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THE LOGICAL CHARACTER OF NUMBER: REPLY TO ABEL LASALLE CASANAVE

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Abstract: In §1 I discuss Dedekind and Frege on the logical and structural analysis of natural numbers and present my view that the logical analysis of the notion of number involves a combination of their analyses. In §2 I answer some of the specific questions that Abel raises in connection with Chapter 9 of *Logical Forms*.

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Abel rightly says that I favor a conception of analysis as clarification. In fact, I agree that analysis involves a search for essence – at least in most cases. The question¹ that Abel raises at the end of his comments is quite pertinent to my aims and I will try to answer it here. Before I do, however, I should point out that my book is not specifically about problems in the philosophy of mathematics, but has a more general nature. So even though I discuss various issues related to

¹ “In what does the logical analysis of the concept of number consist, for Oswaldo Chateaubriand?”

mathematics, these are not sustained discussions in which I am trying to present a systematically worked out view. This is reflected in my discussion in Chapter 9, which is the basis for most of Abel's comments. Parts of my discussion in that chapter are comments on Frege's views; other parts are comments on Plato's views; yet other parts are speculations to which I am not committed; and still other parts are actual views that I hold. It is not always easy to tell what is one thing and what is another, and the tensions that Abel discerns in my discussion are due to this. I will try to clarify some of the ambiguities below².

1. THE NATURE OF NUMBER

I still consider Dedekind's essay the deepest analysis of the structure of the natural numbers. It is, in my view, a paradigm of conceptual analysis, and I never cease to be amazed by its elegance and by how much is accomplished in the space of mere 50 or 60 pages. Nothing that has been written on the nature of number gets even close to Dedekind's essay – especially if we join it to his letter to Keferstein – as a mathematical conceptual analysis of the structure of the number series. In which way does it fall short of being a complete logical analysis? I think that there are three main weaknesses in this regard.

One is the account of what is logical in the preface to the first edition. Dedekind says at the outset:³

In speaking of arithmetic (algebra, analysis) as merely a part of logic I mean to imply that I consider the number-concept entirely independent of the notions or intuitions of space and time – that I rather consider it an immediate product of the pure laws of thought.

² In connection with the subjects that I will discuss here, including the relation between Dedekind and Frege, there is a very detailed and interesting discussion in Dummett (1991).

³ The quotations from Dedekind *Was sind und was sollen die Zahlen?* are from the translation in Ewald (1996).

And a little later:

If we scrutinize closely what is done in counting a set or number of things, we are led to consider the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing, an ability without which no thinking is possible. Upon this unique and therefore absolutely indispensable foundation ... the whole science of numbers, must, in my opinion, be established.

Although I quite agree with these ideas of Dedekind – and he does a masterful presentation of the concepts of set, relation, function, etc. – his book does not contain a deep analysis of the logical as such. His attitude is essentially that of a mathematician: given these basic ideas, let us develop the theory. And he does.

Another related point, it seems to me, is that there isn't a clear logical account of cardinality attributions. There is a theory of cardinality, to be sure, but it is not quite clear what is the logical character of a cardinality attribution. If I say that there are ten people in the room, what is logical about that? The answer that we derive from Dedekind is that we can count the set of people in the room by establishing a one-one correspondence with Z_{10} – the set of positive integers smaller than or equal to 10. This is fine as far as it goes, but in my view something is missing.

The third weakness is Dedekind's proof of Theorem 66 that there are infinite sets. Although this proof has been much criticized, I am actually rather sympathetic to it. I think that it, or something like it, can be used as an argument for the existence of *potential* infinities, but I do not think that it can be used to establish the existence of *actual* infinities. I think, for example, that an intuitionist can argue more or less *à la* Dedekind for the potential infinity of the natural numbers, but the argument will depend essentially on his intuition of time. *Logical* infinities, on the other hand, independent of intuitions of space and time, must in my view be actual infinities.

It is in connection with the first two of these points that I emphasize Frege's ideas. The depth of Frege's analysis of the logical is as impressive as the depth of Dedekind's analysis of the number structure; especially because Frege went at it from so many different directions – ontological, epistemological, linguistic, formal, axiomatic, etc. One might disagree here and there but there is no question that he did bring an enormous amount of light into the nature of the logical. In particular, he had a very deep insight into the logical structure of reality and the logical nature of cardinality. This insight is his distinction between properties of things and properties of properties that he claims in "Function and Concept" to be "founded deep in the nature of things" (p. 31). He realized that cardinality attributions are attributions to properties of things and not to the things themselves and that these higher-order cardinality properties are purely logical. These are the properties *Nullness*, *Oneness*, *Twoness*, etc. that I emphasize. They are essentially the Platonic forms that are structured as the natural numbers – and according to (Wedberg's interpretation of) Plato they *are* the (pure) numbers. But Frege had the idea that numbers are *objects* rather than properties, because they can be referred to by definite descriptions and names that appear in subject position in sentences. This is an idea that I criticize at length in my book, and it leads Frege to introduce extensions as objects and to characterize numbers as those objects that are the extensions of the cardinality properties. This is one place where I see Frege's analysis as going awry.

Although Frege already had a very insightful logical analysis of the structure of the number series in *Begriffsschrift* using properties (functions), the decision in *Grundlagen* to bring in extensions as objects – on the ground that numbers *must be* objects – compromised his whole enterprise. As I see it, the problem is not so much that Basic Law V leads to contradiction, but the *character* of this law – and of Hume's Principle, which is essential for Frege's proof that there are infinitely many numbers. When we read them intuitively, both of these principles seem so simple and so clearly true that it is hard to see how the first could lead

to contradiction and the second could lead to the existence of infinitely many objects. The basic insights behind these principles are:

(S_E) The concept F has the same extension as the concept G iff
 $\forall x(Fx \leftrightarrow Gx)$

(S_N) The concept F applies to the same number of things as the concept
 G iff $F \sim G$,

where ' $F \sim G$ ' means that F can be put into one-one correspondence with G . What could be more natural than that?⁴

The problem arises, in both cases, from introducing *singular terms* 'the extension of F ', 'the extension of G ', 'the number that belongs to F ', 'the number that belongs to G ' and formulating the principles as:

(V) the extension of $F =$ the extension of G iff $\forall x(Fx \leftrightarrow Gx)$.

(HP) the number that belongs to $F =$ the number that belongs to
 G iff $F \sim G$.

For now, as if by magic, it turns out that these singular terms denote *objects*, and extensions and numbers begin to appear in all their infinite

⁴ (S_E) has the air of a mere terminological convention: when the concepts F and G apply to the same things, then we will say that they have the same extension. (S_N) makes a more interesting statement, especially when we consider concepts that apply to infinitely many things, but still seems ontologically very innocuous. If one reads '*has the same number as*' or '*the number of F 's is the same as the number of G 's*' for '*applies to the same number of things as*', then there is a stronger ontological suggestion.

multiplicity. I do not deny that there are extensions and numbers, but I disagree that they can be shown to exist by such linguistic maneuvers⁵.

So just as I think that the considerations in Dedekind's theorem 66 do not prove the existence of an actual infinity of objects, I also think that the considerations in the formulation of Hume's Principle do not prove the existence of an actual infinity of objects. Hence, I do not consider that the reconstruction of Frege's arithmetic in second-order logic by means of Hume's Principle establishes that the concept of number has a purely logical character⁶.

My view is that one should work with the properties themselves and that it is a logical axiom that there are infinitely many logical properties – including the properties *Nullness*, *Oneness*, *Twoness*, etc. These are the numbers, and their logical character was clearly revealed by Frege's work. With my argument in p. 425 that properties can be both subject and predicate I try to counteract Frege's claim that since numbers are referred to by expressions appearing in subject position, then they must be objects. I think, therefore, that the logical analysis of number is the combination of Frege's analysis and Dedekind's⁷.

A problem with this approach is that a cardinality property such as *Nullness*, or *Oneness*, is not an ontological unit, but will appear all over the hierarchy of properties. And even though in my account properties can accumulate, it is not possible to accumulate all the *Nullness* properties

⁵ In other words, although one may postulate that for every concept F there is an object that is the extension of F and that there is an object that is the number that belongs to F , this does not seem to follow from our intuitive understanding of (S_E) and (S_N) – nor does it follow that these postulations have a logical character.

⁶ This is not to deny the logical interest of the work that has been done in this direction both by the defenders and by the opponents of this view (Wright, Boolos, Hale, Heck, etc.).

⁷ To these I would also add Peano, whose (second-order) axiomatization gives a more intrinsic characterization of the number structure.

into a single *Nullness* property. It is in this connection that I make a reference to Dedekind in p. 319. After the first passage I quoted above Dedekind continues:

My answer to the problems propounded in the title of this paper is, then, briefly this: numbers are free creations of the human mind; they serve as a means of apprehending more easily and more sharply the difference of things.

My suggestion in p. 339 (note 38) – admittedly somewhat enigmatic and certainly not worked out – is that *in the most general sense* numbers might have the character of intersubjective abstractions based on the nature of things. And I do not think that it follows from this that the number concept is not a logical concept.

2. SOME REMARKS ON CHAPTER 9

I was never completely satisfied with Chapter 9 because too many different things are discussed in it. I tried to revise it in various ways, even to split it up into different discussions, but I did not manage to restructure it to my satisfaction. So I let it stand even though it does not have a main line of argument. I will not discuss all the different aspects of Chapter 9 here, but I will comment on those that are related to Abel's questions.

One of his main questions is whether I view mathematical objects as having an irreducible non-logical character. I do not, but I indulge in a fair amount of speculation about it. At the very beginning I say that for Frege “a proper account of the ontology of arithmetic must have the numbers as objects”⁸. In p. 313 I go back to this issue and start the long speculative discussion about it that goes on until p. 317. The way I thought about this when I wrote the chapter was as follows. Frege

⁸ P. 297. In p. 314 I refer back to this remark in a way that might perhaps suggest that I am agreeing with Frege.

postulates extensions at level 0 and the interaction between these extensions and the properties in the hierarchy led to the contradictions that affected his system. So what should one do if one wants numbers to be objects? My initial idea was along the lines of Columbus' solution to the problem of getting an egg to stand on end: if one wants to have sets as level 0 objects, why not postulate something like (the objects of) Zermelo-Fraenkel set theory and be done with it? This led me to the discussion of the pure set structures and of the difficulty of talking about structures without objectifying their content in some way. Which led, in turn, to the discussion of Plato and Gödel in notes 23-30. Let me comment briefly on some of these issues.

I think that Plato's distinction between ideal numbers and mathematical numbers (as interpreted by Wedberg) is very interesting and connects in several ways with what I was saying above about the nature of numbers. The mathematician uses exemplars of the structure of ideal numbers, and for the purposes of mathematics *any* exemplar will do as a representation of the structure of numbers. That is the reason why we find the number structure imbedded in so many mathematical structures and theories. Yet it is hard to maintain that these exemplars *are* the numbers, or that there is nothing to number aside from these exemplars. This is essentially Quine's position which I reject⁹. Although the structure consisting of the ideal numbers *Nullness*, *Onewness*, *Twoness*, etc. may not be *the* number structure either, it captures the fundamental character of numbers as individual cardinality properties – and each element of this structure may be considered a self-subsistent entity in a sense possibly akin to Frege's. But what is the structure *as such*? Now I think that it is the *successor relation* itself – *i.e.*, considered in intension –

⁹ As Abel mentions, I discuss Quine's position at some length in my thesis. Some of this discussion is also in my paper "Ockham's Razor", which is a preliminary version of Chapter 24.

although when I wrote those remarks on structures I was still trying to find a way to come to grips with them¹⁰.

Another aspect of the discussion in Chapter 9 is to understand better the notions of extension and of set. The notion of extension that I characterize in terms of states of affairs (pp. 311-13) is a quite adequate representation of many intuitions that we have about the extension of properties. The pure set structures discussed in pp. 313-17, on the other hand, are not at all good candidates for extensions, but they may be good candidates for mathematical “objects”. And since these pure set structures are pure cardinality structures or iterations of pure cardinality structures, one could say that in this sense all mathematics can be reduced to the study of number¹¹. Moreover, if cardinality is a logical notion, this gets us very close to a logical account of mathematics. In fact, in p. 314 I say that these cardinality structures could be taken to be logical properties rather than objects, but since there is “a generalized feeling that a structure is something like an object” I go on to examine them from this point of view.

As I said at the beginning many of these ideas are preliminary ones that I did not attempt to develop systematically in the book, but I hope to develop them in a not too distant future.

¹⁰ As I mention in several places (*e.g.*, note 37 p. 40) Frege argued that the axioms of Geometry define a higher-order concept. In the same sense, we may formulate the Peano axioms as a big predicate that characterizes the successor relation along the lines suggested in (c) of note 5 (p. 207) for the axiom of induction (but with only ‘ $\exists xy$ ’ as argument). For this to be a purely intensional characterization we must interpret the quantifiers intensionally rather than objectually, but this is not something that I develop in the book. (Though I did give a talk about it in the VI Colóquio Conesul de Ciências Formais in 2002.)

¹¹ There are some remarks along these lines in Tarski (1986, p. 151). In this connection see also §1 of my reply to Frank Sautter.

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