

BOOK REVIEW

Michael Potter, *Set Theory and Its Philosophy*. Oxford: Oxford University Press, 2004. xiii + 345 pp.

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This book contains a comprehensive presentation of set theory that discusses its mathematical, philosophical, and historical aspects. The author intends to render the main technical and conceptual problems of set theory accessible to both mathematicians and philosophers.

The book has four parts and three appendices. In the first part we find a survey of some conceptual problems, such as the nature of the axiomatic method; the use of second-order axioms in the study of set theory; the distinction between sets and classes, among others. Furthermore, the author introduces the theory ZU that will be used throughout the book. ZU is a first-order theory that has a domain structured in cumulative levels. It has three axioms: the axiom of creation (p. 61), the axiom of infinity (p. 70), and all instances of the separation scheme (p.42). It has individuals but nothing is assumed in relation to them.

Potter shows, in part II, how it is possible to define the structures of natural, rational, and real numbers from ZU, as well as how to attain results involving Souslin lines, Baire lines, and archi-

medean ordered fields. The topics covered in this part of the book are the same as those that we find in introductory books on set theory; the only difference is that Potter presents us with more advanced problems.

In the next part, the author works out a theory on the size of infinite sets, and this is done in the usual way with the study of arithmetical properties of cardinal and ordinal numbers. There is no philosophical discussion in this part, only small historical notes at the end of each chapter. Here again, we come across some results that are not usually found in introductory treatments of transfinite arithmetic. Thus, the author proves the Cantor-Bendixson theorem (p. 183), Shönflies theorem (p. 200), and establishes Zermelo's categoricity theorem (p. 188), among others.

In part IV, Potter discusses some problems related to the axiom of ordinals, the reflection principle, the axiom of choice (in part II Potter has only assumed the axiom of countable choice, p. 161), the axiom of constructibility, limitation of size principles, among others. He considers, for instance, the theory ZfU that is the theory whose axioms are those of ZU together with the axiom of ordinals (p. 218). In chapter fourteen the author presents different versions of the axiom of choice, and explores the relations between them. In the next chapter, Potter goes back to cardinal arithmetic, establishing several results involving cardinal numbers and the axiom of choice. Thus, he proves that the axiom of choice is equivalent to the assertion that any two cardinals are comparable, and the proposition that establishes that the axiom of choice is equivalent to the assertion that $a + b = ab$ for any infinite cardinal numbers a and b . The first theorem was proved by Hartog in 1915, and the second by Tarski in 1924. The last theorem proved in this part is Sierpinski's theorem that establishes that the generalized continuum hypothesis entails the axiom of choice (p. 280).

The appendices provide a survey of technical and philosophical problems that involve ZFC, transitive sets, the axiom scheme of

replacement, classes, virtual classes, the abstraction scheme, classes and quantification, impredicativity, etc., and no technical results are established in them.

The main virtue of Potter's book is presenting almost all important conceptual problems and technical results of set theory in one volume. His decision of not treating the underling logic in the usual way, with the definitions of formation and inference rules, has the advantage of presenting important theorems without wasting time with minor details. Moreover, his analysis of the axiomatic method (pp. 6-11), the use of second-order axioms (pp. 13-15), and the conceptual problems related to the axiom of choice (pp. 238-259), among others, provides a clear picture of some of the main philosophical problems of set theory.

However, the greatest problem with Potter's approach is that it deals with a huge number of problems, possible solutions, historical problems, theorems, etc., and in the end, it becomes very difficult to understand what is really at stake. So, if on one hand we have the advantage of having a great catalogue of the main results of set theory, on the other hand these problems are treated superficially. This is so especially in the conceptual part of the book. A good example of this is the discussion of the problem of reducing arithmetic to set theory (p. 150). Here we have one of the most important conceptual problems involving set theory, and Potter discusses it in two paragraphs.

Furthermore, the author usually presents all possible solutions to a problem and frequently it is very difficult to understand what his position is (see, for instance, his discussion of the paradoxes of set theory, pp. 26-7). His analysis of the use of diagrams in proofs (p. 84) and the notion of *ordered pair* (p. 64) are other examples of how obscure and superficial a problem can become when not treated properly. In the former case, the discussion is so compact that it is difficult to grasp what the point is. Potter's conceptual analysis of *ordered pair* is so concise that he does not refer to Carnap's conception of *explication*,

which is crucial for understanding Quine's position, for instance. These are only some examples (more can easily be found) of his way of presenting the majority of philosophical problems in his book.

The mathematical results Potter establishes also have problems. The author's formulations of some theorems are very obscure, and he does not provide all the information that is necessary in order to grasp their full meaning. Thus, his statement of the Lindström theorem (p. 13) is incorrect. We must add that a language L is first order equivalent if and only if, beyond the Löwenheim-Skolem property, it is also complete or compact, and it is closed under all the first-order syntactic operations. Moreover, the author does not mention the fact that there are several ways of formulating the Löwenheim-Skolem theorem. He does not mention the fact that there are the *upward* and the *downward* versions of it. The absence of this distinction is another fact that makes his discussion of the Löwenheim-Skolem paradox defective (pp. 114-16, and pp. 240-241).

A similar problem appears, for instance, with the Schröder-Bernstein theorem (p. 156). This theorem is one of the main results of set theory and Potter gives a proof that is so compact that it is difficult, at first sight, to understand its meaning. The author does not inform the reader that there are other proofs of this theorem (some, but not all, need the assistance of the axiom of choice), and he does not provide sufficient information that could help the reader to understand the importance of this theorem.

In the beginning of his book, Potters says that it "was written for two groups of people, philosophically informed mathematicians (...), and philosophers with a serious interest in mathematics" (p. v). It is not clear what Potter means when he uses these expressions. However, after reading this book, it seems natural to conclude that Potter's book is too philosophical for the mathematicians, and too mathematical for the philosophers.