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BOUNDED COMMUTATIVE B-C-K LOGIC AND ŁUKASIEWICZ LOGIC

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A mi querida amiga Itala, en sus jóvenes 60 años

Abstract: In [9] it is proved the categorical isomorphism of two varieties: bounded commutative BCK-algebras and MV-algebras. The class of MV-algebras is the algebraic counterpart of the infinite valued propositional calculus Ł of Łukasiewicz (see [4]). The main objective of the present paper is to study that isomorphism from the perspective of logic. The B-C-K logic is algebraizable and the quasivariety of BCK-algebras is the equivalent algebraic semantics for that logic (see [1]). We call *commutative B-C-K logic*, briefly *cBCK*, to the extension of B-C-K logic associated to the variety of commutative BCK–algebras. Moreover, we present the extension $\mathcal{B}oc$ of *cBCK* obtained by adding the axiom of "boundness". We prove that the deductive system $\mathcal{B}oc$ is equivalent to Ł. We observe that *cBCK* admits two interesting extensions: the logic $\mathcal{B}oc$, treated in this paper, which is equivalent to the system Ł of Łukasiewicz, and the logic $\mathcal{C}o$ that is naturally associated to the system $\mathcal{B}al^o$ of ℓ-groups (see [10], [5]) . This constructions establish a link between Ł and $\mathcal{B}al^o$, that would be a logical approach to the categorical relationship between MV–algebras and ℓ-groups (see [4]).

Key-words: B-C-K-logic. BCK-algebras. MV-algebras. Łukasiewicz logic.

1. BCK-ALGEBRAS AND B-C-K LOGIC

The notion of BCK–algebra was introduced by Iseki [6], [7]. A BCK–algebra is a system $\langle A, *, 0\rangle$ of type $(2, 0)$, where the operation $*$ has the properties of set-theoretical difference. We can define an implication in each BCK-algebra by

$$y \to x = x * y.$$

So, we can see $*$ as the dual of implication of B-C-K-logic.

The class of BCK–algebras is defined by a set of identities and quasi–identities, so it is a quasi–variety. In fact, Wronski [11] has shown that does not form a variety.

Definition 1. *(Iseki and Tanaka [7]) The system $\langle A, *, 0\rangle$ is a BCK–algebra if the following identities and quasi–identity hold.*

(IT1) $((x * y) * (x * z)) * (z * y) = 0.$

(IT2) $(x * (x * y)) * y = 0.$

(IT3) $x * x = 0.$

(IT4) $0 * x = 0.$

(IT5) $x * y = 0$, $y * x = 0$ *implies* $x = y$.

If $\langle A, *, 0\rangle$ is a BCK–algebra, then it is known that $\langle A, \leq\rangle$ is a poset with the order defined by
$x \leq y$ if and only if $x * y = 0$.

The *B-C-K logic* is defined as follows.

Language

The only connective considered is $\to$.

Axioms

(B) $(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$,

(C) $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$,

(K) $\varphi \to (\psi \to \varphi)$,

Inference Rules

The only rule considered is Modus Ponens (MP):

$$\frac{\varphi\ ,\ \varphi \to \psi}{\psi}$$

Note (see [1]) that preceding axioms imply

(I) $\varphi \to \varphi$,

Theorem 1. *(see [1]) The B-C-K logic is algebraizable with equivalence formulas* $\{\varphi \to \psi, \psi \to \varphi\}$ *and defining equation* $\varphi \approx (\varphi \to \varphi)$. *The class of BCK–algebras is the equivalent algebraic semantics for B-C-K logic.*

The class of BCK- algebras is thus defined by the identities obtained making the expressions of axioms (B), (C), (K) equal to $\mathbf{0}$ and the quasiidentity: $x \to y \approx \mathbf{0}$, $y \to x \approx \mathbf{0}$ implies $x \approx y$.

2. COMMUTATIVE BCK-ALGEBRAS AND LOGIC cBCK

It was proved by Yutani [12] that the class of *commutative BCK–algebras* has the following equational basis.

(Y1) $(x * y) * z = (x * z) * y$

(Y2) $x * (x * y) = y * (y * x)$

(Y3) $x * x = 0$

(Y4) $x * 0 = x$

A commutative BCK–algebra is a lower semi-lattice with respect to the order above defined, where the infimum is given by

$$x \wedge y = x * (x * y).$$

We will call the following deductive system *commutative B-C-K logic*, briefly *cBCK*.

Language

The only connective considered is $\rightarrow$.

Axioms

(B) $(\varphi \rightarrow \psi) \rightarrow ((\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi))$,

(C) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$,

(K) $\varphi \rightarrow (\psi \rightarrow \varphi)$,

(In) $((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi)$

Inference Rules

The only rule considered is (MP).

Theorem 2. *The deductive system cBCK is algebraizable and the variety of commutative BCK–algebras is the equivalent algebraic semantics for that logic.*

Proof. Being an extension of an algebraizable logic, this logic is also algebraizable with the same defining equations $\varphi \approx \varphi \rightarrow \varphi$ and equivalence formulas $\{\varphi \rightarrow \psi\ ,\ \psi \rightarrow \varphi\}$. The corresponding class of algebras is determined by the equations and quasiequations that result of algebraization process in theorem 1 (that are equivalent to Iseki-Tanaka conditions, [7]) plus the condition

$$(x \rightarrow y) \rightarrow y \approx (y \rightarrow x) \rightarrow x$$

that is, (Y2) in terms of $\rightarrow$.

It is known that Iseki-Tanaka conditions plus (Y2) are equivalent to Yutani conditions Y1–Y4. □

3. MV-ALGEBRAS AND THE INFINITE-VALUED Łukasiewicz LOGIC

In 2000, Cignoli, D'Ottaviano and Mundici ([4]) presented a deep algebraic approach of the infinite-valued sentential calculus Ł of Łukasiewicz. In [2], [3] Chang studied this calculus and established that the variety of MV–algebras is the algebraic counterpart of Ł.

An MV–algebra (Chang, [2] Mangani, [8]) is a system $\boldsymbol{A} = \langle A; \oplus, \neg, \mathbf{0} \rangle$ satisfying the following equations.

MV1 $\quad x \oplus (y \oplus z) = (x \oplus y) \oplus z$

MV2 $\quad x \oplus y = y \oplus x$

MV3 $\quad x \oplus \mathbf{0} = x$

MV4 $\quad \neg\neg x = x$

MV5 $\quad x \oplus \neg\mathbf{0} = \neg\mathbf{0}$

MV6 $\quad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

In every MV–algebra we can define the constant $\mathbf{1}$ and the binary operator $\rightarrow$ by the formulas: $\mathbf{1} := \neg\mathbf{0}, \quad x \rightarrow y := \neg x \oplus y$

The following is the definition of *infinite-valued propositional calculus Ł* of Łukasiewicz.

Language

The language is of type $(1, 2)$, given by two connectives $\neg$ and $\rightarrow$.

Axioms

(B) $(\varphi \rightarrow \psi) \rightarrow ((\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow \psi))$,

(K) $\varphi \rightarrow (\psi \rightarrow \varphi)$, (In) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$,

(Ne) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$

Inference Rules

The only rule is Modus Ponens.

From Prop. 4.3.4, ch. 4, [4] we see that

(C) $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$

is a theorem of Ł.

4. BOUNDED COMMUTATIVE BCK-ALGEBRAS AND LOGIC $\mathcal{B}oc$

The logic $\mathcal{B}oc$ is the extension of commutative B-C-K logic obtained by adding the axiom of "boundness". The algebraization of this logic provides the class of commutative bounded BCK–algebras.

A *bounded commutative BCK–algebra* is a commutative BCK-algebra $\langle A, *, \mathbf{0}, \mathbf{1}\rangle$ with a maximun element 1 such that the following identity holds.

(Ma) $x * \mathbf{1} = \mathbf{0}$.

The system $\mathcal{B}oc$ of *bounded commutative B-C-K logic* is defined as follows.

Language

Let us consider the language $\mathcal{L} = \{\to, \top\}$ of type $(2, 0)$.

Axioms and Inference Rules

(B) $(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$,

(C) $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$,

(K) $\varphi \to (\psi \to \varphi)$,

(In) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

(Bo) $\top \to \varphi$

Theorem 3. *The deductive system $\mathcal{B}oc$ is algebraizable and the variety of bounded commutative BCK–algebras is the equivalent algebraic semantics for that logic.*

Proof. The system $\mathcal{B}oc$ is an extension of *cBCK* logic, so is algebraizable with the same defining equations $\varphi \approx \varphi \to \varphi$ and equivalence formulas $\{\varphi \to \psi \ , \ \psi \to \varphi\}$. As a consequence of theorem 2, the corresponding class of algebras is determined by conditions (Y1), ..., (Y4) of Yutani, plus the condition corresponding to axiom (Bo), that is, $\top \to x \approx \mathbf{0}$. □

Theorem 4. *Let $\mathcal{F}_{\mathcal{B}oc}$ be the set of formulas of $\mathcal{B}oc$, let the binary relation $\equiv$ over $\mathcal{F}_{\mathcal{B}oc}$ be defined by $\varphi \equiv \psi$ iff $\vdash \varphi \to \psi$ and $\vdash \psi \to \varphi$. The system $\mathcal{F}_{\mathcal{B}oc}/\equiv \; = \langle \mathcal{F}_{\mathcal{B}oc}/\equiv ; *, \mathbb{O}, \mathbb{U}\rangle$ is a bounded commutative BCK- algebra, where operations are defined in the quotient by*

$[\varphi] * [\psi] = [\psi \to \varphi]$, $\mathbb{O} = [\varphi \to \varphi]$, $\mathbb{U} = [\top]$.

Proof. It suffices to prove that (Ma) holds. In fact,

$[\varphi] * \mathbb{U} = [\top \to \varphi] = \mathbb{O}$. □

5. EQUIVALENCE BETWEEN Ł AND $\mathcal{B}oc$

The rest of the paper is devoted to show that $\mathcal{B}oc$ is equivalent to Ł with an adequate translation of connectives. This relationship seems to be the logical approach to the definitional equivalence proved by Mundici in [9] between the varieties of bounded commutative BCK-algebras and MV–algebras.

We can remark that the interpretation of "truth" in B-C-K logic is **0**, but in Ł is **1**.

Theorem 5. *The logic Ł is equivalent to $\mathcal{B}oc$.*

Proof. The axioms (B), (K) and (In) are common to both systems and Prop. 4.3.4, ch. 4, [4] gives a proof in Ł of condition (C) from (B), (K) and (In). So, it suffices to show that (Bo) is a theorem of Ł considering the translation $\top := \neg(\varphi \to \varphi)$ and that (Ne) follows from (B), (C), (K), (In) and (Bo), if we define $\neg\varphi := \varphi \to \top$.

In first place, it is known that (I) is a B-C-K theorem (see section 2) and the proposition 4.3.4 mentioned above also states that $\vdash_{Ł} \varphi \leftrightarrow \neg\neg\varphi$. So, $\vdash \neg\neg(\varphi \to \varphi)$. We have also:

1. $\vdash (\neg\neg(\varphi \to \varphi)) \to (\neg\psi \to \neg\neg(\varphi \to \varphi))$ (instance of (K))
2. $\vdash (\neg\psi \to \neg\neg(\varphi \to \varphi)) \to (\neg(\varphi \to \varphi) \to \psi)$ (instance of (Ne))

Therefore, by (MP)

1. $\vdash \neg(\varphi \to \varphi) \to \psi$, that is, (Bo).

On the other hand, we give a proof of (Ne) from the axioms (B), (C), (K), (In) and (Bo) as follows.

We first prove $\vdash ((\varphi \to \top) \to \top) \leftrightarrow \varphi$ (that is, $\vdash (\neg\neg\varphi) \leftrightarrow \varphi$).

1 $\vdash (\psi \to \varphi) \to (\psi \to \varphi)$ (instance of (I))

2 $\vdash \psi \to ((\psi \to \varphi) \to \varphi)$ (by (C) and (MP))

3 $\vdash (\top \to \varphi) \to (((\top \to \varphi) \to \varphi) \to \varphi)$ (by (2))

4 $\vdash ((\top \to \varphi) \to \varphi) \to \varphi$ (by (3), (Bo) and (MP))

5 $\vdash ((\varphi \to \top) \to \top) \to ((\top \to \varphi) \to \varphi)$ (instance of (In))

6 $\vdash ((\varphi \to \top) \to \top) \to \varphi$ (by (4) and (5), by hypothetical syllogism, that holds in B-C-K logic.)

The converse
$\vdash \varphi \to ((\varphi \to \top) \to \top)$
is an instance of (2), above.
Now, (Ne) follows from
$\vdash (\neg\varphi \to \neg\psi) \to ((\neg\neg\psi) \to (\neg\neg\varphi))$ (instance of (B)),
$\vdash (\neg\neg\varphi) \to \varphi$, and
$\vdash \psi \to (\neg\neg\psi)$,
by adequate application of hypothetical syllogism. □

References

[1] BLOK, W.J., PIGOZZI, D. "Algebraizable Logics". *Memoirs of the A.M.S.*, 77, Nr. 396, 1989.

[2] CHANG, C.C. "Algebraic Analysis of Many-Valued Logics". *Transactions of the A.M.S.*, 88, pp. 467-490, 1958.

[3] ———. "A New Proof of the Completeness of the Łukasiewicz Axioms". *Transactions of the A.M.S.*, 93, pp. 74-80, 1959.

[4] CIGNOLI, R., D'OTTAVIANO, I.M.L., MUNDICI, D. *Algebraic Foundations of many-valued Reasoning.* **Trends in Logic**, Studia Logic Library, vol. 7, Kluwer Ac. Publ., 2000.

[5] GALLI, A., LEVIN, R., SAGASTUME, M. "The Logic of Equilibrium and Abelian Lattice Ordered Groups". *Arch. Math Logic*, 43, pp. 141-158, 2004.

[6] ISEKI, K. "An Algebra Related with a Propositional Calculus". *Proc. Japan Acad.*, 42, pp. 26-29, 1966.

[7] ISEKI, K., TANAKA, S. "An Introduction to the Theory of BCK-algebras". *Math. Japon*, 23, pp. 1-26, 1978.

[8] MANGANI, P. "Su Certe Algebre Connesse con Logiche a Piú Valori". *Bollettino Unione Matematica Italiana*, 8, pp. 68-78, 1973.

[9] MUNDICI, D. "MV-algebras are Categorically Equivalent to Bounded Commutative BCK-algebras". *Math. Japonica*, 31(6), pp. 889-894, 1986.

[10] SAGASTUME, M., "Conical Logic and l-groups Logic". *JANCL*, 15(3), pp. 265-283, 2005.

[11] WRONSKI, A. "BCK-algebras do Not Form a Variety". *Math. Japonica*, 28, pp. 211-213, 1983.

[12] YUTANI, H. "Quasi-commutative BCK-algebras and Congruence Relations". *Math. Seminar Notes*, 5, pp. 469-480, 1977.