let* $a \in L$. *We say that*

[30] It is enough to verify that meets (or joins) exist for all subsets, for one property implies its dual.

a) ***a* is pseudo-complemented in *L*** *if*

$$\neg a \;\; =_{def} \;\; \max \{x \in L : x \wedge a = \bot\}$$

exists in L; in this case $\neg a$ *is called the* **pseudo-complement or negation of *a*** *in L.*

b) ***a* is clopen in *L*** *if it is pseudo-complemented and* $a \vee \neg a = \top$*; in this case,* $\neg a$ *is called the* **complement of *a*** *in L. Write* $\mathfrak{B}(L)$ *for the set of clopens in L. Note that* $\top, \bot \in \mathfrak{B}(L)$[31].

Lemma A.11 *Let L be a distributive lattice with* $\bot$ *and* $\top$*. For* $a \in L$ *consider the system of equations in one unknown*

(♯) $$x \wedge a = \bot \quad \textit{and} \quad x \vee a = \top.$$

If (♯) *has a solution in L, then its* **unique** *solution is* $\neg a$*. In particular, a is clopen in L.*

Proof. Let y, z be solutions of (1). Then, the distributive laws yield

$$y = y \wedge \top = y \wedge (z \vee a) = (y \wedge z) \vee (y \wedge a) = (y \wedge z) \vee \bot = y \wedge z.$$

Since the argument is symmetrical in y and z, we can also show that $z = y \wedge z = y$, establishing uniqueness. Let z be the *unique* solution of (1) in L. If $x \in L$ is such that $x \wedge a = \bot$, then

$$x = x \wedge \top = x \wedge (z \vee a) = (x \wedge z) \vee (x \wedge a) = (x \wedge z) \vee \bot = x \wedge z,$$

and so $x = x \wedge z \leq z$. Since $z \wedge a = \bot$, we get

$$z = \max \{x \in L : x \wedge a = \bot\} = \neg a,$$

completing the proof. □

We can now define a fundamental concept:

Definition A.12 *A distributive lattice with* $\bot$ *and* $\top$ *is a* **Boolean algebra (BA)** *if all of its elements are clopen. A* **complete Boolean algebra (cBa)** *is a BA that is complete as a lattice.*

Remark A.13 If L is a distributive lattice with $\bot$ and $\top$, note that with the partial order induced by L, $\mathfrak{B}(L)$ is a BA and, in fact, the *largest Boolean algebra* that is a sublattice of L. □

[31] In general, there might be no others.

III Ordinals and Cardinals

Since *ordinals* will be important in some of our constructions, we shall briefly comment on the concept. References are [Ku], [Le] and [Mi3], just to cite a few.

Definition A.14 *A poset $\langle P, \leq \rangle$ is* **well-ordered** *(and $\leq$ is a well-ordering) if all non-empty $S \subseteq P$ have a least element, written min S. Hence, if $\emptyset \neq S \subseteq P$, there is $x \in S$ such that $x \leq s$, for all $s \in S$.*

John von Neumann had the idea of constructing a complete sample of well-orderings in the universe using the basic binary relations in Set Theory: $\in$ and $\subseteq$. The pertinence relation $\in$ is considered the *strict* part of the ordering, while $\subseteq$ is its "less than or equal" counterpart.

An **ordinal** is a set α with the following properties:

[ord 1]: (Transitivity) $\beta \subseteq \alpha \Leftrightarrow \beta = \alpha$ or $\beta \in \alpha$;

[ord 2]: (Regularity) $\forall\, S \subseteq \alpha,\ S \neq \emptyset \Rightarrow \exists\, \beta \in S$ with $\beta \cap S = \emptyset$.

The statement [ord 2], when applying to all sets in the universe, is what is known as the *Axiom of Regularity*, normally included among the axioms of Set Theory.

Example A.15 1. For an integer $n \geq 0$, define

$$* \ \underline{0} = \emptyset; \qquad\qquad * \ \underline{n+1} = \underline{n} \cup \{\underline{n}\}.$$

Each $\underline{n}$ is an ordinal, identified, in Set Theory with the natural number n.

2. Let $\omega = \bigcup_{n \geq 0} \underline{n}$. Then, ω is an ordinal, the copy of the natural numbers employed in Set theory.

3. If α is an ordinal, then $\alpha + 1 = \alpha \cup \{\alpha\}$ is also an ordinal, the **successor** of α. $\diamond$

Write *Ord* for the class $-$ *it is not a set* $-$ of all ordinals. Write ZFC for Zermelo-Fraenkel Set Theory, with the Axioms of Regularity and Choice.

If α, $\beta \in Ord$, then we have

[ord 3]: $\alpha \subseteq \beta$ iff $\alpha \in \beta$ or $\alpha = \beta$.

[ord 4]: $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

[ord 5]: If $T \neq \emptyset$ is a set of ordinals, then $\bigcap T$ and $\bigcup T$ are ordinals.

[ord 6]: Any non-empty set of ordinals has a least element.

[ord 7]: Any non-empty set T of ordinals has a **least upper bound**, written sup T.

Ordinals can be divided into two categories: **successors** and **limits**. Successor ordinals were defined in A.15.(3). An ordinal is a limit ordinal if it is not a successor. Thus,

[ord 8]: α is a limit ordinal iff $\alpha = \bigcup_{\beta \in \alpha} \beta$.

Since ordinals are well-ordered, we can use induction to construct objects and proofs. In the general, one must deal with the induction steps at *successors and limits*. This generalizes of the usual induction on the natural numbers, in which we have only to deal with successors.

A **cardinal** is an *initial ordinal*, in the following sense: given an ordinal α, consider the set of all ordinals that can be put in bijective correspondence with α. The least such ordinal ([ord 6]) is called the **cardinal of $\boldsymbol{\alpha}$**. For instance, ω (A.15.(2)) is a cardinal, because it is the least infinite countable ordinal. The infinite cardinals can be arranged in a sequence

$$\omega, \omega_1, \omega_2, \ldots, \omega_n, \ldots, \omega_\omega, \ldots$$

which has no upper bound. The first uncountable cardinal, ω_1, consists of all countable and finite ordinals. The same pattern applies to the whole sequence of cardinals.

IV Topology

This appendix presents some basic notions in Topology. Whenever proofs are not provided, they can be found in [Bu], [En] or [Ke].

Definition A.16 *A* **topological space** *is a pair* $\langle X, \mathcal{O} \rangle$ *where* X *is a set and* $\mathcal{O}$ *is a subset of* 2^X *such that*

[*top* 1]*:* $\emptyset$*,* $X \in \mathcal{O}$*;*

[*top* 2]*: $\mathcal{O}$ is closed under finite intersections;*
[*top* 3]*: $\mathcal{O}$ is closed under arbitrary unions.*
The elements of $\mathcal{O}$ are called **opens** *and $\mathcal{O}$ a* **topology** *on X. A subset of X is* **closed** *if its complement is open.*

It is readily verified that the set of topologies on X − a subset of 2^{2^X} −, is closed under arbitrary intersections. Hence, if $\mathcal{U}$ is any collection of subsets of X, the intersection of all topologies containing $\mathcal{U}$ is also a topology on X, the **topology generated by $\mathcal{U}$ on X**.

Example A.17 Let $\langle X, \mathcal{O} \rangle$ be a topological space and A be a subset of X. Define

$$\mathcal{O}_{|A} = \{C \subseteq A : \exists\, U \in \mathcal{O} \text{ such that } C = U \cap A\}.$$

Then, $\mathcal{O}_{|A}$ is a topology on A, called the **induced or subspace topology** on A. □

In general, families of subsets of a set X, closed under arbitrary unions or intersections, give rise to interesting operations on 2^X. Here are two examples:

Example A.18 Let $\langle X, \mathcal{O}(X) \rangle$ be a topological space. Define operations

$$\text{int} : 2^X \longrightarrow 2^X \quad \text{and} \quad \overline{\cdot} : 2^X \longrightarrow 2^X$$

as follows:

$$\begin{cases} \text{int}\, A &= \bigcup \{V \in \mathcal{O} : V \subseteq A\}; \\ \overline{A} &= \bigcap \{F \subseteq X : F \text{ is closed and } A \subseteq F\}, \end{cases}$$

called, respectively, the **interior** and **closure** of A in the space X. These operations have the following properties, where A, $B \subseteq X$:

(0) (i) $\text{int}\ A \subseteq A$ and A is open iff $A = \text{int}\ A$.

(ii) $A \subseteq \overline{A}$ and A is closed iff $\overline{A} = A$.

Hence, the open sets are the *fixed points* of the interior operation; and the closed sets are the *fixed points* of the closure operation.

(1) Increasing: $A \subseteq B \Rightarrow \text{int}\ A \subseteq \text{int}\ B$ and $\overline{A} \subseteq \overline{B}$;

(2) Idempotent: $\text{int}(\text{int}\ A) = \text{int}\ A$ and $\overline{(\overline{A})} = \overline{A}$.

(3) $\text{int}\, A \cup \text{int}\, B \subseteq \text{int}\, (A \cup B)$ and $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.[32]

(4) $\text{int}\, (A \cap B) = \text{int}\, A \cap \text{int}\, B$ and $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

It is straightforward to check that for all $A \subseteq X$

(5) $\overline{A} = \{p \in X : \forall\, V \in \mathcal{O},\ \ p \in V \ \Rightarrow\ V \cap A \neq \emptyset\}$. □

In many contexts it is important to be able to separate sets by opens. A sample of separation axioms is presented in

Definition A.19 *Let $\langle X, \mathcal{O} \rangle$ be a topological space.*

a) X is $\boldsymbol{T_0}$ *if for all $p, q \in X$, $p \neq q \ \Rightarrow\ \overline{\{p\}} \neq \overline{\{q\}}$.*

b) X is $\boldsymbol{T_1}$ *if all its points are closed.*

c) X is $\boldsymbol{T_2}$ *or* **Hausdorff** *if for all $p \neq q$ in X, there are disjoint opens U, V with $p \in U$ and $q \in V$.*

d) X is **regular** *if it is T_1 and for all pairs consisting of a point p and closed set F such that $p \notin F$, there are disjoint opens U, V such that $p \in U$ and $F \subseteq V$.*

e) X is **normal** *if it is T_1 and for all disjoint closed sets, F, F', there are disjoint opens U, V such that $F \subseteq U$ and $F' \subseteq V$.*

It is easily established that normal $\Rightarrow$ regular $\Rightarrow$ Hausdorff $\Rightarrow$ $T_1 \Rightarrow T_0$.

Definition A.20 *Let $\langle X, \mathcal{O} \rangle$ be a topological space and A, $B \subseteq X$. A is* **dense** *in B if $B \subseteq \overline{A}$. Write $\mathfrak{D}(\mathcal{O})$ for the collection of* **dense open sets in $\boldsymbol{X}$**.

Remark A.21 Note that because of item (5) in A.18, a set is dense in X iff it has non-empty intersection with every non-empty open in X. In particular, if X is a non-empty space, $\mathfrak{D}(\mathcal{O})$ is a **filter** in $\mathcal{O}$, as in Definition A.40. □

In a certain sense, dual to denseness is the concept of *irreducibility*:

Definition A.22 *Let F be a closed set in a topological space X and $p \in X$.*

[32] In general, interior does not preserve unions and closure does not preserve intersection.

a) p is a **generic point** *for F iff* $\overline{\{p\}} = F$.

b) F is **irreducible** *if it cannot be written as a union of two closed sets distinct from itself.*

c) An **irreducible component** *of X is a maximal*[33] *closed irreducible subset of X.*

Remark A.23 a) The closure of any point is an irreducible closed set. However, there are spaces containing irreducible closed sets which **are not** the closure of a point.

b) Irreducibility is not preserved by intersections, even if finite. □

Lemma A.24 *The following conditions are equivalent for a closed set F in a space X:*

(1) *F is irreducible;*

(2) *If* F_1, F_2 *are closed sets such that* $F = F_1 \cup F_2$, *then* $F_1 \subseteq F_2$ *or* $F_2 \subseteq F_1$.

(3) *If V is an open set and* $V \cap F \neq \emptyset$, *then* $V \cap F$ *is dense in F.*

(4) *If U, V are opens having non-empty intersection with F, then* $U \cap V \cap F \neq \emptyset$.

Proof. It is clear that (1) and (2) are equivalent.

(2) ⇒ (3): Let V be an open set such that $V \cap F \neq \emptyset$. We have $F = \overline{V \cap F} \cup (V^c \cap F)$, and so (2) entails $\overline{V \cap F} = F$ or $V^c \cap F = F$. But the latter equation entails $F \subseteq V^c$, that is, $V \cap F = \emptyset$, a contradiction.

(3) ⇒ (4): If $U \cap F$, $V \cap F \neq \emptyset$, (3) implies $\overline{U \cap F} = \overline{V \cap F} = F$. Let $p \in F$; by A.18.(5), if W is an open set containing p, then $W \cap U \cap F \neq \emptyset$. Let $q \in W \cap U \cap F$; since $q \in \overline{V \cap F}$ and $W \cap U$ is an open neighborhood of q, A.18.(5) implies that $W \cap U \cap V \cap F \neq \emptyset$.[34].

(4) ⇒ (1): Suppose that $F = F_1 \cup F_2$, where F_1, F_2 are closed; then $F \cap F_1^c \cap F_2^c = \emptyset$ and so (4) implies $F \cap F_1^c = \emptyset$ or $F \cap F_2^c = \emptyset$. Hence, $F = F_1$ or $F = F_2$, as needed. □

[33] With respect to inclusion.

[34] We have shown that the intersection of a pair dense opens is a dense open, verifying condition [fil 2] in A.21.(1).

Our next theme is the notion of *compactness*, a generalization of finiteness.

Definition A.25 *A subset A of a topological space X is* **compact** *if every open covering of A*[35] *has a finite sub-covering. Some authors prefer the term* quasi-compact *when X* **is not** *Hausdorff* $(A.19.(c))$. *A subset of X is* **relatively compact** *if its closure is compact.*

Lemma A.26 *Let $\langle X, \mathcal{O} \rangle$ be a topological space and $A \subseteq X$.*

a) The following are equivalent:

(1) *A is compact;*

(2) (The finite intersection property (fip)) *Let $\{F_\lambda : \lambda \in \Lambda\}$ be a collection of closed sets such that for all finite $\alpha \subseteq \Lambda$, $A \cap \bigcap_{\lambda \in \alpha} F_\lambda \neq \emptyset$. Then, $A \cap \bigcap_{\lambda \in \Lambda} F_\lambda \neq \emptyset$.*

b) If $F \subseteq A$, A is compact and F is closed, then F is compact.

c) The finite union of compacts is compact.[36]

d) In a Hausdorff space, all compacts are closed. Hence, in a Hausdorff space, the intersection of compacts is compact.

e) A compact Hausdorff space is normal. □

Our last strictly topological theme is *connectedness*.

Definition A.27 *A subset D of a topological space X is* **disconnected** *if there are open sets A, B such that*

$* \; A \cap B \cap D = \emptyset; \quad * \; A \cap D,\ B \cap D \neq \emptyset; \quad * \; D \subseteq A \cup B.$

A subset of X is **connected** *if it is not disconnected. A subset C of X is a* **connected component** *of X if it is a maximal connected subset of X, that is, it is connected and is not contained in any strictly larger connected subset of X.*

Remark A.28 a) The empty set is connected (hail the laws of logic!).
b) The intersection of connected sets might be disconnected. There are simple examples in the plane ($\mathbb{R}^2$).

[35] An open covering is a collection of opens whose union contains A (of course!).

[36] However, the intersection of compacts might not be compact; the analogy with "finite" has its limits.

c) By their very definition and item (b) in Lemma A.29 below, connected components are always closed, but not necessarily open. As an example, we mention any infinite product of finite discrete spaces; the connected components are its points.

d) By (4) in Lemma A.24, an irreducible closed set (A.22.(b)) is connected. □

Lemma A.29 *Let X be a topological space.*

a) The closure of a connected set is connected.

b) The union of connected subsets of X with non-empty intersection is connected.[37]

c) Distinct connected components of X are closed and disjoint.

d) Every point in X is contained in a unique connected component of X. In particular, X is the disjoint union of its connected components.

□

Remark A.30 If X, Y are topological spaces, a map $f : X \longrightarrow Y$ is **continuous** iff

[cont] For all $W \in \mathcal{O}(Y)$, $f^{-1}(W) \in \mathcal{O}(X)$.

It is straightforward that continuous maps preserve compactness and connectedness, i.e., the continuous image of a compact or a connected set is compact or connected, respectively. □

[37] This is not the most general statement concerning unions, but it will suffice for our purposes.

V Frames and Topology

Let X be a topological space and let $\langle\, \mathcal{O}(X), \subseteq \,\rangle$ be its topology, partially ordered by inclusion. For $\{V_i : i \in I\} \subseteq \mathcal{O}(X)$, set

$$\bigwedge_{i\in I} V_i \;=\; \text{int}\left(\bigcap_{\in I} V_i\right) \quad \text{and} \quad \bigvee_{i\in I} V_i \;=\; \bigcup_{i\in I} V_i,$$

which are clearly in $\mathcal{O}(X)$. Moreover, these are precisely the meet and join of the V_i, in the inclusion partial order **in $\mathcal{O}(X)$**. Hence, with $\emptyset = \bot$ and $X = \top$, $\langle\, \mathcal{O}(X), \subseteq \,\rangle$ is *a complete lattice.* Note that if $U_1, \ldots, U_n$ is a finite subset of $\mathcal{O}(X)$, then A.18.(4) yields

$$\bigwedge_{i=1}^{n} U_i \;=\; \bigcap_{i=1}^{n} U_i.$$

Moreover, it is easily verified that for all U, V_i, $i \in I$, in $\mathcal{O}(X)$

$$[\wedge, \bigvee] \qquad U \wedge \bigvee_{i\in I} V_i \;=\; \bigvee_{i\in I} U \wedge V_i,$$

and so $\mathcal{O}(X)$ is a **frame** [38], i.e., a complete lattice satisfying the $[\wedge, \bigvee]$-distributive law stated above. In particular (see Remark A.9), every frame is a distributive lattice. We may define *negation* and *implication* in the frame $\mathcal{O}(X)$, as follows:

Definition A.31 *If $\langle\, X, \mathcal{O} \,\rangle$ is a topological space and U, $V \in \mathcal{O}$, define* **implication** *by*

$$[\rightarrow] \quad U \rightarrow V \;=\; \text{int}\,(U^c \cup V) \;=\; \bigvee \{W \in \mathcal{O} : W \wedge U \leq V\}.$$

In fact, this last formula defines implication in any *frame. Define* **negation** *and* **equivalence** *in $\mathcal{O}$ by*

$[\neg]$ $\neg U = U \rightarrow \emptyset = \text{int}\, U^c$;

$[\leftrightarrow]$ $U \leftrightarrow V = (U \rightarrow V) \cap (V \rightarrow U)$.

Remark A.32 The class of frames is much larger than those that arise from topological spaces, but the latter subclass is of central importance among all frames. □

The fundamental rules for implication, negation and equivalence are collected in the next result. All are the best possible statements that can be ascertained in general.

Lemma A.33 *If $\langle\, X, \mathcal{O} \,\rangle$ is a topological space and U, V, $W \in \mathcal{O}$, then*

[38] Also *locale*, *complete Heyting algebra* or *complete pseudo-Boolean algebra.*

a) $U \subseteq V \to W$ *iff* $U \cap V \subseteq W$.

b) $U \cap (U \to V) \;=\; U \cap V$.

c) $U \cap V \;=\; \emptyset$ *iff* $U \subseteq \neg V$. *In particular,* $U \cap \neg U \;=\emptyset$.

d) $U \subseteq V \quad \Rightarrow \quad \begin{cases} (1) & W \to U \;\subseteq\; W \to V; \\ (2) & V \to W \;\subseteq\; U \to W; \\ (3) & (U \to V) \;=\; X. \end{cases}$

e) $U \subseteq \neg\neg U \;=\; \mathrm{int}\, \overline{U}$.

f) $U \subseteq V \quad \Rightarrow \quad \begin{cases} (1) & \neg V \;\subseteq\; \neg U; \\ (2) & \neg\neg U \;\subseteq\; \neg\neg V. \end{cases}$

g) $U \cap V = \emptyset$ *iff* $U \cap \neg\neg V = \emptyset$.

h) $W \subseteq (U \leftrightarrow V)$ *iff* $W \cap U = W \cap V$.

Proof. a) Let $U \in \mathcal{O}$; then
$U \subseteq V \to W$ iff $U \subseteq \mathrm{int}(V^c \cup W)$ iff $U \subseteq V^c \cup W$ iff $U \cap V \subseteq W$,
as desired. *The remaining assertions follow from this* **very important** *adjunction property.*

b) From $U \to V \subseteq U \to V$, (a) implies $U \cap (U \to V) \subseteq V$. Hence, $U \cap (U \to V) \subseteq U \cap V$. For the reverse inclusion, note that $(U \cap V) \cap U = U \cap V \subseteq V$, and use (a) to get $U \cap V \subseteq (U \to V)$. Item (c) is just (a) in the case of the implication whose consequent is $\emptyset$. Items (d) and (e) follow from the same technique; for example

$$U \subseteq \neg\neg U \quad \text{iff} \quad U \subseteq (U \to \emptyset) \to \emptyset \quad \text{iff} \quad U \cap (U \to \emptyset) \subseteq \emptyset,$$

which is a consequence of (c). The last equality in (e) is straightforward computation. Item (f).(1) comes from (d).(2) with $W = \emptyset$, while (f).(2) is a consequence of (f).(1).

g) Since $V \subseteq \neg\neg V$, ($\Leftarrow$) is clear. For the converse, if $U \cap V = \emptyset$, the first part of (c) yields $U \subseteq \neg V$. Taking the meet on both sides with $\neg\neg V$, the second part of (c) entails $U \cap \neg\neg V \subseteq \neg V \cap \neg\neg V = \emptyset$, as needed. Item (h) is an immediate consequence of (a). □

Remark A.34 a) All the usual topological spaces ($\mathbb{R}$, $\mathbb{R}^n$, etc.) will provide examples of opens such that $\neg U \;\neq\; U^c$ and $\neg\neg U \;\neq\; U$.

b) If $\mathcal{O}(X) = 2^X$, that is, if X has the discrete topology (all points are open), then

$$\neg U = U^c \quad \text{and} \quad U \to V = U^c \cup V,$$

the classical negation and implication. This is because in this topology **all subsets are open** (and closed). Moreover, the topological meets and joins defined above reduced to the familiar intersections and unions. Thus, the operations in $\mathcal{O}(X)$ *generalize* the standard ones in 2^I. □

Definition A.35 *An open U in X is*

a) **regular** *iff $U = \neg\neg U$*[39]*. Write $Reg(X)$ for the set of regular opens in X.*

b) **clopen** *if U is also closed. Write $\mathfrak{B}(X)$ for the set of clopens in X*[40]*.*

Regarding negation and double negation, we have

Lemma A.36 *Let $\langle X, \mathcal{O} \rangle$ be a topological space. For U, $V \in \mathcal{O}$*

a) $\neg\neg\neg U = \neg U$.

b) $\neg(U \to V) = \neg\neg U \cap \neg V$.

c) $\neg(U \cup V) = \neg U \cap \neg V$.

d) $\neg(U \cap V) = \neg\neg(\neg U \cup \neg V)$.

e) $V \in Reg(X) \quad \Rightarrow \quad (U \to V) \in Reg(X)$.

f) $\neg\neg(U \cap V) = \neg\neg U \cap \neg\neg V$.

g) $\neg\neg(U \to V) = \neg\neg U \to \neg\neg V$.

h) $\neg\neg(U \cup V) = \neg\neg(\neg\neg U \cup \neg\neg V)$.

i) $(U \cup \neg U) \in \mathfrak{D}(\mathcal{O})$*, i.e., it is a dense open set in X.*[41]

j) $U \in \mathfrak{D}(\mathcal{O})$ *iff* $\neg U = \emptyset$ *iff* $\neg\neg U = X$.

k) $\neg U = U^c$ *iff* U *is clopen.* □

[39] Equivalently, U is the interior of a closed set.

[40] This matches precisely the definition of *clopen in the lattice* $\mathcal{O}(X)$, as in A.11; see also A.36.(k).

[41] $\mathfrak{D}(\mathcal{O})$ is the collection of dense opens in X, as in A.20.(b).

The next remark identifies certain Boolean algebras naturally associated to $\mathcal{O}(X)$.

Remark A.37 Let $\langle X, \mathcal{O} \rangle$ be a topological space. As noted in footnote 40 of A.35.(b),

$$\mathfrak{B}(X) = \{C \subseteq X : C \text{ is clopen in } X\}$$

is the BA of clopens of the frame $\mathcal{O}(X)$. Moreover, in $\mathfrak{B}(X)$, the finitary operations of join, meet and complement are precisely the usual ones of union, intersection and complement, respectively. Hence, $\mathfrak{B}(X)$ is a subalgebra of 2^X and of $\mathcal{O}$.

For the regular opens, $Reg(X)$ (A.35.(a)) it is a different story. Clearly, $\mathfrak{B}(X) \subseteq Reg(X)$. Moreover, $Reg(X)$ can be structured as a **complete Boolean algebra**[42] with the following operations: the partial order on $Reg(X)$ is set-theoretical inclusion, while join, meet and negation are given, for U, U_λ, $\lambda \in \Lambda$, in $Reg(X)$, by the rules

$$\bigvee_{\lambda\in\Lambda} U_\lambda \;=_{def}\; \mathrm{int}\left(\overline{\bigcup_{\lambda\in\Lambda} U_\lambda}\right); \qquad \bigwedge_{\lambda\in\Lambda} U_\lambda \;=_{def}\; \mathrm{int}\left(\overline{\bigcap_{\lambda\in\Lambda} U_\lambda}\right);$$
$$\neg U \;=\; \mathrm{int}\, U^c.$$

By A.36.(i), $U \cup \mathrm{int}(U^c) = U \cup \neg U$ is a dense open set. Thus,

$$U \vee \neg U \;=\; \mathrm{int}\,(\overline{U \cup \mathrm{int}\, U^c}) \;=\; \mathrm{int}\, X = X,$$

verifying the law of the excluded middle, characteristic of BAs. Furthermore, if U, $V \in Reg(X)$, A.36.(f) yields $U \wedge V \;=\; U \cap V$, and so finite meets of regular opens is just set-theoretic intersection. □

The basic rules relating negation and the operations $\bigvee$, $\bigwedge$ in a frame Ω are described in

Lemma A.38 *Let Ω be a frame and $S \subseteq \Omega$. Then*

a) $\neg(\bigvee S) \;=\; \bigwedge_{s\in S} \neg s$.

b) $\neg\neg \bigwedge_{s\in S} \neg\neg s \;=\; \bigwedge_{s\in S} \neg\neg s$.[43]

c) $\neg\neg(\bigvee S) \;=\; \neg\neg \bigvee_{s\in S} \neg\neg s \;=\; \neg \bigwedge_{s\in S} \neg s$.

[42] But not, in general, a subalgebra of 2^X or of $\mathcal{O}$!

[43] The meet of regular elements is regular.

Henceforth, $\mathcal{O}(X)$ is considered a frame with the structure defined above.

A.39 Frame Morphisms. The notion of **morphism of frames** comes from Topology. If $f : X \longrightarrow Y$ is a continuous map – as defined in Remark A.30 –, it induces a map

$$f^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X), \quad f^*(W) = f^{-1}(W),$$

that preserves finite meets (i.e., intersections) and arbitrary joins (i.e., unions). Generalizing this situation, a map $h : \Omega_1 \longrightarrow \Omega_2$, where Ω_i are frames, is a **frame morphism** or a **$[\wedge, \bigvee]$-morphism** if it preserves finite meets and arbitrary joins. It should be registered that there are frame-morphisms between topologies that *do not* come from continuous maps. However, in our applications here, frame morphisms of topologies shall be induced by continuous maps. □

VI Filters and Topology

Definition A.40 *Let $\langle X, \mathcal{O}(X) \rangle$ be a non-empty topological space. A* **filter** *in $\mathcal{O}(X)$ is a non-empty subset $\mathcal{F}$ of $\mathcal{O}(X)$, satisfying the following conditions for all A, $B \subseteq I$:*

[fil 1]: $A \in \mathcal{F}$ and $A \subseteq B \;\Rightarrow\; B \in \mathcal{F}$;

[fil 2]: $A, B \in \mathcal{F} \;\Rightarrow\; A \cap B \in \mathcal{F}$.

A filter is **proper** *if it is distinct from $\mathcal{O}(X)$.*

A proper filter is

a) **principal** *if it is of the form $[U] =_{def} \{A \in \mathcal{O}(X) : U \subseteq A\}$, for some $U \in \mathcal{O}(X)$;*[44]

b) **prime** *if for all A, $B \in \mathcal{O}(X)$, $A \cup B \in \mathcal{F} \Rightarrow A \in \mathcal{F}$ or $B \in \mathcal{F}$;*

c) **maximal or an ultrafilter** *if the only filter properly containing it is $\mathcal{O}(X)$.*

Remark A.41 a) All the definitions above apply to the situation where $\mathcal{O}(X) = 2^X$, that is, to the discrete topology on X. This is the *classical* theory of filters; a filter in 2^X is called **a filter on X**.

[44]This is consistent with the notation to be introduced for partial orders in A.3.

b) We observed in A.21 that $\mathfrak{D}(\mathcal{O})$, the family of dense open sets in X, is a filter in X. Note that if the topology on X is discrete, then $\mathfrak{D}(\mathcal{O}) = \{X\}$, i.e., the only dense open in X is X itself.

c) If $\mathcal{F}$ is a filter in $\mathcal{O}$, A.33.(b) and the fact that $\mathcal{F}$ is closed under finite meets entail

$$(\sharp) \qquad U \in \mathcal{F} \text{ and } (U \to V) \in \mathcal{F} \quad \Rightarrow \quad V \in \mathcal{F}. \qquad \square$$

Example A.42 Let X be a topological space and let $x \in X$. Then,

$$\nu_x = \{U \in X : U \text{ is open and } x \in U\}$$

is a filter in $\mathcal{O}(X)$, the *filter of open neighborhoods of x in X*. With the order opposite of inclusion, that is, $U \leq V$ iff $V \subseteq U$, *ν_x is a lattice* because ν_x is closed under finite intersections and unions (it is a filter). The poset $\langle \nu_x, \leq \rangle$ is important in Analysis, Topology and Geometry, being at the root of the notion of **germ** of a map and of **stalk** of a presheaf. $\square$

Proposition A.43 *Let $X \neq \emptyset$ be a topological space, $S \subseteq \mathcal{O}(X)$ and $\mathcal{F}$ a filter in $\mathcal{O}(X)$.*

a) $X \in \mathcal{F}$; $\mathcal{F}$ is proper iff $\emptyset \notin \mathcal{F}$.

b) The set of filters in $\mathcal{O}(X)$ is closed under intersection. The filter generated by $S \subseteq \mathcal{O}(X)$ is

$$\begin{aligned} fil(S) &= \bigcap \{\mathcal{G} : \mathcal{G} \text{ is a filter in } \mathcal{O}(X) \text{ and } S \subseteq \mathcal{G}\} \\ &= \{A \in \mathcal{O}(X) : \exists \text{ a finite } \alpha \subseteq S \text{ such that } \bigcap \alpha \subseteq A\}. \end{aligned}$$

c) The union of a right-directed[45] *family of filters is a filter.*

d) The following are equivalent:

(1) *$fil(S)$ is a proper filter in $\mathcal{O}(X)$;*

(2) (Finite intersection property (fip)) *For all finite $\alpha \subseteq S$, $\bigcap \alpha \neq \emptyset$.*

e) The following are equivalent, for a proper filter $\mathcal{F}$:

(1) *$\mathcal{F}$ is prime and $\mathfrak{D}(\mathcal{O}) \subseteq \mathcal{F}$;*[46]

[45] Under inclusion.

[46] $\mathfrak{D}(\mathcal{O})$ is the filter of dense opens in X; see A.21.

(2) *For all* $A \in \mathcal{O}(X)$, $A \in \mathcal{F}$ *or* $\neg A \in \mathcal{F}$; [47]

(3) $\mathcal{F}$ *is an ultrafilter.*

f) (Stone separation) *If* $\emptyset \neq \alpha \subseteq \mathcal{O}(X)$ *verifies the following conditions:*

(i) *For all* $A, B \subseteq X$, $A, B \in \alpha \Rightarrow A \cup B \in \alpha$;

(ii) $\mathcal{F} \cap \alpha = \emptyset$,

there is a prime filter $\mathcal{P}$ *in* $\mathcal{O}(X)$, *such that* $\mathcal{F} \subseteq \mathcal{P}$ *and* $\alpha \cap \mathcal{P} = \emptyset$.

g) If S *has the fip*[48], *then there is an ultrafilter* $\mathcal{U}$ *in* $\mathcal{O}(X)$, *such that* $S \subseteq \mathcal{U}$.

h) If $A \in \mathcal{O}(X) \setminus \mathcal{F}$, *then there is a prime filter* $\mathcal{P}$ *in* $\mathcal{O}(X)$ *such that* $\mathcal{F} \subseteq \mathcal{P}$ *and* $A \notin \mathcal{P}$.

Proof. This is a collection of statements that apply to distributive lattices, or Heyting algebras when negation is involved. We refer the reader to [BD] or [RS]. □

Example A.44 The filter of open neighborhoods of 0 in the real line $\mathbb{R}$ is an example of a *prime filter* in $\mathcal{O}$ which is not maximal. There are many such examples. □

For the classical case *of filters on a set* as presented in A.41.(a), we have

Corollary A.45 *Let* I *be a set and let* $\mathcal{F}$ *be a proper filter on* I.

a) The following are equivalent:

(1) $\mathcal{F}$ *is prime;* (2) $\mathcal{F}$ *is an ultrafilter;*

(3) *For all* $A \subseteq I$, $A \in \mathcal{F}$ *or* $A^c \in \mathcal{F}$.

b) An ultrafilter contains a finite set iff it is principal. □

VII Quotients by Filters

In this appendix, when proofs are not provided, they can be found in [BD] or [RS].

[47] But not both, since $\mathcal{F}$ is proper! Moreover, $\neg A$ is the negation of A in the frame $\mathcal{O}(X)$, as in section V.

[48] See item (d).(2) of the statement.

Definition A.46 *Let* $\langle X, \mathcal{O} \rangle$ *be a topological space. If* $\mathcal{F}$ *is a filter in* $\mathcal{O}$*, define a relation* $\sim_F$ *on* $\mathcal{O}$*, by*

$$A \sim_F B \quad \text{iff} \quad \exists\, U \in \mathcal{F} \text{ such that } A \cap U = B \cap U.$$

Proposition A.47 *Let* $\langle X, \mathcal{O} \rangle$ *be a topological space and* $\mathcal{F}$ *a filter in* $\mathcal{O}$*.*

a) The relation $\sim_F$ *is a equivalence relation on* $\mathcal{O}$ *and for* $U, V \in \mathcal{O}$,

$$U \sim_F V \quad \text{iff} \quad (U \leftrightarrow V) \in \mathcal{F},$$

where $\leftrightarrow$ *is equivalence in* $\mathcal{O}$ *(see* $[\leftrightarrow]$ *in* A.31).

b) The relation $\sim_F$ *is a* **congruence** *with respect to the* finite *lattice operations on* $\mathcal{O}$*, that is, if* A, B, C, D *are in* $\mathcal{O}$*, then*

$$A \sim_F B \text{ and } C \sim_F D \;\Rightarrow\; \begin{cases} A \cup C & \sim_F \quad B \cup D; \\ A \cap C & \sim_F \quad B \cap D; \\ A \rightarrow C & \sim_F \quad B \rightarrow D; \\ \neg A & \sim_F \quad \neg B. \end{cases}$$

c) The set of equivalence classes of $\mathcal{O}$ *by* $\sim_F$, $\mathcal{O}/\mathcal{F} = \{A/\mathcal{F} : A \in \mathcal{O}\}$, *can be given the quotient structure from* $\mathcal{O}$*, that is*

$*$ *join (corresponding to union):* $A/\mathcal{F} \vee B/\mathcal{F} =_{def} (A \cup B)\mathcal{F}$;

$*$ *meet (corresponding to intersection):* $A/\mathcal{F} \wedge B/\mathcal{F} =_{def} (A \cap B)\mathcal{F}$;

$*$ *negation (corresponding to complement):* $\neg(A/\mathcal{F}) =_{def} \neg A/\mathcal{F}$;

$*$ *implication:* $A/\mathcal{F} \rightarrow B/\mathcal{F} \;=\; (A \rightarrow B)/\mathcal{F}$.

With these operations, $\mathcal{O}/\mathcal{F}$ *is a Heyting algebra, the* **quotient algebra of** $\boldsymbol{\mathcal{O}}$ **by** $\boldsymbol{\mathcal{F}}$*. The natural* **quotient map**

$$\pi_{\mathcal{F}} : \mathcal{O} \longrightarrow \mathcal{O}/\mathcal{F}, \quad U \mapsto U/\mathcal{F},$$

is a morphism, that is, it preserves all the finitary operations in $\mathcal{O}$*.*

d) If $\mathcal{O} = 2^X$*, then* $2^X/\mathcal{F}$ *is a Boolean algebra, the* **quotient algebra of** $\mathbf{2^X}$ **by** $\boldsymbol{\mathcal{F}}$*.*

e) The map $\mathcal{G} \subseteq \mathcal{O}/\mathcal{F} \longmapsto \pi_{\mathcal{F}}^{-1}(\mathcal{G})$ *is an increasing bijection between the filters in* $\mathcal{O}/\mathcal{F}$ *and the filters in* $\mathcal{O}$ *that contain* $\mathcal{F}$*. In particular,* $\mathcal{G}$ *is a proper filter in* $\mathcal{O}/\mathcal{F}$ *iff* $\pi_{\mathcal{F}}^{-1}(\mathcal{G})$ *is a proper filter in* $\mathcal{O}$*.* □

Remark A.48 The partial order in $\mathcal{O}/\mathcal{F}$ is related to the inclusion in $\mathcal{O}$, as follows:

$$\begin{aligned} A/\mathcal{F} \leq B/\mathcal{F} \ \text{ iff } \ & A/\mathcal{F} \wedge B/\mathcal{F} = A/\mathcal{F} \ \text{ iff } \ (A \cap B)/\mathcal{F} = A/\mathcal{F} \\ \text{iff } \ & \exists\, U \in \mathcal{F} \text{ such that } A \cap B \cap U = A \cap U \\ \text{iff } \ & \exists\, U \in \mathcal{F} \text{ such that } A \cap U \subseteq B \cap U. \\ \text{iff } \ & \exists\, U \in \mathcal{F} \text{ such that } A \cap U \subseteq B. \end{aligned}$$

Hence, the class of A is below the class of B in the quotient algebra iff there is a set U in $\mathcal{F}$ such that the part of A inside U is contained in B. It follows that the quotient map $\pi_{\mathcal{F}} : \mathcal{O} \longrightarrow \mathcal{O}/\mathcal{F}$ (A.47.(c)) is increasing. Clearly, this applies just as well to the quotient of 2^X by a filter on X. □

The quotient of $\mathcal{O}(X)$ by $\mathfrak{D}(\mathcal{O})$ is an important construct. The main result is

Theorem A.49 (Glivenko) *Let $\langle X, \mathcal{O} \rangle$ be a topological space and $\mathfrak{D} = \mathfrak{D}(\mathcal{O})$ be the filter of dense elements in X. Then,*

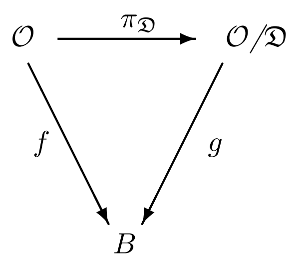
a) $\mathcal{O}/\mathfrak{D}$ is a complete Boolean algebra.

b) The map $\mathfrak{r} : Reg(X) \longrightarrow \mathcal{O}/\mathfrak{D}$, given by $\mathfrak{r}(U) = U/\mathfrak{D}$, is a Boolean algebra **isomorphism** *of $Reg(X)$ onto $\mathcal{O}/\mathfrak{D}$.*

c) The quotient map $\pi_{\mathfrak{D}}$ preserves[49] *arbitrary joins, i.e., for all $S \subseteq \mathcal{O}$, $\pi_{\mathfrak{D}}(\bigvee S) = \bigvee_{s \in S} \pi_{\mathfrak{D}}(s)$.*

d) If B is a complete Boolean algebra and $\mathcal{O} \xrightarrow{f} B$ is a map preserving the finitary operations and arbitrary joins, then there is a unique *Boolean algebra morphism, $g : \mathcal{O}/\mathfrak{D} \longrightarrow B$, such that g preserves arbitrary meets and joins and the following diagram is commutative:*

[49] Besides the finitary operations; see A.47.(c).



e) The map $\mathcal{F} \longmapsto \pi_{\mathfrak{D}}^{-1}(\mathcal{F})$ *is a natural bijective correspondence between the ultrafilters in* $\mathcal{O}/\mathfrak{D}$ *and the ultrafilters in* $\mathcal{O}$. □

VIII Logic and *L*-structures

We assume that the reader is familiar with the basic notions of first-order Logic and Model Theory. General references on this topic are [CK], [Ho], [Kl1] and [BS].

Let L be a first-order language with equality. For an integer $n \geq 1$,
$*$ $rel(n)$ is the set of n-ary relations in L; $*$ $op(n)$ is the set of n-ary function symbols in L;
$*$ Ct is the set of constants in L.

If $\phi(v_1, \ldots, v_n)$ is a formula in L, we follow the usual convention that the free variables in ϕ are **among** the $v_1, \ldots, v_n$. A **sentence** is a formula without free variables. As usual, write

[↔] $$\phi \leftrightarrow \psi \quad \text{for} \quad (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi).$$

Definition A.50 *The set* $\mathcal{T}(L)$ *of* **terms** *of* L *consists of the finite strings of* L*'s alphabet constructed by the following rules*
$*$ *Variables and constants are terms;*
$*$ *If* $\omega \in op(n)$ *and* $\tau_1, \ldots, \tau_n$ *are terms in* L, $\omega(\tau_1, \ldots, \tau_n)$ *is a term.*

The following definition isolates some standard fragments of L.

Definition A.51 *If S is a subset of logical symbols in L,*

$$S \subseteq \{\wedge, \vee, \neg\,, \rightarrow, \exists, \forall\},$$

$L(S)$ is the set of formulas which are logically equivalent to the family of formulas constructed from the atomic ones using only the logical symbols in S. As examples, let ϕ be a formula in L.

a) ϕ is **positive quantifier-free** *if $\phi \in L(\wedge, \vee)$; ϕ is* **quantifier-free** *if $\phi \in L(\wedge, \vee, \neg\,, \rightarrow)$;*

b) ϕ is **positive** *if $\phi \in L(\wedge, \vee, \exists, \forall)$;*

c) ϕ is **existential** *if $\phi \in L(\wedge, \vee, \rightarrow, \neg\,, \exists)$;*

d) ϕ is **universal** *if $\phi \in L(\wedge, \vee, \rightarrow, \neg\,, \forall)$;*

e) ϕ is $\forall\exists$ if it is logically equivalent to a formula $\forall \overline{x}\; \exists \overline{y}\; \psi$, where ψ is quantifier free.

Because of frequent use,

f) Write $L_\exists$ for $L(\wedge, \vee, \neg\,, \rightarrow, \exists)$ and $L_\forall$ for $L(\wedge, \vee, \neg\,, \rightarrow, \forall)$.

Remark A.52 The meaning of expressions such as *positive existential*, *positive* $\forall\exists$, etc., should be clear from the examples in A.51. □

The last theme of this section is a proof-theoretic version of the Classical and Intuitionistic Predicate Calculi with equality. In [Fi], [Kl1], [Kl2] and [Pr] the reader will find a discussion of several proof theory versions of Intuitionism.

A.53 The following is a Hilbert style axiomatization of the (Heyting) Intuitionistic Predicate Calculus. For formulas ϕ, ψ, χ in L,

1. $\phi \rightarrow (\psi \rightarrow \phi)$;
2. $(\phi \rightarrow \psi) \rightarrow ((\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\phi \rightarrow \chi))$;
3. $\phi \rightarrow (\psi \rightarrow \phi \wedge \psi)$;
4. $\phi \wedge \psi \rightarrow \phi$;
5. $\phi \wedge \psi \rightarrow \psi$;
6. $\phi \rightarrow (\phi \vee \psi)$;
7. $\psi \rightarrow (\phi \vee \psi)$;
8. $(\phi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \vee \psi \rightarrow \chi))$;
9. $(\phi \rightarrow \psi) \rightarrow ((\phi \rightarrow \neg\,\psi) \rightarrow \neg\,\phi)$;
10. $\neg\,\phi \rightarrow (\phi \rightarrow \psi)$.

11. If τ is a term free for v in ϕ, [50] $\begin{cases} \text{11.a. } \forall v\ \phi \to \phi(\tau); \\ \text{11.b. } \phi(\tau) \to \exists v\ \phi, \end{cases}$

where $\phi(\tau)$ denotes the substitution of every occurrence of v in ϕ by τ.

12. **Deduction rules**:

$$\text{Modus Ponens: } \frac{\phi,\ \phi \to \psi}{\psi} \qquad \begin{cases} \forall\text{-rule} \quad : \quad \dfrac{\phi \to \psi(v)}{\phi \to \forall v\, \psi(v)} \\ \exists\text{-rule} \quad : \quad \dfrac{\psi(v) \to \phi}{\exists v\, \psi(v) \to \phi,} \end{cases}$$

where in the $\forall$-rule and the $\exists$-rule v must not occur free in ϕ.

The axioms for equality are the usual ones, including the Leibniz substitution rule

[L] If τ is a term in L, free for a variable v in ϕ, then $\phi(v) \wedge (v = \tau) \ \to\ \phi(\tau)$.

In fact, it is enough to assume that [L] holds just for functions and relations symbols in L.

The first ten schemata, together with **Modus Ponens** formalize the Intuitionistic Propositional Calculus. To obtain the Classical Calculi, add (or replace axiom 10 by)

$$10^C.\ \neg\neg\,\phi \to \phi.$$

If $\Gamma \cup \{\phi\}$ is a set of formulas in L,

$$\Gamma \vdash_{\mathcal{H}} \phi \quad \text{and} \quad \Gamma \vdash_C \phi,$$

mean that ϕ is a logical consequence of Γ in the Intuitionistic or Classical Predicate Calculus, respectively, *holding constant all free variables in the formulas of* Γ.[51] With this restriction we have the **Deduction Theorem**, that is,

[DT] $$\Gamma, \phi \vdash_{\mathcal{H}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathcal{H}} \phi \to \psi,$$

with a similar statement holding for $\vdash_C$. □

Proposition A.54 *If ϕ, ψ are formulas in L, then*

a) $\vdash_{\mathcal{H}} \phi \to \neg\neg\,\phi$.

[50] That is, by replacing v by τ in ϕ, no variable in τ becomes bound.

[51] This means that the $\forall$-rule and the $\exists$-rule in (12) are not applied with respect to a free variable in a formula of Γ, except preceding the first occurrence of an element of Γ in the proof ([Kl1], §21, page 107ff). No wonder it is easier to deal with *sentences*.

b) $\vdash_{\mathcal{H}} \neg(\neg\neg\phi) \leftrightarrow \neg\phi$.

c) $\vdash_{\mathcal{H}} \neg\neg(\phi \wedge \psi) \quad \leftrightarrow \quad (\neg\neg\phi \wedge \neg\neg\psi)$.

d) $\vdash_{\mathcal{H}} \neg\neg(\phi \rightarrow \psi) \quad \leftrightarrow \quad (\neg\neg\phi \rightarrow \neg\neg\psi)$.

e) $\vdash_{\mathcal{H}} \neg\neg(\phi \vee \psi) \quad \leftrightarrow \quad \neg\neg(\neg\neg\phi \vee \neg\neg\psi)$. □

For the quantifiers we have

Proposition A.55 *If ϕ is a formula in L,* [52]

a) $\vdash_{\mathcal{H}} \neg\exists v\ \phi \quad \leftrightarrow \quad \forall v\ \neg\phi$.

b) $\vdash_{\mathcal{H}} \neg\neg\forall v\ \neg\neg\phi \quad \leftrightarrow \quad \forall v\ \neg\neg\phi$.

c) $\vdash_{\mathcal{H}} \neg\neg\exists v\ \phi \quad \leftrightarrow \quad \neg\neg\exists v\ \neg\neg\phi \quad \leftrightarrow \quad \neg\forall v\ \neg\phi$.

d) $\vdash_{\mathcal{H}} \neg\forall v\ \neg\neg\phi \quad \leftrightarrow \quad \neg\neg\exists v\ \neg\phi$. □

Definition A.56 *Let L be a first-order language with equality. A* ***L*-structure** *is a non-empty set M together with assignments*

∗ *A n-ary relation, $R^M \subseteq M^n$, the interpretation of $R \in rel(n)$ in M;*[53]

∗ *A n-ary function, $\omega^M : M^n \longrightarrow M$, the interpretation of $\omega \in op(n)$ in M;*

∗ *$c^M \in M$, the interpretation of $c \in Ct$ in M.*

When M is clear from context, we may omit its mention from the notation of interpretation.

Remark A.57 Since we shall be constantly using finite sequences, we adopt current conventions to handle these, namely, if A is a set and $f : A \longrightarrow B$ is a map

∗ $\overline{a} \in A^n$ denotes a n-sequence in A, $\overline{a} = \langle a_1, \ldots, a_n \rangle$;

∗ For $\overline{a} \in A^n$, then $f\overline{a} = \langle fa_1, \ldots, fa_n \rangle \in B^n$. □

A.58 Interpretation of Terms. If M is a L-structure, each term $\tau(v_1, \ldots, v_n)$ in L $(A.50)$ gives rise to a map, $\tau^M : M^n \longrightarrow M$, its **interpretation**, defined by induction on complexity as follows:

[52] The similarity with A.38 is no coincidence; moreover, all rules follow from (a).

[53] The interpretation of equality is **always** the identity, that is, the diagonal of the product $M \times M$.

$*$ For $\tau = c \in Ct$, $\tau^M = c^M$;

$*$ For $\tau = v_n$, $\tau^M =$ the n^{th} projection from M^n to M;

$*$ If $\tau = \omega(\tau_1(v_1, \ldots, v_n), \ \ldots, \ \tau_k(v_1, \ldots, v_n))$, where $\omega \in op(k)$, and $\overline{m} \in M^n$, then $\tau^M(\overline{m}) = \omega^M(\tau_1^M(\overline{m}), \ \ldots, \ \tau_k^M(\overline{m}))$. □

A **theory**, T, in L is a set of sentences (formulas without free variables). A L-structure M is **model of** $\boldsymbol{T}$ iff for all $\sigma \in T$, we have $M \models \sigma$.

Of course, any time we are dealing with classical satisfaction, the corresponding proof-theoretic apparatus is the Classical Predicate Calculus with equality.

We recall the some of the types of morphisms that are current in Model Theory.

Definition A.59 *Let M, N be L-structures and $f : M \longrightarrow N$ be a map.*

a) f is a $\boldsymbol{L}$**-morphism** *iff for all* **atomic** *formulas $\phi(v_1, \ldots, v_n)$ in L and $\overline{m} \in M^n$, $M \models \phi[\overline{m}] \ \Rightarrow \ N \models \phi[f\overline{m}]$.*

b) f is an **embedding** *iff for all* **atomic** *formulas $\phi(v_1, \ldots, v_n)$ in L and $\overline{m} \in M^n$, $M \models \phi[\overline{m}] \ \Leftrightarrow \ N \models \phi[f\overline{m}]$.*

Since L has equality, any embedding is injective.

c) f is an **elementary embedding** *iff for all formulas $\phi(v_1, \ldots, v_n)$ and $\overline{m} \in M^n$, $M \models \phi[\overline{m}] \ \Leftrightarrow \ N \models \phi[f\overline{m}]$.*

d) $\boldsymbol{M}$ **is elementarily equivalent to** $\boldsymbol{N}$*, written $M \equiv N$, if for all* **sentences** *σ in L, $M \models \sigma$ iff $N \models \sigma$.*

L-structures and L-morphisms constitute a category, written $\boldsymbol{L}$ ***mod***.

Remark A.60 a) For a map $f : M \longrightarrow N$, M, N L-structures, to be a L-morphism, it is necessary and sufficient that it preserve relations, operations and constants, i.e., for $n \geq 1$ and $\overline{a} \in A^n$,

$*$ For a n-ary relation R in L, $M \models R[\overline{a}] \Rightarrow N \models R[f\overline{a}]$;

$*$ For a n-ary operation ω in L, $f(\omega^M(\overline{a})) = \omega^N(f\overline{a})$;

$*$ For a constant c in L, $f(c^M) = c^N$.

b) If $f : M \longrightarrow N$ is an embedding, it is common practice to identify M with its image inside N via f and write $M \subseteq N$, read ***M* is a substructure of *N***. Similarly, if f is an elementary embedding, with the above identification ***M* is an elementary substructure of *N***, written $M \preceq N$.

c) In the Classical Predicate Calculus, every formula is equivalent to one in $L_\exists$. It is then straightforward to check that the following are equivalent, where $f : M \longrightarrow N$ is a L-morphism :

(1) f is an elementary embedding;

(2) For all formulas $\phi(v_1, \ldots, v_n)$ in $L_\exists$ and $\overline{x} \in M^n$,
$M \models \phi[\overline{x}] \quad \Rightarrow \quad N \models \phi[f(\overline{x})]$.

(3) For all formulas $\phi(v_1, \ldots, v_n)$ in $L_\exists$ and $\overline{x} \in M^n$,
$N \models \phi[f(\overline{x})] \quad \Rightarrow \quad M \models \phi[\overline{x}]$. □

IX The Gödel Transform

The transform in the title, due to K. Gödel, can be very useful in dealing with model-theoretic questions in an Intuitionistic setting. Many examples can be found in [Br].

Definition A.61 *Let L be a first-order language with equality. We define a map, $(\cdot)^G : L \longrightarrow L$, called the* **Gödel transform**, *by induction on complexity of formulas, as follows:*

(1) *If ϕ is atomic,* $\phi^G = \neg\neg\,\phi$;

(2) $(\phi \wedge \psi)^G = \phi^G \wedge \psi^G$;

(3) $(\phi \vee \psi)^G = \neg\neg\,(\phi^G \vee \psi^G)$;

(4) $(\phi \rightarrow \psi)^G = \phi^G \rightarrow \psi^G$;

(5) $(\neg\,\phi)^G = \neg\,\phi^G$;

(6) $(\exists v\, \phi)^G = \neg\neg\,\exists v\ \phi^G$;

(7) $(\forall v\, \phi)^G = \forall v\ \phi^G$.

Clearly, ϕ^G has the same free and bound variables as ϕ. If Γ is a set of formulas in L, define $\Gamma^G = \{\phi^G : \phi \in \Gamma\}$.

Lemma A.62 *For all formulas ϕ in L,*

a) $\vdash_C \phi \leftrightarrow \phi^G$.[54] *b)* $\vdash_{\mathcal{H}} \neg\neg\,\phi^G \leftrightarrow \phi^G$.

c) If ϕ is an axiom of the Classical *Predicate Calculus* (*A*.53)*, then* $\vdash_{\mathcal{H}} \phi^G$.

Proof. Item (a) is a consequence of the fact that in classical logic ϕ is equivalent to $\neg\neg\,\phi$.

b) Since $\vdash_{\mathcal{H}} \psi \to \neg\neg\,\psi$ (A.54.(a)), it is enough to check that $\vdash_{\mathcal{H}} \neg\neg\,\phi^G \to \phi^G$. We proceed by induction on complexity, using "=" in place of $\leftrightarrow$ to ease readability. If ϕ is atomic, the result follows from A.54.(b). The induction steps for negation, conjunction and implication follow from the definition in A.61 and the corresponding items in A.54, that is, (b), (c) and (d). For conjunction, the definition of the Gödel transform together with (b) and (e) in A.54, yield

$$\neg\neg\,(\phi \vee \psi)^G = \neg\neg\,(\neg\neg\,(\phi^G \vee \psi^G)) = \neg\neg\,(\phi^G \vee \psi^G) = (\phi \vee \psi)^G.$$

The same technique goes through the induction step for the existential quantifier. Finally, induction and A.55.(b) yield[55] $\neg\neg\,(\forall v\,\phi)^G = \neg\neg\,\forall v\ \phi^G = \neg\neg\,\forall v\ \neg\neg\,\phi^G = \forall v\ \neg\neg\,\phi^G = \forall v\ \phi^G$, establishing (b).

c) With the same numbering as in A.53, inspection shows that axioms (1), (2), (3), (4), (5), (9) and (10) remain valid after taking the Gödel transform. For (10_C), we have

$$(\neg\neg\,\phi \to \phi)^G = \neg\neg\,\phi^G \to \phi^G,$$

which is intuitionistically valid by (b). For Axiom (6), we have

$$(\phi \to (\phi \vee \psi))^G = \phi^G \to (\phi \vee \psi)^G = \phi^G \to \neg\neg\,(\phi^G \vee \psi^G).$$

Since $\vdash_{\mathcal{H}} \chi \to \neg\neg\,\chi$ and axiom (6) implies $\phi^G \to (\phi^G \vee \psi^G)$, we get $\phi^G \to \neg\neg\,(\phi^G \vee \psi^G)$, as needed. The same argument applies to axiom (7). For (8), first recall that for all formulas ψ, χ in L

$$\vdash_{\mathcal{H}} \psi \to \chi \quad \text{and} \quad \vdash_{\mathcal{H}} \chi \leftrightarrow \neg\neg\,\chi \quad \Rightarrow \quad \vdash_{\mathcal{H}} \neg\neg\,\psi \to \chi. \qquad (\sharp)$$

By axiom (8), we have

$$(\phi^G \to \chi^G) \to ((\psi^G \to \chi^G) \to (\phi^G \vee \psi^G \to \chi^G)). \qquad (\sharp\sharp)$$

[54] $\vdash_C$ corresponds to proof in classical logic as in A.53. As above, $\leftrightarrow$ is equivalence.

[55] Intuitionistically, equivalents may be substituted for each other in any formula to yield equivalent formulas.

However, the Gödel transform of (8) is

$$(\phi^G \to \chi^G) \to ((\psi^G \to \chi^G) \to (\neg\neg(\phi^G \vee \psi^G) \to \chi^G)). \qquad (\sharp\sharp\sharp)$$

By (b), $\vdash_{\mathcal{H}} \chi^G \leftrightarrow \neg\neg\chi^G$, and so ($\sharp$) shows that ($\sharp\sharp$) entails ($\sharp\sharp\sharp$), as needed. Since the Gödel transform does not change free or bound occurrences of variables, it is clear that is preserves axiom 11.a. For 11.b, we may apply it to ϕ^G to obtain $\phi^G(\tau) \to \exists v\, \phi^G$, which implies, $\phi^G(\tau) \to \neg\neg\exists v \phi^G = (\phi(\tau) \to \exists v\, \phi)^G$, ending the proof. $\square$

Theorem A.63 (Gödel) *If* $\Gamma \cup \{\phi\}$ *is a set of formulas in* L*, then* $\Gamma \vdash_C \phi \quad \Leftrightarrow \quad \Gamma^G \vdash_{\mathcal{H}} \phi^G$.

Proof. Since ϕ^G is classically equivalent to ϕ (A.62.(a)) and any intuitionistic proof is a classical proof it is sufficient to show ($\Rightarrow$). Let $\{\psi_1, \ldots, \psi_n\}$ be a classical proof of ϕ from Γ. We shall verify that $\{\psi_1^G, \ldots, \psi_n^G\}$ is an intuitionistic proof of ϕ^G from Γ^G, by induction on length and the reason for including each ψ_k in the original proof. Clearly, this holds true if any of the ψ_i are in Γ. The case in which ψ_k is an axiom was taken care of by A.62.(c). It remains to check the passage through the deduction rules in A.53.(12). We use equality in place of equivalence to ease presentation. One should keep in mind that the Gödel transform preserves bound and free occurrences of all variables.

$*$ ψ_m follows from an application of *Modus Ponens.* In this case, there are k, $l < m$ such that ψ_l is $(\psi_k \to \psi_m)$. By induction, we have ψ_k^G and

$$\psi_l^G = (\psi_k \to \psi_m)^G = \psi_k^G \longrightarrow \psi_m^G,$$

and an application of *Modus Ponens* yields ψ_m^G, as needed.

$*$ ψ_m follows by an application of the $\forall$-rule A.53.(12). Therefore, for some $k < m$, ψ_k is $(\chi \to \phi)$ and ψ_m is $(\chi \to \forall v\, \phi)$, where v is not free in χ. By induction, we have

$$(\chi \to \phi(v))^G = \chi^G \to \phi^G(v),$$

where v is not free in χ^G. Hence, the $\forall$-rule yields

$$\chi^G \to \forall v\, \phi^G = (\chi \to \forall v \phi)^G = \psi_m^G,$$

as needed.

$*$ ψ_m follows from an application of the $\exists$-rule A.53.(12). As above, for some $k < m$, ψ_k is $(\phi(v) \to \chi)$ and ψ_m is $(\exists v\ \phi \to \chi)$, where v is not free in χ. By induction, we have

$$(\phi(v) \to \chi)^G = \phi^G(v) \to \chi^G,$$

with v not occurring free in χ^G. An application of the $\exists$-rule yields $\exists v\ \phi^G \to \chi^G$. Now items (a) and (d) in A.54, imply, in view of A.62.(b)

$$\begin{aligned} \neg\neg(\exists v\ \phi^G \to \chi^G) &= \neg\neg\exists v\ \phi^G \to \neg\neg\chi^G = \neg\neg\exists v\ \phi^G \to \chi^G \\ &= (\exists v\, \phi)^G \to \chi^G = (\exists v\, \phi \to \chi)^G = \psi_m^G, \end{aligned}$$

completing the proof. □

Remark A.64 In general, ϕ^G **is not** even (intuitionistically) equivalent to $\neg\neg\phi$. As an example, consider $\phi \equiv \forall v(R(v) \vee \neg R(v))$. Note that

$$\phi^G = \forall v\ \neg\neg(\neg\neg R(v) \vee \neg R(v)).$$

By Theorem A.63, ϕ^G is an intuitionistic tautology, while $\neg\neg\phi$ *is not* (see 5.3). However there is a significant fragment of L for which this is true, as shown by the next result. □

Lemma A.65 *If ϕ is a formula in $L_\exists$, then $\vdash_{\mathcal{H}} \phi^G \leftrightarrow \neg\neg\phi$.*

Proof. Recall $L_\exists$ consists of the formulas constructed from the atomic using the *all* the propositional connectives and $\exists$ (A.51). Just proceed by induction using A.54 and A.55. □

X Products and Reduced Products

Let M_i, $i \in I$, be a family of L-structures and $M = \prod_{i \in I} M_i$ be their set-theoretical product. M becomes a L-structure as follows: for $f_1, \ldots, f_n \in M^n$,

$*$ If R is a n-ary relation in L, then

$$M \models R[f_1, \ldots, f_n] \quad \text{iff} \quad \forall\, i \in I, \quad M_i \models R[f_1(i), \ldots, f_n(i)];$$

$*$ If ω is a n-ary operation in L, then for each $i \in I$,

$$\omega(f_1, \ldots, f_n)(i) = \omega(f_1(i), \ldots, f_n(i));$$

$*$ If $c \in Ct$, $c^M = \langle c^{M_i} \rangle$, that is, the I-sequence whose i^{th}-coordinate is the interpretation of c in M_i.

It is straightforward that the canonical projections, $\pi_i : M \longrightarrow M_i$, $\pi_i(f) = f(i)$, are L-morphisms. With this structure, *M is the product of the M_i in the category **L mod**.*

Definition A.66 *Let M_i, $i \in I$, be a family of L-structures and M be their product, as above. Let $\phi(v_1, \ldots, v_n)$ be a formula in L and $\overline{f} \in M^n$. The* **Feferman-Vaught value** *of ϕ is the map*

$$\mathfrak{v}\phi : M^n \longrightarrow 2^I, \quad \text{given by} \quad \mathfrak{v}\phi(\overline{f}) = \{i \in I : M_i \models \phi[\overline{f}(i)]\}$$

where $\overline{f}(i) = \langle f_1(i), \ldots, f_n(i) \rangle$. The Feferman-Vaught value of **equality** *is written*

$$[\![f = g]\!] = \{i \in I : f(i) = g(i)\}.$$

If $\overline{f}$, $\overline{g} \in M^n$, we extend the preceding notation by setting $[\![\overline{f} = \overline{g}]\!] = \bigcap_{k=1}^{n} [\![f_k = g_k]\!]$.

Let M_i, $i \in I$, be a family of L-structures and let F be a proper filter in I.

Definition A.67 *Define a relation E_F on $M = \prod_{i \in I} M_i$ by*

$$f \; E_F \; g \quad \text{iff} \quad [\![f = g]\!] \in F.$$

If $\overline{f}$, $\overline{g} \in M^n$, then $\overline{f} \; E_F \; \overline{g}$ means $\forall\, 1 \le k \le n$, $f_k \; E_F \; g_k$.

Lemma A.68 *Notation as above, if $\overline{f}$, $\overline{g} \in M^n$, then $\overline{f} \; E_F \; \overline{g}$ iff $[\![\overline{f} = \overline{g}]\!] \in F$.* □

Lemma A.69 *a) E_F is a L-congruence, that is, an equivalence relation, such that for all terms $\tau(v_1, \ldots, v_n)$ and atomic formulas $\phi(v_1, \ldots, v_n)$ in L, and $\overline{f}$, $\overline{g} \in M^n$*

$$\overline{f} \; E_F \; \overline{g} \quad \Rightarrow \quad \begin{cases} (1) \quad [\![\tau(\overline{f}) = \tau(\overline{g})]\!] \in F; \\ (2) \quad \mathfrak{v}\phi(\overline{f}) \sim_F \mathfrak{v}\phi(\overline{g}), \end{cases}$$

where $\sim_F$ is the congruence generated by F in 2^I, as in Definition A.46.
b) For $\overline{f}$, $\overline{g} \in M^n$, if $\overline{f} \; E_F \; \overline{g}$ and $\phi(v_1, \ldots, v_n)$ is an atomic formula in L, then

$$\mathfrak{v}\phi(\overline{f}) \in F \quad \text{iff} \quad \mathfrak{v}\phi(\overline{g}) \in F.$$

Proof. a) For (1), by induction on complexity of terms, it is enough to check that if $\omega \in op(n)$ and $\overline{f}\ E_F\ \overline{g}$, then $[\![\omega(\overline{f}) = \omega(\overline{g})]\!] \in F$. By Lemma A.68, $\overline{f}\ E_F\ \overline{g}$ iff $[\![\overline{f} = \overline{g}]\!] \in \mathcal{F}$. But note that

$$[\![\overline{f} = \overline{g}]\!] \subseteq [\![\omega(\overline{f}) = \omega(\overline{g})]\!]$$

and so, F being a filter, it follows that $[\![\omega(\overline{f}) = \omega(\overline{g})]\!] \in F$. To prove (2), recall that an atomic formula in L is a formula of the type $\phi(v_1, \ldots, v_n) \equiv R(\tau_1(v_1, \ldots, v_n), \ldots, \tau_k(v_1, \ldots, v_n))$, where R is a k-ary relation in L. Now note that $\mathfrak{v}\phi(\overline{f}) \cap \bigcap_{l=1}^{k} [\![\tau_l(\overline{f}) = \tau_l(\overline{g})]\!] = \mathfrak{v}\phi(\overline{g}) \cap \bigcap_{l=1}^{k} [\![\tau_l(\overline{f}) = \tau_l(\overline{g})]\!]$, and so (1), the closure of F under finite intersections and A.46 entail $\mathfrak{v}\phi(\overline{f}) \sim_F \mathfrak{v}\phi(\overline{g})$. Item (b) is straightforward from (a) and the fact that F is a filter. $\square$

Henceforth, we shall write the class of an element $f \in M = \prod_{i\in I} M_i$ under $\sim_F$ as f/F. Similarly, if $\overline{f} \in M^n$, then $\overline{f}/F = \langle f_1/F, \ldots, f_n/F \rangle$.

Let $M/F = \prod_{i\in I} M_i/F$ be the set of equivalence classes of M under E_F. We make M/F into an L-structure as follows:

$*$ If $c \in Ct$, its interpretation is $\langle c^{M_i} \rangle/F$;

$*$ If $\omega \in op(n)$, its interpretation is the map $\overline{f}/F \longmapsto \langle \omega(\overline{f}(i)) \rangle/F$;

$*$ If $R \in rel(k)$ and $\overline{f} \in M^n$, then $M/F \models R[\overline{f}/F]$ iff $\mathfrak{v}R(\overline{f}) \in F$.

By Lemma A.69, this definition is independent of representatives. Moreover,

Corollary A.70 *For all atomic formulas $\phi(v_1, \ldots, v_n)$ in L and $\overline{f} \in \left(\prod_{i\in I} M_i\right)^n$,*

$$\prod_{i\in I} M_i/F \models \phi[\overline{f}/F] \quad \textit{iff} \quad \mathfrak{v}\phi(\overline{f}) \in F. \qquad \square$$

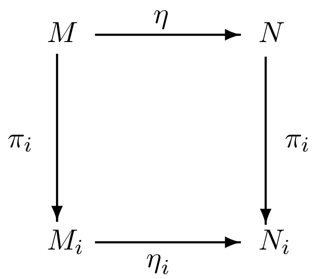
Definition A.71 *The L-structure $\prod_{i\in I} M_i/F$ is the* **reduced product** *of the family M_i by F. When* ***F* is an ultrafilter** *this reduced product is called an* **ultraproduct** *of the M_i. If all M_i are the same structure, these constructions are referred to as* **reduced power** *and* **ultrapower** *by F, respectively, written M^I/F.*

Remark A.72 Let M_i, N_i, $i \in I$, be families of L-structures. Let $\eta_i : M_i \longrightarrow N_i$, $i \in I$, be L-morphisms.

a) The η_i's induce a *unique* L-morphism

$$\eta : \prod_{i \in I} M_i \longrightarrow \prod_{i \in I} N_i, \text{ given by } \eta(f) = \langle\, \eta_i(f(i))\, \rangle_{i \in I},$$

that makes the following diagram commute, for all $i \in I$,



where $M = \prod_{I \in I} M_i$ and $N = \prod_{i \in I} N_i$ and each π_i is the canonical projection. The map η is the **product** of the η_i, written $\prod \eta_i$.

b) If F is a filter on I, the η_i induce a L-morphism of reduced products

$$\eta/F : M/F \longrightarrow N/F, \text{ given by } \eta/F(f/F) = \eta(f)/F.$$

There is a fundamental result, due to J. Łós, characterizing satisfaction in a ultraproduct:

Theorem A.73 (Łós) *Let M_i, $i \in I$, be a family of L-structures and F an ultrafilter in I. If $\phi(v_1, \ldots, v_n)$ is a formula in L and $\overline{f} \in \left(\prod_{i \in I} M_i\right)^n$, then*

$$\prod_{i \in I} M_i/F \models \phi[\overline{f}/F] \quad \textit{iff} \quad \mathfrak{v}(\overline{f}) \in F.$$

Proof. Induction on complexity of formulas; the ultrafilter property in Corollary A.45 is essential to get through the induction step involving negation. One can also consult [BS] or [CK]. □

Corollary A.74 *Let M be a L-structure, I a set and let F an ultrafilter on I.*

a) The diagonal map $\Delta(m) = \langle\, m\, \rangle/F$, from M into M^I/F, is an elementary embedding.

b) $M \equiv M^I/F$, that is, M is elementarily equivalent to any of its ultrapowers.

Remark A.75 Let $\langle I, F \rangle$ be an *filter pair*, that is, F is a proper filter in I; this pair determines a **covariant functor**,

$$(\cdot)^I/F : \boldsymbol{L\,mod} \longrightarrow \boldsymbol{L\,mod},$$

given, notation as in A.72, by:

$$M \longmapsto M^I/F \quad \text{and} \quad M \xrightarrow{f} N \longmapsto M^I/F \xrightarrow{f^I_F} N^I/F.$$

In particular, if $\langle I, F \rangle$ is an *ultrafilter pair*, the ultrapower construction using this pair is a covariant functor from $\boldsymbol{L\,mod}$ to $\boldsymbol{L\,mod}$. □

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