Abstract: In [9] it is proved the categorical isomorphism of two varieties: bounded commutative $BCK$-algebras and $MV$-algebras. The class of $MV$-algebras is the algebraic counterpart of the infinite valued propositional calculus $L$ of Lukasiewicz (see [4]). The main objective of the present paper is to study that isomorphism from the perspective of logic. The B-C-K logic is algebraizable and the quasivariety of $BCK$-algebras is the equivalent algebraic semantics for that logic (see [1]). We call commutative $B-C-K$ logic, briefly $cBCK$, to the extension of B-C-K logic associated to the variety of commutative $BCK$–algebras. Moreover, we present the extension $Boc$ of $cBCK$ obtained by adding the axiom of “boundness”. We prove that the deductive system $Boc$ is equivalent to $L$. We observe that $cBCK$ admits two interesting extensions: the logic $Boc$, treated in this paper, which is equivalent to the system $L$ of Lukasiewicz, and the logic $Co$ that is naturally associated to the system $Bal^\omega$ of $\ell$-groups (see [10], [5]). This constructions establish a link between $L$ and $Bal^\omega$, that would be a logical approach to the categorical relationship between $MV$–algebras and $\ell$-groups (see [4]).

Key-words: B-C-K-logic. BCK-algebras. MV-algebras. Lukasiewicz logic.
1. BCK-ALGEBRAS AND B-C-K LOGIC

The notion of BCK–algebra was introduced by Iseki [6], [7]. A BCK–algebra is a system \( \langle A, *, 0 \rangle \) of type \((2, 0)\), where the operation \(*\) has the properties of set-theoretical difference. We can define an implication in each BCK-algebra by
\[
y \rightarrow x = x * y.
\]

So, we can see \(*\) as the dual of implication of B-C-K-logic.

The class of BCK–algebras is defined by a set of identities and quasi–identities, so it is a quasi–variety. In fact, Wronski [11] has shown that does not form a variety.

**Definition 1.** (Iseki and Tanaka [7]) The system \( \langle A, *, 0 \rangle \) is a BCK–algebra if the following identities and quasi–identity hold.

\begin{align*}
(IT1) & \quad ((x * y) * (x * z)) * (z * y) = 0. \\
(IT2) & \quad (x * (x * y)) * y = 0. \\
(IT3) & \quad x * x = 0. \\
(IT4) & \quad 0 * x = 0. \\
(IT5) & \quad x * y = 0, y * x = 0 \text{ implies } x = y.
\end{align*}

If \( \langle A, *, 0 \rangle \) is a BCK–algebra, then it is known that \( \langle A, \leq \rangle \) is a poset with the order defined by
\[
x \leq y \text{ if and only if } x * y = 0.
\]

The B-C-K logic is defined as follows.

**Language**

The only connective considered is \( \rightarrow \).

**Axioms**

\[(B) \quad (\varphi \rightarrow \psi) \rightarrow ((\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi)),\]
(C) \((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))\),

(K) \(\varphi \rightarrow (\psi \rightarrow \varphi)\),

**Inference Rules**

The only rule considered is Modus Ponens (MP):

\[
\begin{array}{c}
\varphi, \varphi \rightarrow \psi \\
\hline
\psi
\end{array}
\]

Note (see [1]) that preceding axioms imply

(I) \(\varphi \rightarrow \varphi\),

**Theorem 1.** (see [1]) The B-C-K logic is algebraizable with equivalence formulas \(\{\varphi \rightarrow \psi, \psi \rightarrow \varphi\}\) and defining equation \(\varphi \approx (\varphi \rightarrow \varphi)\). The class of BCK–algebras is the equivalent algebraic semantics for B-C-K logic.

The class of BCK-algebras is thus defined by the identities obtained making the expressions of axioms (B), (C), (K) equal to 0 and the quasiidentity: \(x \rightarrow y \approx 0\), \(y \rightarrow x \approx 0\) implies \(x \approx y\).

**2. COMMUTATIVE BCK-ALGEBRAS AND LOGIC cBCK**

It was proved by Yutani [12] that the class of *commutative BCK–algebras* has the following equational basis.

(Y1) \((x \ast y) \ast z = (x \ast z) \ast y\)

(Y2) \(x \ast (x \ast y) = y \ast (y \ast x)\)

(Y3) \(x \ast x = 0\)

(Y4) \(x \ast 0 = x\)
A commutative BCK–algebra is a lower semi-lattice with respect to the order above defined, where the infimum is given by

\[ x \land y = x \ast (x \ast y). \]

We will call the following deductive system *commutative B-C-K logic*, briefly cBCK.

**Language**

The only connective considered is \( \rightarrow \).

**Axioms**

(B) \((\phi \rightarrow \psi) \rightarrow ((\chi \rightarrow \phi) \rightarrow (\chi \rightarrow \psi))\),

(C) \((\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\phi \rightarrow \chi))\),

(K) \(\phi \rightarrow (\psi \rightarrow \phi)\),

(In) \(((\phi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \phi) \rightarrow \psi)\)

**Inference Rules**

The only rule considered is (MP).

**Theorem 2.** The deductive system cBCK is algebraizable and the variety of commutative BCK–algebras is the equivalent algebraic semantics for that logic.

**Proof.** Being an extension of an algebraizable logic, this logic is also algebraizable with the same defining equations \( \phi \approx \phi \rightarrow \phi \) and equivalence formulas \( \{ \phi \rightarrow \psi , \psi \rightarrow \phi \} \). The corresponding class of algebras is determined by the equations and quasiequations that result of algebraization process in theorem 1 (that are equivalent to Iseki-Tanaka conditions, [7]) plus the condition

\[(x \rightarrow y) \rightarrow y \approx (y \rightarrow x) \rightarrow x\]

that is, (Y2) in terms of \( \rightarrow \).
It is known that Iseki-Tanaka conditions plus (Y2) are equivalent to Yutani conditions Y1–Y4.

3. MV-ALGEBRAS AND THE INFINITE-VALUED Lukasiewicz LOGIC

In 2000, Cignoli, D’Ottaviano and Mundici ([4]) presented a deep algebraic approach of the infinite-valued sentential calculus L of Lukasiewicz. In [2], [3] Chang studied this calculus and established that the variety of MV–algebras is the algebraic counterpart of L.

An MV–algebra (Chang, [2] Mangani, [8]) is a system \( A = (A; \oplus, \neg, 0) \) satisfying the following equations.

\( \text{MV1} \quad x \oplus (y \oplus z) = (x \oplus y) \oplus z \)

\( \text{MV2} \quad x \oplus y = y \oplus x \)

\( \text{MV3} \quad x \oplus 0 = x \)

\( \text{MV4} \quad \neg \neg x = x \)

\( \text{MV5} \quad x \oplus \neg 0 = \neg 0 \)

\( \text{MV6} \quad \neg (\neg x \oplus y) \oplus y = \neg (\neg y \oplus x) \oplus x \)

In every MV–algebra we can define the constant 1 and the binary operator → by the formulas: \( 1 := \neg 0, \quad x \rightarrow y := \neg x \oplus y \)

The following is the definition of infinite-valued propositional calculus \( L \) of Lukasiewicz.

Language

The language is of type (1, 2), given by two connectives \( \neg \) and →.

Axioms

\( (B) \quad (\varphi \rightarrow \psi) \rightarrow ((\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi)) \),

\( (K) \quad \varphi \rightarrow (\psi \rightarrow \varphi) \),
(In) \(((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)\),

(Ne) \((\neg \varphi \to \neg \psi) \to (\psi \to \varphi)\)

**Inference Rules**

The only rule is Modus Ponens.

From Prop. 4.3.4, ch. 4, [4] we see that

(C) \((\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))\)

is a theorem of L.

4. BOUNDED COMMUTATIVE BCK-ALGEBRAS AND LOGIC $B_{oc}$

The logic $B_{oc}$ is the extension of commutative B-C-K logic obtained by adding the axiom of “boundness”. The algebraization of this logic provides the class of commutative bounded BCK–algebras.

A *bounded commutative BCK–algebra* is a commutative BCK-algebra \(\langle A, \ast, 0, 1 \rangle\) with a maximum element 1 such that the following identity holds.

(Ma) \(x \ast 1 = 0\).

The system $B_{oc}$ of *bounded commutative B-C-K logic* is defined as follows.

**Language**

Let us consider the language \(L = \{\to, \top\}\) of type (2, 0).

**Axioms and Inference Rules**

(B) \((\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))\),

(C) \((\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))\),

(K) \(\varphi \to (\psi \to \varphi)\),

(In) \(((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)\)

(Bo) \(\top \to \varphi\)
Theorem 3. The deductive system \( \text{Boc} \) is algebraizable and the variety of bounded commutative BCK–algebras is the equivalent algebraic semantics for that logic.

Proof. The system \( \text{Boc} \) is an extension of \( cBCK \) logic, so is algebraizable with the same defining equations \( \varphi \approx \varphi \rightarrow \varphi \) and equivalence formulas \( \{ \varphi \rightarrow \psi, \psi \rightarrow \varphi \} \). As a consequence of theorem 2, the corresponding class of algebras is determined by conditions (Y1), ..., (Y4) of Yutani, plus the condition corresponding to axiom (Bo), that is, \( \top \rightarrow x \approx 0 \).

Theorem 4. Let \( \mathcal{F}_{\text{Boc}} \) be the set of formulas of \( \text{Boc} \), let the binary relation \( \equiv \) over \( \mathcal{F}_{\text{Boc}} \) be defined by \( \varphi \equiv \psi \iff \vdash \varphi \rightarrow \psi \text{ and } \vdash \psi \rightarrow \varphi \). The system \( \mathcal{F}_{\text{Boc}} / \equiv = (\mathcal{F}_{\text{Boc}} / \equiv ; *, \circ, U) \) is a bounded commutative BCK- algebra, where operations are defined in the quotient by

\[
[\varphi] * [\psi] = [\psi \rightarrow \varphi], \quad \circ = [\varphi \rightarrow \varphi], \quad U = [\top].
\]

Proof. It suffices to prove that (Ma) holds. In fact,

\[
[\varphi] * U = [\top \rightarrow \varphi] = 0.
\]

5. EQUIVALENCE BETWEEN L AND Boc

The rest of the paper is devoted to show that \( \text{Boc} \) is equivalent to \( L \) with an adequate translation of connectives. This relationship seems to be the logical approach to the definitional equivalence proved by Mundici in [9] between the varieties of bounded commutative BCK-algebras and MV–algebras.

We can remark that the interpretation of “truth” in B-C-K logic is 0, but in \( L \) is 1.

Theorem 5. The logic \( L \) is equivalent to \( \text{Boc} \).

Proof. The axioms (B), (K) and (In) are common to both systems and Prop. 4.3.4, ch. 4, [4] gives a proof in \( L \) of condition (C) from (B), (K) and (In). So, it suffices to show that (Bo) is a theorem of \( L \) considering the translation \( \top := \neg(\varphi \rightarrow \varphi) \) and that (Ne) follows from (B), (C), (K), (In) and (Bo), if we define \( \neg \varphi := \varphi \rightarrow \top \).

In first place, it is known that (I) is a B-C-K theorem (see section 2) and the proposition 4.3.4 mentioned above also states that \( \vdash L \phi \leftrightarrow \neg\neg\phi \). So, \( \vdash \neg\neg(\phi \rightarrow \phi) \). We have also:

1. \( \vdash (\neg\neg(\phi \rightarrow \phi)) \rightarrow (\neg\psi \rightarrow \neg\neg(\phi \rightarrow \phi)) \) (instance of (K))
2. \( \vdash (\neg\psi \rightarrow \neg\neg(\phi \rightarrow \phi)) \rightarrow (\neg(\phi \rightarrow \phi) \rightarrow \psi) \) (instance of (Ne))

Therefore, by (MP)

1. \( \vdash (\neg(\phi \rightarrow \phi) \rightarrow \psi) \), that is, (Bo).

On the other hand, we give a proof of (Ne) from the axioms (B), (C), (K), (In) and (Bo) as follows.

We first prove \( \vdash ((\phi \rightarrow T) \rightarrow T) \leftrightarrow \phi \) (that is, \( \vdash (\neg\neg\phi) \leftrightarrow \phi \)).

1 \( \vdash (\psi \rightarrow \phi) \rightarrow (\psi \rightarrow \phi) \) (instance of (I))
2 \( \vdash \psi \rightarrow ((\psi \rightarrow \phi) \rightarrow \phi) \) (by (C) and (MP))
3 \( \vdash (T \rightarrow \phi) \rightarrow (((T \rightarrow \phi) \rightarrow \phi) \rightarrow \phi) \) (by (2))
4 \( \vdash ((T \rightarrow \phi) \rightarrow \phi) \rightarrow \phi \) (by (3), (Bo) and (MP))
5 \( \vdash ((\phi \rightarrow T) \rightarrow T) \rightarrow (((\phi \rightarrow T) \rightarrow T) \rightarrow (T \rightarrow \phi) \rightarrow \phi) \) (instance of (In) )
6 \( \vdash ((\phi \rightarrow T) \rightarrow T) \rightarrow \phi \) (by (4) and (5), by hypothetical syllogism, that holds in B-C-K logic.)

The converse

\( \vdash \phi \rightarrow ((\phi \rightarrow T) \rightarrow T) \)

is an instance of (2), above.

Now, (Ne) follows from

\( \vdash (\neg\phi \rightarrow \neg\psi) \rightarrow ((\neg\neg\psi) \rightarrow (\neg\neg\phi)) \) (instance of (B)),
\( \vdash (\neg\neg\phi) \rightarrow \phi \), and
\( \vdash \psi \rightarrow (\neg\neg\phi) \),

by adequate application of hypothetical syllogism. \( \square \)
References


