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ARE WE CLOSER TO A SOLUTION OF THE CONTINUUM PROBLEM?

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Abstract: The Continuum Hypothesis has motivated a considerable part of the development of axiomatic set theory for over a century. We present, in a very schematic way, some of the results that give information related to Cantor's Continuum Problem.

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1. THE CONTINUUM HYPOTHESIS

The notion of infinite appears in mathematics in many different ways. The notion of limit or endless processes of approximations have been considered since ancient times, but it was in the decade of 1870 that the systematic study of infinite collections as completed totalities was initiated by George Cantor, originating what is now known as set theory. Cantor's original motivations, related to certain problems of mathematical analysis, led him to consider properties of sets of real numbers, in particular their size, or number of elements.

Given two collections of objects, we say they have the same size, or the same number of elements, if we can establish a one to one correspondence between them, that is, if to each element of the first collection we can assign a unique element of the second, in such a way that every element of the second collection gets assigned to exactly one element of the first collection. In mathematical terms, this is called a bijection between the two collections. We will thus say that two sets A and B are equipotent, or that they have the same cardinality, if there is a bijection from A to B. It is easy, to verify that the set $\mathbb{N}$ of natural numbers $(0, 1, 2, \ldots)$ has the same cardinality as the set of even numbers (just set the correspondence sending the number n to the number $2n$). Some interesting collections of real numbers have also the same cardinality, for example, the set of rational numbers, and the set of algebraic real numbers (those real numbers which are roots of a polynomial with integer coefficients) can be also put in a bijective correspondence with $\mathbb{N}$.

Cantor proved that there is no bijection between the points of the real line $\mathbb{R}$ and the set of natural numbers; there are, thus, at least two different kinds of infinite sets: those infinite sets which have the same cardinality as $\mathbb{N}$, called countable, and the others, like the set $\mathbb{R}$ of real numbers, which are said to be uncountable. Cantor also noted that the interval $[0, 1]$ and $\mathbb{R}$ have the same cardinality, as well as the sets $\mathbb{R}^k$, $\mathbb{R}^{\mathbb{N}}$ and $\{0, 1\}^{\mathbb{N}}$, the set of all sequences of zeroes and ones. All of these sets are uncountable.

A very natural question to ask is if every uncountable set of real numbers has the same cardinality as $\mathbb{R}$. Cantor's continuum hypothesis is precisely that statement: every uncountable set of real numbers has the same cardinality as the whole set of real numbers. Cantor worked without success to prove this hypothesis, and, as we will see, still today the problem remains unsolved.

Hilbert considered this problem so important that he put it first in the list of problems he presented to the International Congress of Mathematicians held in Paris in 1900. It continues to be one of the most important problems of mathematics, and it can be said that it has guided the development of set theory for over a century.

We will try to present here, in a very schematic way, some of the most important developments related to the continuum hypothesis. The book (Jech 2003) is a good reference to consult for those interested in further reading on this subject and set theory in general; Kanamori (1994) is also an excellent source of information, specially on topics related to large cardinals.

2. AXIOMS OF SET THEORY

Cantor introduced transfinite ordinals and cardinals to deal with infinite collections. When we count the elements in a collection, we assign natural numbers to the elements of the collection. If we count real numbers, for example, the elements of the unit interval [0.1], we run out of natural numbers before we end the counting, so we need new numbers to continue, the infinite ordinal numbers. The ordinals, thus, start with the natural numbers and go on indefinitely:

$$0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega, \omega + \omega + 1, \ldots$$

When we count the elements of a set, we order these elements, the first one, the second, etc. ω is the first infinite ordinal, it is the number assigned to the first element counted after we run out of natural numbers, then comes $\omega + 1$, obtained adding one new element, adding another we obtain $\omega + 2$, and so on. In the limit we have $\omega + \omega$, The ordinals obtained starting from ω and applying the operations of adding one more element, and passing to the limit are the countable ordinals. Any set counted with a countable ordinal has the same cardinality as $\mathbb{N}$, which was called $\aleph_0$ by Cantor. It turns out that the set of all countable ordinals is not countable, and its cardinality is the next infinite cardinal, called $\aleph_1$. The process can be continued, if ω_1 is the ordinal that comes after all the countable ordinals; then, adding one more element we get $\omega_1 + 1$, then $\omega_1 + 2$, and so on. The limit is $\omega_1 + \omega$, and in this way we get the ordinals that count sets which have the same cardinality as ω_1. The cardinality of this set of ordinals is $\aleph_2$, the next cardinal after $\aleph_1$. The transfinite cardinals are

$$\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\omega, \aleph_{\omega+1}, \ldots, \aleph_{\omega_1}, \ldots$$

For each ordinal α there is a cardinal $\aleph_\alpha$, and these are all different "sizes" of infinite sets. An infinite cardinal $\aleph_\beta$ is a successor cardinal if $\beta = \alpha + 1$ for some ordinal α, otherwise it is a limit cardinal.

Cantor conjectured that every set can be put in one to one correspondence with an ordinal. In other words, that every set can be well ordered, which means totally ordered in such a way that every non-empty subset has a minimal element. This statement is called the well ordering principle. If it holds, then the real numbers, in particular, can be well ordered and there is a unique cardinal $\boldsymbol{c}$ equipotent with the set of real numbers; $\boldsymbol{c}$ is the cardinality of the set of real numbers. The continuum hypothesis just means that $\boldsymbol{c}$ is $\aleph_1$, the least uncountable cardinal. Since the set of real numbers is equipotent with the set

of sequences of sequences of zeroes and ones, $\boldsymbol{c}$, the cardinality of the set of real numbers is the same as $2^{\aleph_0}$. Thus, the continuum hypothesis is sometimes expressed as $2^{\aleph_0} = \aleph_1$.

Soon after, around the turn of the century, several paradoxes were discovered in the theory of sets, the most famous of which is Russell's paradox. Bertrand Russell showed that the intuitive idea that every property determines a set (the collection of objects having the property) leads to contradictions. Consider the collection B of all sets x with the property that x is not an element of x. If B is not an element of B then, by its own definition, B is an element of B; and vice-versa, if B is an element of B it is because B is not an element of B. Therefore this collection B cannot be a set. The collection of all sets is another collection which cannot be a set, as well as the collection of all ordinals.

Ernst Zermelo, in 1908, proposed an axiomatization of set theory to avoid these paradoxes. This axiomatization, modified later by Abraham Fraenkel, is the theory ZF, and together with the axiom of choice it is called the theory of sets ZFC (Zermelo-Fraenkel with Choice).

The axioms of ZFC are the following.

1. Axiom of extensionality. If X and Y have the same elements, they are equal.

2. Axiom of pairs. Given X and Y, there is a set $\{X, Y\}$ whose elements are exactly X and Y.

3. Axiom of union. For every set X, there is a set $Y = \cup X$ which is the union of the elements of X.

4. Axiom of power set. For every set X, there is a set $Y = \mathcal{P}(X)$, the set of all subsets of X.

5. Axiom of infinity. There is an infinite set.

6. Axiom of replacement. If F is a definable function, for every set X there is a set $Y = \{F(x) : x \in X\}$.

7. Axiom of regularity. Every non-empty set has an $\in$-minimal element.

8. Axiom of choice. Every family of non-empty sets has a choice function.

The well ordering principle, which states that every set can be well ordered, is equivalent to the axiom of choice.

The ordinals are identified in the theory ZFC with some specific sets, in such a way that each ordinal is the set of its predecessors. Starting with $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, $\dots n + 1 = n \cup \{n\}$, etc, we get the natural numbers. Then, $\omega = \{0, 1, 2, \dots\}$ is the set of all natural numbers, $\omega + 1 = \omega \cup \{\omega\}$; and in general, every ordinal α has a successor $\alpha + 1 = \alpha \cup \{\alpha\}$. The limit of a set of ordinals is their union.

The cardinals are those ordinals that are not equipotent with any of its elements; for example, the finite ordinals are cardinals, $\omega = \aleph_0$, $\omega_1 = \aleph_1$, etc.

The axioms of ZFC describe the universe of all sets, and they imply that the universe can be organized in a cumulative hierarchy defined by induction.

$V_0 = \emptyset$, is the empty set.

$V_{\alpha+1} = \mathcal{P}(V_\alpha)$, the set of all the subsets of V_α, and

$V_\lambda = \bigcup_{\xi<\lambda} V_\xi$, the union of all the sets V_ξ with ξ less than λ, if λ is a limit ordinal.

The V_α's constitute an increasing collection of sets, and from the axioms of ZFC follows that every set is in V_α for α sufficiently large. Thus, $V = \bigcup_{\alpha \in Or} V_\alpha$, the union of all the V_α's is the universe of all sets. As a matter of fact, the axiom of regularity is equivalent to "all sets are in V".

For every $n \in \omega$, V_n is finite, V_ω is infinite, but all its elements are finite, and moreover, any element of an element of V_ω is finite, and the same for any element of an element of an element of V_ω, and so on. V_ω is the collection of all the hereditarily finite sets. In particular, every natural number n belongs to V_ω, so $\omega \subset V_\omega$; but ω itself is not an element of V_ω, it is an element of $V_{\omega+1}$, as is the case for any subset of ω.

3. TRUTH, PROOF, UNDECIDABILITY

The truth of a mathematical statement is established by means of deductive methods. We know that a statement φ is true if we can prove it starting from basic principles applying the rules of logic. How do we know if there is a proof for a given statement φ? Even if such a proof exists, it might be very hard to find. It could happen that there is no proof of φ. For example, if there is a proof of the negation of φ then φ cannot be provable, unless mathematics is inconsistent. But it can happen that neither φ nor its negation have proofs. In this case it is said that φ is undecidable.

In 1931, Kurt Gödel proved his famous incompleteness theorem, which says that for any axiomatic system which satisfies a few reasonable requirements, there are statements which are undecidable, that is, neither the statement nor its negation are provable in the system. The requirements that must be satisfied are, first, that the system is not contradictory; second, that the axioms can be algorithmically recognized, in other words that there is an effective procedure to decide if a given statement is one of the axioms; and finally, that elementary arithmetic can be developed within the system. The theory ZFC has the last two properties, so if we assume that it is consistent, it follows that there are mathematical statements that are undecidable in ZFC.

Another remarkable result of Gödel, known as the second incompleteness theorem, implies that the consistency of ZFC cannot be proved from the axioms of ZFC. To prove his incompleteness theorems, Gödel devised a way to express questions about mathematical theories by means of arithmetical statements. In particular, there is a formula of the language of arithmetic which expresses "there is no proof of $0 = 1$ from the axioms of ZFC". This formula says that ZFC is consistent, and Gödel showed that it is not provable in ZFC. The second incompleteness theorem is more general, it says that the same holds for any formal theory satisfying the conditions mentioned above.

The continuum hypothesis is one of those statements undecidable in ZFC; assuming ZFC is consistent, there is no proof of the continuum hypothesis nor of its negation from the axioms of ZFC. The next section is devoted to describe how this has been established.

4. THE INDEPENDENCY OF CONTINUUM HYPOTHESIS

A way to prove that a statement is not provable in a certain axiomatic system is to find a model of the axioms in which the statement is false. Gödel proved in 1936 that if there is a model of ZFC, then there is a model of ZFC in which the continuum hypothesis holds. This implies that the negation of CH is not provable in ZCF, unless the theory ZFC is itself inconsistent. Gödel's model is called the constructible universe, and it is denoted by L. This model L is obtained like V, defining by transfinite induction a hierarchy of sets, for each ordinal α, a set L_α. For $\alpha = 0$, $L_0 = \emptyset$, to define $L_{\alpha+1}$ we do not take all the subsets of L_α like it is done for V, but only the subsets of L_α which are definable in L_α, and if α is a limit ordinal, then L_α is the union of all the L_β with $\beta < \alpha$.

The idea is that at every stage of the construction we put in the model just the sets that are strictly necessary. This will give as a result that we only include the minimal possible quantity of real numbers, which is $\aleph_1$. Thus, since $\aleph_1$ is the first uncountable cardinal, in L, every uncountable set of real numbers has cardinality $\aleph_1$.

The axiom of choice is not used in the construction of L, and by the way the L_α's are defined, every set in L can be well ordered. This gives that the axiom of choice holds in L, and therefore, if ZF is consistent, then ZFC is consistent as well.

Once we have a model of ZFC where CH holds, we know that the negation of CH is not provable in ZFC. So, to complete the proof of the undecidability of the continuum hypothesis, it remains to find a model of ZFC where CH does not hold, which would show that CH itself is not provable in ZFC. This was done by Paul Cohen in 1963. Cohen invented a very powerful method to build models of set theory known as forcing. Starting with a model M of ZFC and a partial order in M, the method of forcing consists in using the partial order to add some new elements to M, in order to get a larger model preserving some of the properties of M and changing others. The properties of the new model are determined by the partial order used. This technique is reminiscent of the process of going from a certain field to an algebraic extension which contains a root for a polynomial with no roots in the original field.

Let us try to describe how the method works to add a new real number to a model M. First, we can identify real numbers with subsets of ω, or their characteristic functions, functions from ω into $\{0,1\}$. In M, consider the set $\mathbb{P}$ of all functions from a finite subset of ω taking values in $\{0,1\}$. This set can be partially ordered putting $q \leq p$ if q extends p as a function. A subset D of $\mathbb{P}$ is dense if for every $p \in \mathbb{P}$ there is $q \in D$ such that $q \leq p$. A subset G of $\mathbb{P}$ is a generic filter over M if it has the following three properties:

(i) For any $p, q \in \mathbb{P}$, if $p \in G$ and $p \leq q$, then $q \in G$,

(ii) If $p, q \in G$, then there is $r \in G$ such that $r \leq p$ and $r \leq q$ (p and q are compatible), and

(iii) For any dense subset D of $\mathbb{P}$ which belongs to M, $G \cap D \neq \emptyset$

This is usually expressed saying that G is $\mathbb{P}$-generic over M. It is not hard to prove that if M is countable such a generic filter G exists, and that it cannot be an element of M. Since the elements of G are pairwise compatible, $\cup G$, the union of all the elements of G is a function, and by the properties of G, it can be shown that its domain is ω. Therefore $g = \cup G$ is a real number which does not belong to M. This g is called a generic real. A model $M[g]$ can be defined in terms of M and g, in such a way that $M[g]$ is also a model of ZFC, $M \subseteq M[g]$, and $g \in M[g]$. The models M and $M[g]$ have the same ordinals, and $M[g]$ is contained in any other model N satisfying $M \subseteq N$ and $g \in N$. The model $M[g]$ is called a generic extension of the model M. Since the real g is defined from G, and G can be reconstructed from g, the roles of G and g are interchangeable in the construction of the generic extension, thus $M[g] = M[G]$.

The construction of the model $M[g]$ and the proof that it has the desired properties is quite elaborate. Certain elements of the model M are used as "names" for elements of $M[g]$. Which set is the object named by a name τ depends on the generic g, and given g, the model $M[g]$ is the collection of sets named by names in M according to g. A relation $\Vdash$ is defined between elements of $\mathbb{P}$ and formulas of the language of set theory with "names" for elements of $M[g]$. A formula $\varphi(x)$ is satisfied in $M[g]$ by a set a if and only if there is a $p \subset g$ such that $p \Vdash \varphi(\tau)$, where τ is a name for a.

How can this be used to show the independence of the continuum hypothesis? The idea is to add to a model M many new real numbers, say, at least $\aleph_2$ many new reals. This can be achieved as before, but using simultaneously $\aleph_2$ copies of the partial order $\mathbb{P}$. The generic filter G obtained this way will give rise to $\aleph_2$ many distinct real numbers, all of them in the model $M[G]$. One more thing has to be taken care of. Since there are new sets in $M[G]$, in particular there might be new functions, and conceivably, the ordinal that in M is $\aleph_2$, the second uncountable cardinal, could cease to be a cardinal in $M[G]$, for example if one of the new functions is a surjection from ω onto that ordinal, in which case the $\aleph_2$ of M becomes countable in $M[G]$. The properties of the partial order used to construct $M[G]$ are used to show that this is not the case, that if we use this particular partial order, the cardinals of M are still cardinals in $M[G]$. This is due to the fact that this partial order has the property known as the countable chain condition, or *ccc*. To explain this property, first some definitions. As it was mentioned in the definition of a generic filter, two elements a and b of a partially ordered set are compatible if there exists an element c of the partial order such that $c \leq a$ and $c \leq b$. If no such c exists, then a and b are said to be incompatible. An antichain in the partial order is just a set of pairwise incompatible elements. A partial order is *ccc* if every antichain is at most countable. Any generic extension of a model M obtained using a *ccc* partial order has the same cardinals as M.

A very good presentation of the method of forcing and independence proofs in set theory can be found in Kunen (1980).

We have seen that the continuum hypothesis is true in the model L defined by Gödel, and is false in Cohen's model $M[G]$; it is thus undecidable in ZFC. For some mathematicians, this settles the problem: there is no answer to the continuum question. For others, and this is a more common position among set theorists, this only shows that the axioms of set theory are not strong enough to give an answer to the question, and that new natural axioms should be found to really give an answer to the problem. In a very illuminating article, Gödel (1947) expressed this views about this. He thought that new axioms for the theory of sets could settle the continuum problem, and suggested in particular large cardinal axioms. As we will see, the connection between large cardinals and properties of sets of real numbers that has been gradually uncovered, has been shaped into a very deep and exciting theory.

5. LARGE CARDINALS

The axioms of ZFC allow us to organize the universe of sets in an increasing collection of sets, V_α, α an ordinal. It can be verified that V_ω satisfies several of the axioms of ZFC, but it fails to satisfy the axiom of infinity. If α is an ordinal, $\alpha > \omega$, then V_α satisfies the axiom of infinity, but it might not satisfy some other of the axioms. Can we find an ordinal κ such that V_κ satisfies all he axioms of ZFC? In other words, is there a κ such that V_κ is a model of ZFC? Gödel's second incompleteness theorem says that it is impossible to prove the existence of such a κ in ZFC.

A cardinal $\kappa > \omega$ is said to be *weakly inaccessible* if it satisfies the following two conditions:

1. κ is *regular*, which means that the union of fewer than κ sets of cardinality less than κ has cardinality less than κ, and
2. κ is a *limit* cardinal.

If in addition for any set A of cardinality less than κ, $\mathcal{P}(A)$, the set of subsets of A, has also cardinality less than κ, then κ is said to be *(strongly) inaccessible*.

We could say, thus, that an inaccessible cardinal cannot be reached applying the usual set theoretic operations to smaller cardinals. If κ is an inaccessible cardinal, then V_κ satisfies all the axioms of ZFC, so, by Gödel's theorem, the existence of an inaccessible cardinal is not provable in ZFC. We can add to ZFC the axiom "there is an inaccessible cardinal" to obtain a stronger theory. In this theory, the consistency of ZFC can be proved, of course, since it can be proved that there is a model of ZFC, but again, by Gödel's theorem, this theory does not prove its own consistency.

The axiom "there is an inaccessible cardinal" is our first example of a large cardinal axiom. It is an axiom stating the existence of a cardinal so large that its existence can not be proved in ZFC.

In 1930, S. Ulam, concerned with problems related to the theory of Lebesgue measure, formulated the concept of measurable cardinal. The existence of a measurable cardinal is much stronger than the existence of an inaccessible cardinal. In particular, if κ is a measurable cardinal, not only is κ also inaccessible, but there are κ-many inaccessible cardinals below κ.

The original definition of measurable cardinal was given in terms of measures taking values in $\{0, 1\}$, but an equivalent definition is that κ is *measurable* if there is a non-trivial elementary embedding

$$j : V \to M$$

of the universe V into a transitive class M containing all the ordinals such that κ is the first ordinal moved by the embedding.

The class M cannot be the whole universe, as has been shown by Kunen. Axioms stronger than measurability are obtained if the class M is required to be rich. A cardinal κ is λ-*supercompact* if there is an elementary embedding $j : V \to M$ such that κ is the first ordinal moved by j, $j(\kappa) > \lambda$, and $M^\lambda \subseteq M$, which means that all the λ-sequences of elements of M are elements of M. A cardinal κ is said to be a supecompact cardinal if it is λ-supercompact for every λ.

If κ is supercompact, then it is measurable and there are κ many measurable cardinals below κ. The axiom "there is a supercompact cardinal" is therefore very strong. As we will see below, it has many interesting consequences related to basic properties of sets of real numbers. It has been shown by Woodin that to prove many of these consequences, the full force of the axiom is not needed. He formulated the concept of what is now called a Woodin cardinal.

A cardinal κ is a *Woodin cardinal* if for every function $f : \kappa \to \kappa$ there is $\alpha < \kappa$ such that α is closed under f (i.e., $\xi < \alpha$ implies that

$f(\xi) < \alpha$), and an elementary embedding $j : V \to M$ such that α is the first ordinal moved by j and $V_{j(f)(\alpha)} \subseteq M$.

The existence of a Woodin cardinal κ implies that there are many measurable cardinals, but κ itself does not need to be measurable.

6. SETS OF REAL NUMBERS AND LARGE CARDINALS

The area of set theory known as Descriptive Set Theory studies sets of real numbers that can be defined in some way, for example in terms of topological notions. An open interval of the real line $\mathbb{R}$ is a set of the form $(a, b) = \{x \in \mathbb{R} : a < x < b\}$. A subset of $\mathbb{R}$ is open if it is the union of a family of open intervals. A set is closed if its complement is open. The *Borel* sets are those obtained from the open sets (or the closed sets) taking complements and countable unions, in other words the sets in the least σ-algebra containing the open sets. Taking images of Borel sets under continuous functions one goes beyond the Borel sets. A set of real numbers is *analytic* if it is the image of a Borel set under a continuous function. The *projective* sets are those generated from the Borel sets taking images under continuous functions and complements any finite number of times.

The projective sets can be organized in a hierarchy as follows. The symbol $\mathbf{\Sigma^1_1}$ is used to denote the collection of analytic sets. $\mathbf{\Pi^1_1}$ is the collection of complements of analytic sets, or co-analytic sets. $\mathbf{\Sigma^1_2}$ is the collection of images of co-analytic sets under continuous functions, and $\mathbf{\Pi^1_2}$ is the collection of complements of sets in $\mathbf{\Sigma^1_2}$.

Inductively, we define $\mathbf{\Sigma^1_n}$ and $\mathbf{\Pi^1_n}$. $\mathbf{\Sigma^1_{n+1}}$ is the collection of images by continuous functions of sets in $\mathbf{\Pi^1_n}$, and $\mathbf{\Pi^1_{n+1}}$ is the collection of complements of sets in $\mathbf{\Sigma^1_{n+1}}$. For every n, the classes $\mathbf{\Sigma^1_n}$ and $\mathbf{\Pi^1_n}$ are both strictly contained in $\mathbf{\Sigma^1_{n+1}}$ (and also in $\mathbf{\Pi^1_{n+1}}$).

The union of these classes, $\bigcup_n \mathbf{\Sigma}^1_n = \bigcup_n \mathbf{\Pi^1_n}$, is the collection of projective sets. Most sets that appear naturally in mathematics are Borel sets, or sets in the lower levels of the projective hierarchy.

These classes of sets can be defined in similar ways for any euclidean space $\mathbb{R}^n$.

A non-empty set of real numbers is *perfect* if it is closed and does not have isolated points. It can be shown that every non-empty perfect set has the cardinality of $\mathbb{R}$. A set of real numbers has the perfect subset property if it is either countable or contains a perfect subset.

A set $A \subseteq \mathbb{R}$ has the property of Baire if it differs from an open set in a meager set. Recall that a set is *nowhere dense* if its topological closure has empty interior, and a set is *meager* if it is the union of a countable collection of nowhere dense sets. Notice that this definition makes sense also if A is a subset of any topological space.

In ZFC it can be proved that the analytic sets have several regularity properties, for example, they are Lebesgue measurable, have the property of Baire, and any uncountable analytic set contains a perfect subset, and therefore it has the cardinality of $\mathbb{R}$.

Nevertheless, ZFC is not strong enough to show the same for the $\mathbf{\Sigma^1_2}$ sets. In 1969, Solovay showed that if there is a measurable cardinal, then all $\mathbf{\Sigma^1_2}$ sets are Lebesgue measurable, have the property of Baire and the perfect set property.

Shelah and Woodin (1990) proved that if there is a supercompact cardinal, then all the projective sets have the regularity properties mentioned (see also Foreman, Magidor & Shelah (1988)). It is very surprising that the existence of such large cardinals has such strong consequences in the realm of the projective sets. The importance of this result is evident if we consider that all the sets of real numbers that appear normally in mathematics are projective.

The following generalization of the property of Baire has become an important notion in the recent development of set theory. A set $A \subseteq \mathbb{R}^n$ is universally Baire if for every compact Hausdorff space Ω, and every continuous function $f : \Omega \to \mathbb{R}^n$, the preimage of A by f has the property of Baire. Clearly, every universally Baire set has the Baire property, it is enough to consider the identity function. All universally Baire sets are also Lebesgue measurable.

7. GAMES AND REALS. DETERMINACY

Given a set $A \subseteq [0, 1]$ a game G_A is defined for two players, I and II, who alternate playing 0's and 1's to form an infinite sequence. Player I chooses $\epsilon_1 \in \{0, 1\}$, then player II chooses $\epsilon_2 \in \{0, 1\}$, player one plays again to choose $\epsilon_3 \in \{0, 1\}$, and so on. In this way they form a sequence $\langle \epsilon_i : i = 1, 2, \dots \rangle$ with each $\epsilon_i \in \{0, 1\}$. Player I wins G_A if

$$\sum_{i=1}^{\infty} \epsilon_i 2^{-i} \in A,$$

otherwise, player II wins. A strategy is a function defined on the set of all finite sequences of 0's and 1's and taking values in $\{0,1\}$. A strategy σ is a winning strategy for player I if every run of the game in which I uses the function σ to decide what to play is won by I. In other words, if for every sequence $\{\epsilon_i\}$ such that $\epsilon_1 = \sigma(\emptyset)$ and $\epsilon_{2k+1} = \sigma(\langle \epsilon_1, \ldots, \epsilon_{2k} \rangle)$ for $k = 1, 2, \ldots$, the number $\sum_{i=1}^{\infty} \epsilon_i 2^{-i}$ is in A. Analogously, a strategy σ is a winning strategy for II if $\sum_{i=1}^{\infty} \epsilon_i 2^{-i}$ is not in A for every sequence $\{\epsilon_i\}$ such that $\epsilon_2 k + 2 = \sigma(\epsilon_1, \ldots, \epsilon_{2k+1})$. The set A is determined if one of the players has a winning strategy in the game G_A. Obviously, at most one of the players has a winning strategy for a game G_A . Using the axiom of choice it can be shown that there are sets which are not determined, however, such sets are quite complicated, for example they are not Borel sets. In fact, Martin (1970) proved that all Borel sets are determined. He had previously shown (Martin 1975) that if there is a measurable cardinal, then every analytic set is determined.

The axiom of projective determinacy states that all projective sets are determined. This axiom decides all important questions about projective sets, for example it implies that every projective set is Lebesgue measurable and has the property of Baire, and that every uncountable projective set has cardinality $2^{\aleph_0}$. So, under the axiom of projective determinacy, no projective set can be a counterexample to the continuum hypothesis.

The work of Martin, Steel, Woodin and others has brought up a very interesting and deep relationship between determinacy and large cardinals. Martin and Steel proved that if there are infinitely many Woodin cardinals, then every projective set is determined. Woodin has found that projective determinacy is equivalent to the existence of certain type of models with an arbitrarily large finite number of Woodin cardinals.

I. Neeman has shown that if there is a Woodin cardinal, every universally Baire set is determined. The existence of arbitrarily large Woodin cardinals implies that every projective set is universally Baire, and thus that the projective sets have the perfect set property.

8. FORCING AXIOMS

Let $(P, <)$ is a partially ordered set, a subset D of P is *dense* if for every $p \in P$ there is $q \in D$ such that $q \leq p$. A subset F of P is a *filter* if the elements of F are pairwise compatible and if $p \in F$ and $p \leq q$, then $q \in F$. Given a family $\mathcal{D}$ of dense subsets of P, we say that a filter F is $\mathcal{D}$*-generic* if $F \cap D \neq \emptyset$ for every $D \in \mathcal{D}$. Recall that an *antichain* of P is a set of pairwise incompatible elements of P. The partial order P satisfies the countable chain condition (*ccc*) if every antichain of P is at most countable.

The following statement is called *Martin's Axiom* (MA): If $(P, <)$ is a partially ordered set that satisfies the countable chain condition, and $\mathcal{D}$ is a collection of fewer that $\boldsymbol{c}$ dense subsets of P, then there is a $\mathcal{D}$-generic filter on P.

$\mathcal{D}$-generic filters always exist if $\mathcal{D}$ is a countable family of dense subsets of P. Thus Martin's Axiom is implied by the continuum hypothesis. Solovay and Tennenbaum showed that MA is consistent with the negation of the continuum hypothesis. MA has been a quite successful tool used to solve problems in other areas of mathematics, specially in analysis and topology (see, for example, Fremlin (1984)).

$MA_{\aleph_1}$ is the statement: For every *ccc* partial order and every family $\mathcal{D}$ of $\aleph_1$ dense sets, there is a $\mathcal{D}$-generic filter. It has many interesting consequences, for example, it implies that every $\boldsymbol{\Sigma}^1_2$ set of reals is Lebesgue measurable and has the Baire property.

Stronger principles can be obtained extending the class of partial orders considered. The principle known as Martin's Maximum (MM) was introduced in Foreman, Magidor & Shelah (1988). Its statement requires some previous definitions. A subset A of $\aleph_1$ is unbounded if for every ordinal $\alpha < \aleph_1$ there is $\beta \in A$ such that $\alpha < \beta$; and we say that A is closed if every countable subset of A has its supremum in A. A set $S \subseteq \aleph_1$ is stationary if it meets every closed and unbounded subset of $\aleph_1$. A partial order $(P, <)$ is stationary preserving if any stationary subset of $\aleph_1$ remains stationary in any generic extension obtained using $(P, <)$.

Martin's Maximum (MM)is the following statement:

If $(P, <)$ is a stationary preserving partial order and $\mathcal{D}$ is a family of $\aleph_1$ dense subsets of P, then there is a $\mathcal{D}$-generic filter on P.

Foreman, Magidor and Shelah showed that if there is a supercompact cardinal, then there is a model of ZFC satisfying MM.

Martin's Maximum implies that $\boldsymbol{c} = \aleph_2$.

9. WOODIN'S APPROACH

Given an infinite cardinal κ, let $H(\kappa)$ be the set of all sets of cardinality hereditarily less than κ, that is, all sets x such that x, the elements of x, the elements of the elements of x, etc., all have cardinality less than κ. The set $H(\omega)$ is thus the collection of all hereditarily finite sets, and coincides with V_ω. $H(\omega_1)$ is the set of hereditarily countable sets; $H(\omega_1) \subseteq V_{\omega_1}$, but the equality does not hold. If κ is an inaccessible cardinal, then $H(\kappa) = V_\kappa$.

Projective sets correspond to subsets of $H(\omega_1)$ which can be defined in $H(\omega_1)$ by a formula of the language of set theory which only refers to elements of $H(\omega_1)$. So, as we already know, there are questions about $H(\omega_1)$ that are unsolvable in ZFC. Projective determinacy can be considered as the correct axiom to settle all structural questions about the structure $H(\omega_1)$.

Woodin (2001) explains his recent work in the direction of finding an axiom for deciding the theory of $H(\omega_2)$. The continuum hypothesis can be expressed as a statement about $H(\omega_2)$, and this constitutes the main motivation of this work.

There are certain facts of the universe of sets that cannot be changed by forcing. For example, by the absoluteness theorem of Levy, if an existential statement which only refers to sets in $H(\omega_1)$ is valid in a generic extension, then it is also valid in the universe V.

We could consider a generalization of this to $H(\omega_2)$: if an existential statement which mentions only sets in $H(\omega_2)$ is valid in a generic extension $V^{\mathbb{P}}$, then it holds in the universe V, but this is false. Nevertheless, if we restrict ourselves to certain types of generic extensions we get something interesting. If we restrict ourselves to generic extensions $V^{\mathbb{P}}$ obtained by *ccc* partial orders $\mathbb{P}$, then this generalization is equivalent to $MA_{\aleph_1}$. If we consider instead the partial orders which preserve stationary sets then we obtain what is called the Bounded Martin's Maximum BMM. If we assume the consistency with ZFC of the existence of supercompact cardinals, then BMM is consistent with ZFC. Todorcevic (2002) proved, based in work by Woodin and Asperó, that BMM implies that the cardinality of $\mathbb{R}$ is $\aleph_2$ (this result has been very recently improved by Justin Moore, who showed that a weaker principle known as $BPFA$ is enough). Bagaria (1998) argues in favor of considering $ZFC + BMM$ as a natural extension of the theory ZFC, which decides the cardinality of the continuum.

Woodin has proved that if there is a proper class of Woodin cardinals then the theory of $H(\omega_1)$ cannot be changed by forcing. He has also formulated a statement, called Axiom $(*)$, which decides all the important facts about $H(\omega_2)$ and makes the theory of $H(\omega_2)$ generically absolute. Thus, axiom $(*)$ is a candidate for playing with respect to $H(\omega_2)$ the role played by projective determinacy with respect to $H(\omega_1)$.

The facts of $H(\omega_2)$ decided by $(*)$, are precisely determined using a strong logic, called Ω-logic, introduced by Woodin. $(*)$ decides, in Ω-logic, the theory of $H(\omega_2)$. This axiom, which is equivalent to a generalization of BMM if there is a proper class of Woodin cardinals, also implies that the cardinality of the continuum is $\aleph_2$. Assuming the existence of a proper class of Woodin cardinals and that there is a weakly inaccessible cardinal which is a limit of Woodin cardinals then axiom $(*)$ is consistent with ZFC in Ω-logic.

Woodin has shown that under large cardinal hypothesis, any extension of ZFC that decides in Ω-logic all statements of the same formal complexity as CH, must refute the continuum hypothesis. As it was just mentioned, this is the case of the theory obtained adding to ZFC the axiom $(*)$. The Ω-conjecture formulated by Woodin, if shown to be true, would provide a characterization of validity in Ω-logic, and would contribute to view axiom $(*)$ as a natural axiom of set theory.

10. FINAL REMARKS

It is interesting that Gödel conceived an argument to show that $\boldsymbol{c}$, the cardinality of the continuum, is $\aleph_2$ (see Gödel (2001)). Although the argument was not complete, it indicates the direction in which Gödel's intuition pointed.

Woodin asserts that even if he is not completely convinced that his approach will lead to a solution of the continuum problem, it certainly provides convincing evidence that there is a solution. Even is the continuum problem is solved by understanding completely the structure $H(\omega_2)$, this will not necessarily convey information abut the structures $H(\omega_3)$, $H(\omega_4)$, etc., so progress towards resolving the continuum hypothesis will not necessarily help in resolving the generalized continuum hypothesis, the assertion that for every ordinal α,

$$2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

There are other views and other approaches. For example, Foreman (1998) has formulated some axioms of generic large cardinals which settle the continuum hypothesis proving it. The main question is which axioms can be considered intuitively true.

The search for new axioms that would resolve the continuum problem has been rewarding. It has given rise to impressive developments establishing unexpected connections between ideas previously thought to be unrelated, and has also activated philosophical investigations on foundational aspects of mathematics (see, for example, Feferman *et al.* (2000)).

The continuum problem has, no doubt, motivated a large amount of research, with results that have enriched our view and understanding not only of set theory but also of foundational questions of other areas of mathematics.

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