CDD: 511.3

DECIDABILITY AND GÖDEL INCOMPLETENESS IN AF C*-ALGEBRAS*

DANIELE MUNDICI

Department of Mathematics "Ulisse Dini"
University of Florence
Viale Morgagni 67/A
50134 FLORENCE
ITALY

mundici@math.unifi.it

Em homenagem aos 60 anos da
Professora Itala M. Loffredo D'Ottaviano

Abstract: In the algebraic treatment of quantum statistical systems, the claim "Nature does not have ideals" is sometimes used to convey the idea that the C*-algebras describing natural systems are *simple,* i.e., they do not have nontrivial homomorphic images. Using our interpretation of AF C*-algebras as algebras of Łukasiewicz calculus, in a previous paper the claim was shown to be incompatible with the existence of a Gödel incomplete AF C*-algebra for a quantum physical system existing in nature. In this note we survey recent developments on Gödel incompleteness and decidability issues for AF C*-algebras.

Key-words: MV-algebra. AF C*-algebra. Gödel incompleteness. Łukasiewicz calculus. Many-valued logic. Decision problem.

---

*2000 *Mathematics Subject Classification.* Primary: 06D35, 46L80. Secondary: 06F20, 03D15, 03B50.

1 Recapitulation

This paper is a continuation of [29]. For background on quantum mechanics and MV-algebras we refer to [13] and [9] respectively. Observables and pure states of quantum mechanical systems with finitely many degrees of freedom are represented by self-adjoint operators and normalized vectors in a Hilbert space, respectively. This representation is based on the assumption that the operators q and p for position and momentum satisfy the canonical commutation relation $pq - qp = -i\hbar I$ where I is the identity operator and $\hbar$ is Plank constant divided by 2π. Von Neumann's uniqueness theorem states that all irreducible representations satisfying the above relation are equivalent to the Schrödinger representation on the Hilbert space of square integrable functions of the particle. This justifies the standard practice of using the Schrödinger representation on *the* Hilbert space of the system.

By contrast, for a system S with infinitely many degrees of freedom, (e.g., a system arising in quantum statistical mechanics or in relativistic quantum field theory) von Neumann's uniqueness theorem fails in general: S has many inequivalent representations, in correspondence with its macroscopically different classes of states, and one can no longer speak of *the* Hilbert space of S. The C*-algebraic formulation turns out to yield the appropriate framework for S. Here observables are identified with self-adjoint elements of a C*-algebra $A = A_S$, and states with normalized positive linear functionals on A.

In the particular case of a quantum statistical mechanical system S one usually neglects finite-size effects, [34]: while the observables of S in any bounded region j are those of a finite system $S(j)$, S itself is the "thermodynamic limit" of the $S(j)$'s. The observables of S are constructed from the self-adjoint elements of an abstract C*-algebra, given by the following

Definition 1.1 [4, 5, 16] An AF (*approximately finite-dimensional*) C*-algebra is the norm closure of the union of an ascending sequence of finite dimensional C*-algebras, all with the same unit.

AF C*-algebras include Glimm's UHF (uniformly hyperfinite) algebras, which yield the standard tool for the algebraization of certain quantum spin systems [5, 13, 34]. Among them, the Canonical Anticommutation Relation (CAR) algebra provides a mathematical description of the ideal Fermi gas. While UHF algebras are simple (i.e., without nontrivial ideals), general AF C*-algebras A have a rich ideal structure. Following Fell [14], two representations are said to be "physically", or "weakly" equivalent if they have the same kernel. In this way *different primitive ideals (= kernels of irreducible representations) of A correspond to physically inequivalent irreducible representations.* It turns out that the C*-algebras considered in mathematical physics are usually simple—they do not have any (nontrivial) ideal. This state of affairs is summarized by Kastler's claim [17, pp. 851 ff], [19, p.468], "Nature does not have ideals".

2 AF C*-algebras as Lindenbaum algebras of ∞-valued logic

Elliott [12] classified AF C*-algebras A in terms of certain partially defined abelian monoids $D(A)$. Subsequent work by Effros, Handelman, Shen, Goodearl and others [11, 16], showed that $D(A)$ can be replaced by a countable, directed, unperforated, partially ordered abelian group having the Riesz interpolation property together with a distinguished order-unit: the classifying functor is (an order-theoretic refinement of) Grothendieck's group K_0.

In his paper [23] the author established a categorical equivalence Γ between the variety of MV-algebras and the category of unital ℓ-groups. Thus, the composite functor $\Gamma \circ K_0$ yields an MV-algebraic classification of those AF C^*-algebras whose Murray-von Neumann order of projections is a lattice. Here is a precise result:

Theorem 2.1 [23, 31]

(i) For every AF C-algebra A there is at most one associative commutative monotone extension $\oplus$ of Elliott's partial operation, having the property that, for every projection $p \in A$, among all Murray-von Neumann classes $[q]$ such that $[p] + [q] = [I]$ there is a smallest one, denoted $\neg[p]$, namely the class $[I - p]$.*

(ii) *Such a unique extension exists if and only if the Murray-von Neumann order of A is a lattice. In this case the resulting structure* $B(A) = (D(A), 0, \neg, \oplus)$ *is a countable MV-algebra.*

(iii) *Conversely, every countable MV-algebra B is the set of equivalence classes of projections of a unique AF C*-algebra $A = A(B)$ whose Murray-von Neumann order is a lattice, in such a way that Elliott's partial addition in A agrees with the addition operation of B.*

(iv) *The maps $A \mapsto B(A)$ and $B \mapsto A(B)$ are inverses of each other and yield a one-one correspondence between AF C*-algebras whose Murray-von Neumann orders are lattices, and countable MV-algebras.*

(v) *Under this correspondence, commutative AF C*-algebras correspond to Boolean algebras, Glimm's UHF algebras correspond to rational subalgebras of* $[0, 1]$*, finite-dimensional C*-algebras correspond to finite MV-algebras.*

AF C^*-algebras whose Murray-von Neumann order of projections is a lattice include many physically relevant AF C*-algebras, e.g., the CAR algebra and, more generally, all of Glimm's UHF algebras. They also include the Effros-Shen C*-algebras $\mathfrak{F}_\rho$ for ρ irrational, [11, p.65], which—as we shall see below—play an interesting role in topological dynamics, the Behncke-Leptin C*-algebras with a two-point dual [1, 27], and all liminary C*-algebras with separable Boolean spectrum [10] (also see [26] for an important particular case). Trivially, they include all commutative AF C*-algebras, as well as the "universal" AF C*-algebras $\mathfrak{M}$ and $\mathfrak{M}_1$ of [23] and [25].

The suggestive claim that AF C*-algebras are "noncommutative zero-dimensional spaces", [11, p.66], [2, p.53], can be extended by saying that AF C*-algebras are the algebras of infinite-valued logic, [28]: indeed, by Theorem 2.1, every MV-algebra $B(A)$ is the Lindenbaum algebra of some (deductively closed) theory $\Theta = \Theta(A)$ in the infinite-valued calculus of Łukasiewicz. While $\Theta(A)$ is just a countable string of symbols from a finite alphabet, from $\Theta(A)$ one can unambiguously reconstruct the AF C*-algebra A.

Since every theory Θ in the infinite-valued Łukasiewicz calculus uniquely determines an AF C*-algebra $A(\Theta)$, the complexity of the decision problem of Θ impinges upon the algebraic structure of $A(\Theta)$. Thus for instance, $A(\Theta)$ is simple if and only if Θ is maximally consistent, [23].

We naturally say that an AF C*-algebra A is *Gödel incomplete* if $A = A(\Theta)$ for some undecidable and recursively enumerable theory Θ.

In [23, 6.1] it is proved

Theorem 2.2 *Every Gödel incomplete AF C*-algebra A has a nonzero ideal. Thus Gödel incompleteness never affects simple AF C*-algebras.*

We also have [32]:

Theorem 2.3 *Fix an integer $n \geq 1$, and let Θ be a recursively enumerable* prime *theory in the infinite-valued calculus of Łukasiewicz with n variables. In other words, for any two formulas ϕ and ψ, either $\phi \to \psi$ or $\psi \to \phi$ belongs to Θ. Then Θ is decidable.*[1] *In other words, Gödel incompleteness never affects those AF C*-algebras with comparability of projections whose Grothendieck group is finitely generated.*

Most AF C*-algebras existing in the mathematical-physical literature [5, 13, 34] are indeed simple, and hence, by Theorem 2.2 they are not Gödel incomplete.

A different class of AF C*-algebras $A = A(\Theta)$ without Gödel incompleteness arises when Θ is decidable.[2] Such algebras abound in the literature. Thus for instance, references [24, 27] provide many examples of AF C*-algebras $A = A(\Theta)$ where Θ is decidable in polynomial time:

Theorem 2.4 *The following AF C*-algebras have the form $A = A(\Theta)$ where the decision problem of Θ is solvable in polynomial time:*

(i) Each AF C-algebra corresponding to a free MV-algebra, including the universal C*-algebra $\mathfrak{M}$ of* [23].[3]

[1]This result ceases to exist if Θ has infinitely many variables [32].

[2]Note that the decision problem of Θ is the same as the problem of computing identities between Murray-von Neumann equivalence classes of projections in A.

[3]Every AF-algebra whose Murray-von Neumann order of projections is a lattice, is a quotient of the AF-algebra $\mathfrak{M}$ described in [23, Section 8].

(ii) *Every finite-dimensional C*-algebra.*

(iii) *The universal UHF algebra.*

(iv) *The CAR algebra.*

(v) *Each Effros-Shen algebra* $\mathfrak{F}_\sigma$*, for* σ *a quadratic irrational, or* $\sigma = 1/e$.

Summing up, none of the AF C*-algebras A considered in [5, 13, 34, 24, 27] is affected by Gödel incompleteness—and a fortiori, none of these algebras is *essentially* Gödel incomplete. Suppose, however, a certain quantum system S is described by an essentially Gödel incomplete AF C*-algebra $A = A(\Theta)$.[4] Then A has at least two physically inequivalent Hilbert space representations. Since A does not represent a maximum information system, one should rather consider all possible simple algebras $A' = A/M$, where M ranges over maximal ideals M of A. Any such M canonically determines a theory Θ_M. The maximality of M is reflected by Θ_M being maximally consistent. In other words, no consistent theory strictly contains Θ_M. Let $A' = A(\Theta_M)$. The completion process of Θ by Θ_M parallels the surjection of A onto A/M. By definition of essential Gödel incompleteness, for *every* maximal ideal M of A, the theory Θ_M cannot be recursively enumerable. In the completion process $\Theta_M \supseteq \Theta$ the recursive enumerability of Θ is irreparably lost.

3 Decision problems for finitely presented AF C*-algebras

In the previous sections we have considered decision problems for a single AF C*-algebra $A = A(\Theta)$, in terms of the complexity of its associated theory Θ in Łukasiewicz calculus. Decidability and undecidability problems naturally arise also for classes of AF C*-algebras

[4] The AF C*-algebra of all complex-valued continuous functions over the Stone space of the Lindenbaum algebra of Peano arithmetic provides a straightforward example of essentially Gödel incomplete *commutative* AF C*-algebra.

as described by their (Bratteli) diagrams.[5]

In his monograph [2, p.55] Blackadar (using the informal notion of "reasonable algorithm") notes

> ...one major problem restricts the usefulness of the study of AF C*-algebras by diagrams: many quite different diagrams yield isomorphic algebras, and there is no known reasonable algorithm for determining when two diagrams give isomorphic algebras.

As we will see in Theorem 3.1 below, the Turing undecidability of the isomorphism problem for AF C*-algebras does not depend on the fact that Bratteli diagrams (as well as theories in Łukasiewicz logic) yield a presentation of AF C*-algebras in terms of *infinite* strings of symbols. As a matter of fact, in [30] the author associates to every *finite* abstract simplicial complex $\mathcal{C}$ a stable[6] AF C*-algebra $A(\mathcal{C})$. Two abstract simplicial complexes $\mathcal{C}$ and $\mathcal{C}'$ are said to be *C^*-equivalent* if their associated stable AF-algebras $A(\mathcal{C})$ and $A'(\mathcal{C})$ are isomorphic.[7] The main result of [30] is the Gödel-incompleteness (whence, the undecidability) of the isomorphism problem for the subclass of stable AF C*-algebras that are presentable by abstract simplicial complexes: [8]

Theorem 3.1 *The set of pairs of abstract simplicial complexes representing isomorphic stable AF-algebras can be effectively enumerated. On the other hand, there is no algorithm to decide isomorphism of stable AF-algebras associated to abstract simplicial complexes.*

[5] These diagrams yield the first string-theoretic presentation of AF C*-algebras, [4].

[6] Recall that an an AF C*-algebra A is said to be *stable* if it is isomorphic to its tensor product with the compact operators on a separable Hilbert space. Grothendieck's K_0-functor transforms every stable AF C*-algebra A into a countable dimension group $(K_0(A), K_0(A)^+)$, in such a way that isomorphism classes of stable AF-algebras are in 1-1 correspondence with isomorphism classes of countable dimension groups.

[7] Every stable AF-algebra whose Bratteli diagram is generated by an abstract simplicial complex is a quotient of the "universal" AF C*-algebra $\mathfrak{M}$ of [23].

[8] As shown in [30], a stable AF-algebra A arises from an abstract simplicial complex $\mathcal{C}$ if and only if its associated dimension group $(K_0(A), K_0(A)^+)$ is a finitely generated projective ℓ-group having infinitely many maximal ℓ-ideals.

Abstract simplicial complexes are not the only tool for giving finite presentations of classes of AF C*-algebras. Thus for instance, integer matrices are used in [6, 7, 8]. A matrix M is said to be *primitive* if all sufficiently high powers of M have all entries > 0. By an *abacus* we mean a square, nonsingular, integer, primitive matrix. In [6] any abacus M is canonically associated to a stable AF C*-algebra $A(M)$. Two abaci are said to be *C^*-equivalent* if their associated AF C*-algebras are isomorphic. In sharp contrast with the second statement of Theorem 3.1, in [7, 8] it is proved

Theorem 3.2 *There exists an algorithm to decide isomorphism of AF C*-algebras associated to abaci.*

Via Grothendieck's functor K_0 this result also yields a decision procedure for isomorphisms of the ordered simple dimension groups associated to these AF C*-algebras; this class of groups is important for a variety of other problems, especially in symbolic and topological dynamics, [33, 18, 3, 22, 20, 21].

Putting together Theorems 3.1 and 3.2 we have [30]:

Corollary 3.3 *Let φ be an arbitrary function from the set of pairs of abstract simplicial complexes to the set of pairs of abaci. Consider the following conditions:*

$P_1:$ *φ preserves C^*-equivalence;*

$P_2:$ *φ preserves C^*-inequivalence;*

$P_3:$ *φ is Turing-computable.*

Then for any two distinct indexes $i, j \in \{1, 2, 3\}$ there is a function $\varphi_{i,j}$ satisfying properties P_i and P_j, but there is no function φ having simultaneously the three properties.

A new interesting technique yielding finite presentations of AF C*-algebras can be drawn from the constructions of the paper [15]. Here the authors introduce finite presentations of countable abelian ℓ-groups as quotients of free *nonabelian* ℓ-groups. In this way they can give, among others, finite presentations of all Effros-Shen groups $\mathfrak{F}_\rho$, for ρ a recursive real. While the algorithmic aspects of main techniques of

[15] are as yet unexplored, one can naturally pose new decidability and undecidability problems for various important classes of AF C^*-algebras having finite presentations in the sense of [15].

References

[1] BEHNKE, H., LEPTIN, H. "C*-algebras with a two-point dual". *J. Functional Analysis*, 10, pp. 330-335, 1972.

[2] BLACKADAR, B. *K-Theory for Operator Algebras*. New York: Springer-Verlag, 1987.

[3] BOYLE, M., MARCUS, B., TROW, P. "Resolving maps and the dimension group for shifts of finite type". *Mem. Amer. Math. Soc.,* 70, n. 377, 1987.

[4] BRATTELI, O. "Inductive limits of finite dimensional C^*-algebras". *Trans. Amer. Math. Soc.*, 171, pp. 195-234, 1972.

[5] BRATTELI, O., ROBINSON, D.W. *Operator Algebras and Quantum Statistical Mechanics I, II.* Berlin: Springer-Verlag, 1979.

[6] BRATTELI, O., JORGENSEN, P., KIM, K.H., ROUSH, F."Nonstationarity of isomorphism between AF-algebras defined by stationary Bratteli diagrams"'. *Ergodic Theory and Dynamical Systems,* 20, pp. 1639-1656, 2000.

[7] ———. "Decidability of the isomorphism problem for stationary AF-algebras and the associated ordered simple dimension groups". *Ergodic Theory and Dynamical Systems,* 21, pp. 1625-1655, 2001.

[8] ———. Corrigendum to the paper "Decidability of the isomorphism problem for stationary AF-algebras and the associated ordered simple dimension groups". *Ergodic Theory and Dynamical Systems,* 22, p. 633, 2002.

[9] Cignoli, R., D'Ottaviano, I.M.L., Mundici, D.. *Algebraic Foundations of many-valued Reasoning*. Trends in Logic, vol. 7. Dordrecht: Kluwer Academic Publishers, 2000.

[10] Cignoli, R., Elliott, G.A., Mundici, D. "Reconstructing C^*-algebras from their Murray von Neumann orders". *Advances in Mathematics,* 101, pp. 166-179, 1993.

[11] Effros, E.G. *Dimensions and C*-algebras*. CBMS Regional Conf. Series in Math., vol. 46, Amer. Math. Soc. Providence, RI, 1981.

[12] Elliott, G.A. "On the classification of inductive limits of sequences of semisimple finite-dimensional algebras". *J. Algebra*, 38, pp. 29-44, 1976.

[13] Emch, G.G. *Mathematical and Conceptual Foundations of 20th-Century Physics*. Amsterdam: North-Holland, 1984.

[14] Fell, J.M.G. "The dual spaces of C*-algebras". *Trans. Amer. Math. Soc.*, 94, pp. 365-403, 1960.

[15] Glass, A.M.W., Marra, V. "Embedding finitely generated Abelian lattice-ordered groups: Higman's Theorem and a realisation of π". *J. London Math. Soc.,* 68, pp. 545-562, 2003.

[16] Goodearl, K.R. *Notes on Real and Complex C*-Algebras*. Shiva Math. Series, vol. 5. Boston: Birkhäuser, 1982.

[17] Haag, D., Kastler, D. "An algebraic approach to quantum field theory". *J. Math. Physics*, 5, pp. 848-861, 1964.

[18] Handelman, D. "Positive matrices and dimension groups affiliated to C*-algebras and topological Markov chains". *J. Operator Theory,* 6 , pp. 55-74, 1981.

[19] Kastler, D. "Does ergodicity plus locality imply the Gibbs structure?". *Proc. Symp. Pure Math., II*, 38, pp. 467-489, 1982.

[20] Kim, K.H., Roush, F.W. "Some results on decidability of shift equivalence". *J. Combinatorics, Information and System Sci.,* 4, pp. 123-146, 1979.

[21] ———. "Decidability of shift equivalence". *Lecture Notes in Mathematics,* 1342. Springer, pp. 374-424, 1988.

[22] Kitchens, B.P. *Symbolic Dynamics: One-sided, Two-sided and Countable State Markov Shifts.* Berlin: Springer, 1998.

[23] Mundici, D. "Interpretation of AF C*-algebras in Lukasiewicz sentential calculus". *J. Functional Analysis,* 65, pp. 15-63, 1986.

[24] ———. "The Turing complexity of AF C*-algebras with lattice-ordered K_0". *Lecture Notes in Computer Science,* 270, pp. 256-264, 1987.

[25] ———. "Farey stellar subdivisions, ultrasimplicial groups, and K_0 of AF C*-algebras". *Advances in Mathematics,* 68, pp. 23-39, 1988.

[26] ———. "The C*-algebras of three-valued logic". In: *Proceedings Logic Colloquium 1988, Studies in Logic and the Foundations of Mathematics.* Amsterdam: North-Holland, pp. 61-77, 1989.

[27] ———. "Turing complexity of Behncke-Leptin C*-algebras with a two-point dual". *Annals of Mathematics and Artificial Intelligence,* 6, pp. 287-294, 1992.

[28] ———. "Logic of infinite quantum systems". *International Journal of Theoretical Physics,* 32, pp. 1941-1955, 1993.

[29] ———. "Gödel incompleteness and quantum thermodynamic limits". In: *Philosophy of Mathematics Today.* Dordrecht: Kluwer Academic Publishers, pp. 287-298, 1997.

[30] ———. "Simple Bratteli diagrams with a Gödel incomplete isomorphism problem". *Transactions of the American Mathematical Society,* 356, pp. 1937-1955, 2004.

[31] Mundici, D., Panti, G. "Extending addition in Elliott's local semigroup". *Journal of Functional Analysis,* 117, pp. 461-471, 1993.

[32] ———. "Decidable and undecidable prime theories in infinite-valued logic". *Annals of Pure and Applied Logic*, 108, pp. 269-278, 2001.

[33] Palis, J., Takens, F. *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations.* Cambridge University Press, 1993. (Cambridge Studies in Advanced Mathematics, 35).

[34] Sewell, G.L. *Quantum Theory of Collective Phenomena.* Oxford: Clarendon Press, 1986.