FREGE’S VIEWS ON VAGUENESS

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Abstract: The purpose of this paper is to discuss Frege’s view on vagueness, and to draw some relevant consequences of it. By examining what exactly Frege has in mind each time he complains about vagueness and advocates the sharpness requirement, I argue that he shows preoccupation with different kinds of vagueness in different periods of his thought. I also discuss the scope of the sharpness requirement, and argue that it is intended as applying primarily to mathematics and logic. Finally, I try and argue that some of Frege’s remarks on incomplete functions suggest a view that is close in spirit to the contemporary supervaluationist approach to vagueness.


There is an aspect of Frege’s thought that has attracted very little attention in the scholarly literature, namely, his view on vagueness.¹ This is certainly not surprising. He insists on many occasions that concepts (or,

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¹ As far as I know, the literature devoted specifically to Frege’s view on vagueness is restricted to van Heijenoort, 1986 and Burge, 1990.

more generally, functions) should satisfy the requirement of sharp delimitation, i.e., should have a determinate value for every object as argument. But in most of his remarks, Frege expresses a purely negative view of vagueness. He seems to show no interest whatsoever in developing a positive theory of vagueness or of a vague language. This explains in part the lack of interest that most scholars show towards his (few) remarks concerning vague languages and vague concepts.

Philosophical interest in the phenomenon of vagueness is indeed something that flourished only later in the twentieth century. We now know, for example, that there are at least three different kinds of phenomena that philosophers may have in mind when they talk about vagueness, namely, semantic, epistemic and ontological vagueness. Semantic vagueness is, roughly speaking, the failure of terms to have sharp meanings. A proper name like ‘Atlantic Ocean’ is vague because there is no clarity regarding the exact portion of whater to which it refers: there is a multiplicity of portions of whater with precise boundaries to which this name might refer, and in a way it refers to all of them, but it is not committed to any one in particular. We might be unclear regarding the truth-value of the sentence ‘John is an adult’ (knowing that John is 15 years old), and this is so because ‘adult’ has no clear extension (and no clear intension), and probably we should say in this case that the sentence simply has no truth-value. Epistemic vagueness is a different phenomenon, and it has its origin in incomplete knowledge or understanding (or, in some cases, in a deficiency of our senses). We might be unclear regarding the truth-value of a sentence ‘The world’s population right now is smaller than the number of stars in the smallest galaxy in Virgo’. The main source of unclarity here is a limitation in our knowledge: we simply don’t know what the world’s population is right now, just as we don’t know the exact number of stars in the smallest galaxy in Virgo, but there are nevertheless precise numbers.
corresponding to these expressions, and the sentence above is certainly true or false. A higher power of resolution of our instruments could in principle reduce (and maybe eliminate) the unclarity. Ontological vagueness is supposed to be something different: it is the idea that things in the world might themselves be vague, i.e., they might have a vague identity, independently of vagueness in language and of unclear understanding. (Some philosophers advocate the primacy of one kind of vagueness over the others, and some even deny that some of these forms of vagueness exist at all. I shall not argue for any of these views here. I shall take it for granted that there are three ways of understanding vagueness.)

In this paper I do not intend to extract a clear and coherent theory of vagueness or of vague languages from Frege’s writings. I agree with Michael Dummett that he was not primarily interested in offering anything like this. What I intend to do is rather to make some remarks concerning three points. First, I want to ask what exactly Frege has in mind each time he complains about vagueness and advocates the sharpness requirement. Some scholars, like van Heijenoort (1986), and some specialists on vagueness, like Williamson (1994), tend to focus almost exclusively on Frege’s worries about semantic vagueness. Others, like Burge (1991), tend to focus only on Frege’s concern with epistemic vagueness. Almost no one, as far as I know, has recognized any concern with ontological vagueness in Frege. As I shall argue, there is evidence that he actually shows preoccupation with all three different types of vagueness, but with different emphasis at different times: he advocates the exclusion of predicates that are semantically vague in his early writings; he criticizes what he sees as a vague understanding of key mathematical concepts in his middle period; and he shows some preoccupation with the possibility of improper objects and concepts in his mature period. Frege himself does not distinguish clearly, but these
are different elements of a philosophical view on vagueness. Second, I shall access Burge and van Heijenoort’s interpretation according to which Frege presupposes the existence of a sharp concept behind each apparently vague predicate. I want to argue that the evidence makes it more plausible to suppose that this is true only of those concepts belonging to a priori sciences, but not necessarily of those from empirical sciences, and even less of ordinary concepts. Third, I shall argue that there is a way of interpreting Frege’s view on semantic vagueness in mathematics that sees it at least in spirit as compatible with the approach to vagueness nowadays known as supervaluationism, which was developed in the 70s by Kit Fine and van Fraassen, among others.

What is the point of investigating a topic about which Frege says so little? In the first place, understanding what exactly Frege sees as the enemy of logic can throw some light on what the point is of the requirement of sharp delimitation. Second, the evidence goes against a standard interpretation according to which Frege is the paragon of the view that logic can only deal with expressions with sharp meanings, a view of which supervaluationism would be the refutation. This is certainly an oversimplification of Frege’s perspective. Finally, questions of vagueness might have some important implications for the Julius Caesar problem, although I shall not pursue these implications in detail here. Arithmetic is universally applicable, according to Frege, but it is also only applicable to sharp concepts, as he comments on some occasions. If some concepts like *identical with Julius Caesar* are vague, then they should not in principle be a source of worry for the identity of numbers.

1. THE BEGRIFFSSCHRIFT AND SEMANTIC VAGUENESS

Frege’s well known criticism of vagueness and lack of stability of natural language appears at the very beginning in his philosophical
writings. The *Begriffsschft* (1879; henceforth simply *BS*\(^2\)) is considered by most commentators as the *locus classicus* where the ideal of precision implicit in modern logic is formulated. Frege’s remarks about vagueness in the *BS* and in related works from the same period occur in the context of his broader criticism of ordinary language as defective for logical purposes. In the essay “On the Scientific Justification of a Conceptual Notation” (from 1882) he says:

Language is not governed by logical laws in such a way that mere adherence to grammar would guarantee the formal correctness of thought processes. The forms in which inference is expressed is so varied, so loose and vague, that presuppositions can easily slip in unnoticed and then be overlooked when the necessary conditions for the conclusion are enumerated. (*BS* 108/\(CN\) 84-85).\(^3\)

At this point, Frege does not seem to recognize any relevant difference between vagueness and ambiguity, on the one hand, and shifts of meaning according to context on the other. The predominant concern in Frege’s remarks in the introduction of *BS* is the elimination of semantical aspects of natural languages that might be very appropriate for the purposes of ordinary life and communication, but that are also inadequate for the purposes of logic. Frege looks at vagueness as a typically linguistic phenomenon, i.e., as a defect of ordinary language, along with other defects.

\(^2\) See bibliography for the other abbreviations of Frege’s works used in this paper.

\(^3\) The same point would be repeated in a letter to Peano where Frege says that “our vernacular languages are also not made for conducting proofs. And it is precisely the defects that spring from this that have been my main reason for setting up a conceptual notation” (*WB* 183/\(PMC\) 115).
The kind of vagueness with which Frege is mainly concerned in *BS* is hence to be solved by semantical stipulations, i.e., by attaching a definite and sharp meaning to proper names and names of functions. One of the most important features of the newly created symbolic language is that written symbols are to replace words, and are placed, so to speak, in a semantic prison, so that their meaning becomes sharp and rigid. And this effect is achieved, as Frege comments in “On the Scientific Justification of the Conceptual Notation” by exploring several advantages that written symbols have over sounds for logical purposes. The main advantages come, according to him, from the psychological effects that written symbols have upon us, such as permanence, immutability, the “possibility of keeping many things in mind at once”, and the fact that symbols are “set further apart [than the spoken word] from the course of our ideas [Vorstellungsverlauf]” (*BS* 110-1/CN 87).

2. *GRUNDLAGEN AND EPISTEMIC VAGUENESS*

The focus changes considerably in *Grundlagen der Arithmetik* (1884). Here, the predominant worry is clearly with epistemic vagueness, i.e., the blurred understanding of fundamental mathematical concepts. *BS* opened with a complaint about ambiguity and instability of language, but *GLA* opens with a complaint about the poor understanding that most mathematicians have about the concept of number. While the remedy for vagueness in *BS* was semantic stipulation, the remedy for the kind of vagueness that is most damaging in *GLA* is philosophical analysis. That is to say, while the origin of the kind of vagueness that Frege condemns in *BS* is ordinary language, the origin of the kind of vagueness that he criticizes in *GLA* is the lack of a deeper investigation into the nature of mathematical concepts (although Frege thinks that this investigation may also be impaired by logical defects of natural language).
In the opening sections of *GLA* Frege stresses the point that the pressure for sharpening the understanding of fundamental concepts in mathematics comes from the demand for proofs:

Proof is now demanded of many things that formerly passed as self-evident. Again and again the limits to the validity of a proposition have been in this way established for the first time. The concepts of function, of continuity, of limit and of infinity have been shown to stand in need of sharper definition. Negative and irrational numbers, which had long since been admitted into science, have had to submit to a closer scrutiny of their credentials. (*GLA* §1)

Indeed, he contrasts the loose forms of reasoning usually practiced in analysis with the more rigorous forms of reasoning usual in geometry (Frege has the Euclidean system as a model), and blames this lack of commitment to proofs for the foggy understanding that mathematicians have of the most fundamental concepts in analysis (*GLA* §§1-2). Frege’s remarks certainly reflect the process of rigorization of analysis that was taking place in the second half of the nineteenth century. But notice that he is not alluding to a multiplicity of concepts which one might refer with the term ‘function’, for example, but rather to one specific concept, whose borders need to be better understood. Indeed, Frege thinks that the fundamental concepts involved in arithmetic and in logic are objective, i.e., not open to stipulation. (In “Function and Concept” (from 1891) for instance, he will say that the difference between first and second level functions is founded deep in the nature of things (*KS* 142/CP 156).) Frege seems to be concerned that, although terms like ‘function’, ‘continuity’, and so on have one precise meaning, most mathematicians do not know exactly what they are. Incomplete understanding here is both the cause and the product of a mathematical practice in which there is no commitment to proofs.
In *GLA* §28, in criticizing those authors that define numbers as sets or pluralities, Frege talks about the vagueness of this concept, which seems to shift its meaning from author to author. He says:

Moreover, these terms are utterly vague: sometimes they approximate in meaning to heap or group or agglomeration, referring to a juxtaposition in space, sometimes they are so used as to be practically equivalent to Number, only vaguer. No analysis of the concept of Number, therefore, is to be found in a definition of this kind.

In this passage, the complaint is primarily about the semantical vagueness of the term ‘set’, but Frege’s main point is that the corresponding concept is also poorly understood by the authors in question, and this brings it dangerously close to things like heap and agglomeration, that are essentially empirical entities.²

As I said, semantic vagueness is something that the *Begriffsschrift* in itself can at least in principle eliminate; however the epistemic vagueness surrounding fundamental concepts of mathematics is something that only philosophical analysis can eliminate. To illustrate this point, we can remember Frege’s question raised in *GLA* §10 as to whether number is definable or is a primitive notion. Of course the answer to this question yields a sharper understanding of the concept of number, but it is not achieved purely by arbitrarily fixing the sense or the reference of the term ‘number’; more needs to be provided by philosophical analysis.

² In the same vein, he complains in *GLA* §29 that Euclid uses the term ‘monas’ ambiguously, meaning “sometimes an object to be counted, and sometimes a property of such an object” and that the German word ‘Einheit’ is also ambiguous in this way, being therefore adequate to translate the Greek term. Again, Frege is pointing at an ambiguity, but the ambiguity is a symptom of something deeper, namely, the lack of sharp understanding of the corresponding concept, which is objective and presumably sharp.
Concepts are objective entities for Frege, that is to say, they are not created by us, but have an eternal existence, with determinate borders. In a number of places Frege famously expresses his view that concepts have no history, that the so-called history of a concept should rather be understood as the history of the attempts to grasp a concept (GLA vii, KS 122/CP 133). And the discovery of the true borders of a concept might be a task that takes a very long time and demands a huge intellectual effort (GLA vii). In the essay “On the Concept of Inertia” (from 1891) we have a picture of how Frege saw the progressive elimination of epistemic vagueness as a central task of science. In commenting on Lange’s claim that the main motivation for developing concepts in science is the elimination of contradictions, Frege says:

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\text{[C]ontradictions were indeed a driving force behind the search, but not contradictions in the concept; for these always carry with them a sharp boundary: it is known that nothing falls under a contradictory concept [...] The real driving force is the perception of the blurred boundary. In our case too, all efforts have been directed at finding a sharp boundary. (KS 123/CP 134)}
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Notice that the problem pointed out in the passage is the discovery of objective boundaries, and not the creation of arbitrary boundaries for concepts in science. (The particular case discussed by Frege in this context is the concepts of motion and rest in rational mechanics.) Fundamental concepts are already there to be discovered, and our task as scientists is to formulate hypotheses about their real boundaries. These hypotheses might be right or wrong. Frege criticizes Lange for trying to find a border between the concepts of motion and rest where, according to him, there was no border to be found (KS 123/CP 134).
3. GRUNDESETZE AND ONTOLOGICAL VAGUENESS

There are some few passages in Frege’s mature period in which he shows preoccupation with a sort of vagueness intrinsic to objects and concepts (rather than vague reference, or vague understanding of concepts). In a number of places he refers to concepts that are intrinsically incomplete. Frege calls them “concept-like construction” ("begriffsähliche Bildungen") in GGA II §58, and “pseudo-concepts” ("Scheinbegriffe") in GGA II, §62. In “On the Concept of Inertia” he excludes from the field of logic those concepts that are not sharp, but does not go as far as denying their existence. Actually there are comments on things that are concept-like, but can never become a concept already in GLA despite the predominant worry with epistemic vagueness then. In GLA §30, Frege criticizes Baumann’s definition of the number one as a property, i.e., as “[w]hatever we take as a point, or refuse to take as further subdivided into parts, that we even regard as one [...] Every idea is one when isolated [...] into a many”. As Frege points out here, the attribution of the property “one” to an object would be dependent on the way we look at it. But now he complains that a precise science like arithmetic could not be based on such a “hazy concept” ("veschwomene Begriff”). Here the complaint is about an intrinsic feature of this concept – it has hazy boundaries, which are dependent on subjective dispositions.

What about vague objects? In the famously complex GGA I §10, Frege develops an argument to show that the reference of value-range names is less than absolutely clear. As he comments, for all that we know (up to that point in the text), a value-range name can have the intended reference (i.e., the value-range of the corresponding function) or can refer to a truth-value that is univocally associated with this value-range. That is to say, Frege’s point is that, for all that we know, value-range names are ambiguous. Hence, he is pointing primarily at a semantical problem. But
the strategy that he suggests for solving the problem shows a concern for
the identity of value-ranges as objects. He outlines his strategy in the
following famous passage:

How may this indefiniteness be overcome? By its being determined for
every function when it is introduced, what values it takes on for courses-
of-values as arguments, just as for all other arguments.\(^5\) (\textit{GG.A.I} §10)

The strategy seems to target the specification of value-ranges as objects,
since up to this point in the text they are still neither clearly identical nor
clearly distinct from truth-values. It seems to me that Frege had in mind
an application of Leibniz’s principle of identity here, i.e., in order to know
whether two objects are identical or distinct, we have to ask whether the
values of all functions coincide or not for these objects as arguments.
In this particular case, it all comes down to the value of identity
between value-ranges and truth-values. Now of course the way Frege
solves the indeterminacy (i.e., by identifying value-ranges with truth-
values) is authomatically a solution for the ambiguity problem as well,
since now there is only one kind of object as candidate for the reference
of value-range names, namely, the value-ranges themselves. (But, of
course, were Frege to keep truth-values and value-ranges apart, there
would be a solution for the vague-identity problem, but the ambiguity
problem would still persist.) This interpretation is supported by Frege’s
comment in the last paragraph of \textit{GG.A.I} §10:

With this we have determined the courses-of-values so far as is here
possible. As soon as there is a further question of introducing a function
that is not completely reducible to functions known already, we can
stipulate what value it is to have for courses-of-values as arguments; and

\(^5\) “Courses-of-values” is Furth’s translation for the term “\textit{Wertverlauf}”, which
I am translating as “value-ranges” in this paper.

this can then be regarded as much as a further determination of the courses-of-values as of that function.

In this passage he seems to have in mind the determination of entities (objects, functions) and not of the reference of the corresponding names, although he started the section raising the semantical problem.

Another place in which Frege shows preoccupation with the vague identity of value-ranges themselves (rather than with the semantic property of their names or with our incomplete understanding of them) is in those passages in which he reacts to Russell’s paradox. Both in the famous appendix to _GG:A_ and in a letter to Russell dated from September 23, 1902, Frege considers the possibility of value-ranges being what he calls “improper objects” (”unäigentliche Gegenstände”), i.e., objects that do not always yield a definite value when taken as argument by some function. These are objects that do not satisfy the law of excluded middle, according to Frege’s description. Frege seems horrified with this perspective. But, curiously enough, his attitude is not the one commonly seen in the modern literature on vagueness and well represented by Dummett’s famous statement in “Wang’s Paradox” that “the notion that things might actually be vague ... is not properly intelligible” (Dummett, 1978, p. 260). The reason Frege raises for rejecting this possibility is mainly technical. According to him, if value-ranges were improper objects, this would imply the existence of a very complex hierarchy of functions, which would be, or so he seems to believe, very hard to deal with in a logical theory. That is to say, the reason for the rejection of value-ranges as vague objects seems to be primarily technical, and not properly metaphysical.6

6. It is ironic here that in his remarks on value-ranges as “improper objects” Frege is considering Russell’s suggestion communicated in a letter from August 8, 1902, of treating classes “not as objects of the ordinary kind”. The same
4. THE SCOPE OF THE REQUIREMENT OF SHARP DELIMITATION

Frege refers to the sorites paradox a number of times, always in connection with semantic incompleteness of terms like ‘heap’. The first time is in BS §27, in the context of the discussion of the definition of ‘y follows after x in the sequence generated by f ’ (proposition 76). One particular consequence of this definition is proposition 81, according to which a property that is hereditary in an f-sequence will transmit from x to y in case y follows after x in the f-sequence and x instantiates this property. Since the property denoted by ‘heap’ appears to be hereditary in the sequence generated by the procedure of removing a grain, it seems to follow that if x is a heap, then y, which follows from x in the sequence after removing all but one grain, is also a heap. Frege points out that although this reasoning seems correct, it is actually not so, since due to the indeterminacy of the concept ‘heap’, ‘z is a heap’ is not always assertable. Frege’s aim here is to illustrate a natural defect of ordinary language, which prevents the universal applicability of the definitions developed in BS.

Frege mentions the paradox again at least on two occasions (WB 183/PMC 114; NS 168/PW 155), in connection with his criticism of Peano’s conditional definitions, although the point here is slightly different from the one from BS. (He is not complaining now about the semantic incompleteness of ordinary language, but rather about the methodological inadequacy of Peano’s partial definitions, since they do not provide for cases that fail to fulfill the conditions.)

A question to be raised now is whether cases like the one represented by ‘heap’ are seen by him as somehow special or

Russell would later write a famous essay on vagueness (Russell, 1923), in which he rejects ontological vagueness as a product of philosophical confusion.

pathological, or whether they represent a sort of paradigm for the terms of ordinary language, which would be thereby generally infected with vagueness. Burge discusses the passages on the sorites, and argues that there is nothing in them suggesting that Frege is assimilating most words of natural discourse to terms like ‘heap’ (Burge, 1991, p. 35).  

Burge deals with an apparent inconsistency in Frege’s thought: on the one hand, he says that most fundamental concepts in mathematics are poorly understood by the mathematicians of his time. On the other hand, he frequently talks about terms in mathematics as having such and such a Bedeutung. But, according to his own doctrine, a term that is vague has no Bedeutung. Now Burge asks: How can we still maintain that these terms have a Bedeutung if they are vague? The problem that generates the puzzle, Burge argues, is the identification of the Fregean notion of sense with that of linguistic meaning. Terms like ‘number’, ‘function’, and so on, do have an established linguistic meaning that is current among mathematicians, but this meaning is, on the one hand, vague in several respects, and, on the other hand, does not necessarily correspond to the senses associated with the relevant terms. Senses are objective and independent from the linguistic practice of mathematicians. That is to say, according to Burge’s suggestion, behind terms with a vague linguistic meaning, there is for Frege an objective sharp sense, and this sense is what fixes the Bedeutung of these terms. Van Heijenoort goes in the same direction, in that he affirms that “Frege is led to postulate, behind each vague predicate of ordinary language, an exact ‘objective’ predicate, so that logic can operate without a hitch” (van Heijenoort, 1986, p. 37).

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7 A substantial part of Burge’s article is devoted to arguing against the interpretation according to which Frege does not really mean that our expressions have definite senses or Bedeutungen.
I tend to agree with Burge and van Heijenoort as far as mathematics, logic and maybe rational mechanics is concerned. But, as I see it, there is no good evidence that Frege meant this to extend to the terms of empirical sciences, and even less to terms of ordinary language, as van Heijenoort suggests. In some passages like the following Frege talks about the admissibility of concepts in science:

"If we ask, under what conditions a concept is admissible in science, the first thing to stress is that consistency is not such a condition. The only requirement to be made of a concept is that it should have sharp boundaries; that is, for every object it holds that it either falls under the concept or does not do so. (NS 193-4/PIF 179)"

It is tempting to understand Frege’s use of the term ‘science’ here in a broad sense, as referring both to formal and to empirical sciences. But I

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8 The evidence that van Heijenoort presents for the claim quoted above is Frege’s following remark in GLA §26:

The word “white” ordinarily makes us think of a certain sensation, which is, of course, entirely subjective; but even in ordinary everyday speech, it often bears, I think, an objective sense. When we call snow white, we mean to refer to an objective quality which we recognize, in ordinary daylight, by a certain sensation [...] I understand objective to mean what is independent of our sensation, intuition and imagination, and of all construction of mental pictures out of memories of earlier sensations, but not what is independent of the reason.

Van Heijenoort takes this passage in connection with Frege’s view expressed in the same section that “objective here is that which is subject to law, conceptual, adjudicable, what can be expressed in words”, and concludes that Frege must be suggesting that behind predicates like ‘white’ there is a sharply defined property whose definition would not be different from those usual in mathematics. But there is nothing in the passage indicating the connection between objectivity and sharpness that van Heijenoort suggests.

think that this generalization is not clearly correct, although Frege does not do much in his writings to prevent it. The point is that the requirement of sharp delimitation, as Frege understands it, only seems to make sense as an answer to the demand for proofs, as is suggested in the following comment:

All that can be demanded of a concept from the point of view of logic and with an eye to rigour of proof is only that the limits to its applications should be sharp. (*GLA* §74)

The requirement is made only if the concepts in question are to play any role in proofs. But, strictly speaking, proofs are the privileged method only in formal sciences. The centrality of proofs is something distinctive of mathematics and logic, in contrast with empirical sciences, where inductive methods are essential (*GLA* §2). Hence, I think, Frege’s use of the term ‘science’ in those contexts in which he defends the requirement of sharp delimitation can be taken in a restricted sense, i.e., as referring to mathematics, logic and possibly rational mechanics. Or, at least there is no clear indication that he means it in the wider sense.

In the essay “Logic in Mathematics” (from 1914), Frege comments on the development of science and concepts. According to him, a science only reaches a final (ideal) state of completeness and clarity when it is formulated in a formal system. He says:

Science only comes to fruition in a system. We shall never be able to do without systems. Only through a system can we achieve complete clarity and order. No science is in such command of its subject-matter as mathematics and can work it up into such a perspicuous form; but perhaps also no science can be so enveloped in obscurity as mathematics, if it fails to construct a system. (*N3* 261/*PW* 242)
The passage suggests that no other science is more apt to treat its subject in the form of a system than mathematics. But the requirement of dealing with the subject-matter in a system is equivalent with the requirement of sharp delimitation, since Frege opposes the representation in a system with the history of concepts in the same essay (NS 261/PW 241-2). He says that “Mathematics has closer ties with logic than does any other discipline, for almost the entire activity of the mathematician consists in drawing inferences” (NS 219/PW 203). That is to say, no science other than mathematics is more justified in aspiring to come to sharply delimited concepts.

To be fair to Burge and van Heijenoort, there is indeed no indication in Frege’s writings of how far he wanted to generalize on the problems intrinsic to concepts like ‘heap’. But there is no indication whatsoever that he saw a sharp meaning behind most ordinary terms. As I said, Burge suggests that terms like ‘heap’ are considered as special pathological cases by Frege, and discusses a number of passages in which terms are said to have Bedeutung. But all of them are passages in which Frege is talking about mathematical concepts. As Weiner notices, Frege generally avoids talking about terms of ordinary language as having a Bedeutung, and hence as having a sharp boundary (Weiner, 1990, p. 112). There is a comment in GGA II §56 that could be taken as a sign that Frege sees the lack of sharpness as extending to a wider range than the pathological terms like ‘heap’:

E.g. would the sentence ‘Any square root of 9 is odd’ have a comprehensible sense at all if square root of 9 were not a concept with a sharp boundary? Has the question ‘Are we still Christians?’ really got a
However, as I see it, the main point of this passage is to contrast a mathematical concept like \textit{square root of 9} with an ordinary concept like \textit{Christian}. Something that might be important here is that the case for the mathematical concept is formulated in the subjunctive mood of the verb, while the case for \textit{Christian} is formulated in the indicative, which suggests that \textit{Christian} is indeed a vague concept, while \textit{square root of 9} would be a similar case, if it were not sharply delimited. Weiner takes this same passage as evidence for the opposite thesis, i.e., she takes Frege to be demanding that an everyday concept like \textit{Christian} must have sharp boundaries (Weiner, 1990, p. 97). However, she is concentrating on the last sentence of the quotation in isolation, and not, as I am, on the contrast with the numerical concept.

As I said, there is no good evidence that Frege thought that the ideal of precision and sharpness could ever be effectively reached by other sciences besides mathematics, logic and some closer sciences like pure mechanics. But there is a passage in the preface to \textit{BS} where he describes how he thought his formal language could be progressively used in science:

\begin{quote}
I am sure that my conceptual notation can be successfully applied wherever a special value must be placed upon the validity of proofs, as in laying the foundation of the differential and integral calculus.

It appears to me to be still easier to extend the area of application of this formula language to geometry. We should only have to add a few
\end{quote}

\footnote{The German text goes as follows: “\textit{Hätte z. B. der Satz "jede Quadratwurzel aus 9 ist ungerade" wohl überhaupt einen fassbaren Sinn, wenn Quadratwurzel aus 9 nicht ein scharf begrenzter Begriff wäre? Hat die Frage "Sind wir noch Christen?" eigentlich einen Sinn, wenn nicht bestimmt ist, von wem das Predikat Christ mit Wahrheit ausgesagt werden kann, und wenn es abgelehnt werden muss?”}

symbols for the intuitive relations that occur there. In this way, we should acquire a kind of analysis situs.

The transition to pure kinematics and further to mechanics and physics might follow here. In the latter fields, where besides necessity of thought [Denknotwendigkeit], physical necessity [Naturnotwendigkeit] asserts itself, a further development of the mode of notation with the advancement of knowledge is easiest to foresee. But this is no reason to wait until such transformations appear to have become impossible. (By XII/CN 106)

Indeed, in his early years, Frege expressed some optimism regarding the possibilities of applying his Begriffsschrift even to empirical sciences. He comments in “On the Scientific Justification of a Conceptual Notation” that the Begriffsschrift could contribute to the progress in natural science, which would presumably lead to the sharpening of empirical concepts, which in turn would facilitate a better application of the Begriffsschrift to science, in a sort of virtuous circle, in the same way that progress in natural sciences leads to the construction of better instruments, which in turn facilitates further progress in natural sciences (By 113/CN 89). But these passages occur very early in Frege’s writings, when he had not experimented himself with sharpening concepts. But this expression of optimism regarding empirical sciences is absent from his later writings: he tends to focus more and more on the requirement of sharp delimitation for the concepts involved in mathematics and logic only. In his correspondence with Peano, for example, his comments are only in connection with mathematical and logical concepts. We can conjecture that, after dealing with the formal system of GG/A, Frege had a better feeling for how hard it is to achieve real sharpness; even the simple concepts dealt with in his formal system are hard to make sharp, as Frege’s remarks in GG/A §10 show.

If this interpretation is correct, there should be some tension here with Frege’s doctrine of the universal applicability of arithmetic. As we
know, one of the pillars of Frege’s argument in *GLA* (against Kant) for the logicality of arithmetic is that it is universally applicable, i.e., the field of arithmetic extends not only to what is spatial and temporal, but to anything that is thinkable through concepts. But, as Frege comments in his letter to Anton Marty from August 29, 1882, in order to be countable a concept must first have sharp boundaries.\(^\text{10}\) Now if it is true that the requirement of sharp delimitation is primarily made only for concepts of the formal sciences, then we have to conclude that the applicability of arithmetic is actually restricted, primarily to those concepts of the formal sciences (e.g., those concepts that Frege defines in his *GGA*), since these are the only really sharp concepts. Applicability to empirical science would be possible, strictly speaking, only in an ideal situation in which science achieves completion.

5. INCOMPLETE FUNCTIONS AND SUPervaluationism

There are some remarks made by Frege in connection with the requirement of sharp delimitation for functions that create a little puzzle. In “Function and Concept” Frege considers the function addition, and explains the reasons why it should have a determinate value for any object (not just numbers) as arguments:

> It seems to be demanded by scientific rigour that we should have provisos against an expression’s possibly coming to have no meaning; we must see to it that we never perform calculations with empty signs in the belief that we are dealing with objects. People have in the past carried out invalid procedures with divergent infinite series. It is thus necessary to lay down rules from which it follows, e.g., what ‘\(\mathcal{O}+1\)’ is to mean if ‘\(\mathcal{O}\)’ means the Sun. What rules we lay down is a matter of comparative indifference; but it is essential that we do so – that ‘\(a+b\)’ should have a meaning, whatever

\(^{10}\) He uses the example of the concept *bald* since it has no sharp boundaries, it is not countable.
signs for definite objects may be inserted in place of ‘a’ and ‘b’. (Ks 135/CP 148; my emphasis)

That is to say, for those concepts and functions that are not completely determined, i.e., that do not have a value for objects other than those directly relevant for mathematics (numbers), some additional stipulations are necessary, and it is a matter of comparative indifference how these stipulations are carried out. Essentially the same point is made again regarding addition and multiplication in an undated letter to Peano: Frege claims that these functions should have values for any object as argument, and it makes “relatively little indifferent” (“verhältnismässig gleichgültig”) which values we attach to them for objects other than complex numbers (Wb 194/PMC 126). Now two things seem puzzling about these passages. First, concepts and functions in general are objective entities for Frege; hence, they cannot be completed at will, as the passages seem to suggest. Second, if, as the passages suggest, it is a matter of indifference which completions these functions are to receive, then we can conclude that any completion (or almost any) is consistent with the purposes of arithmetic. Let us take as example the case mentioned by Frege, i.e., the function \( x+1 \). It has the standard values for numbers, but no definite value for the sun. Here are some possibilities for completing it:

- \((F1)\) \( \odot+1=\odot \)
- \((F2)\) \( \odot+1=\text{Paris} \)
- \((F3)\) \( \odot+1=\text{Gottlob Frege} \)

Each one of these stipulations generate a different function \((F1, F2, F3 \text{ etc})\). According to Frege, it is imperative to select one of them, but it makes no difference which one of them. But if this is so, what is the point of completing the functions after all? And, if, according to Frege's

extensional criterion of identity, each completion yields a different function, it seems that each completion will give rise to a different arithmetic, since addition is a fundamental concept of this science. In one arithmetic, ‘0+1= 0’ is true and ‘0+1=Paris’ is false, while in other arithmetic, it is the other way round.

Regarding the first puzzle, there is an easy way of harmonizing it with Frege’s claims about objectivity and independence of concepts. Frege’s talk of stipulation should not be taken in the literal sense of constructive completion of a function, but rather as the choice of one of the many complete and objective functions that are compatible with – or that could be seen as a completion of – the incomplete ones. In our previous example, the completion by means of stipulation of the function x+1 should be understood as the choice of one of the functions F1, F2, etc, assuming that these are now complete. Regarding the second puzzle, the only way I can see of harmonizing it with Frege’s other views is as follows. In Frege’s eyes, arithmetic is only concerned with what is essential to all these functions, i.e., it is only interested in what all these many possible sharp functions have in common. If this is so, we can consider F1, F2, etc. as possible specifications of the function x+1, and the values F(0)=1, F(1)=2, F(2)=3, etc. as the initial conditions that all possible complete specifications have to satisfy. Arithmetic is hence the study of what must come out true, independently of which specification we chose. Or we might say that in arithmetical concepts and functions there is an essential part to the meaning that is objective, and there is an unessential part that might be conventional; arithmetic strictly speaking is the theory dealing with what is essential.

If this is correct, then we can easily see that Frege’s picture of arithmetic within the boundaries of a vague language can be quite conveniently described in the supervaluationist jargon: the complete functions F1, F2, etc. are to be seen as complete specifications of the
vague function \( x+1 \), and arithmetic is the unfolding of the super-truths, i.e., those sentences that must come out true in all possible complete specifications. Generalized, this interpretation says that arithmetic is the unfolding of super-truths considering all possible complete specifications of all partial functions that are typically seen as essential to it.

Of course I do not want to go as far as saying that Frege anticipates the whole supervaluationist approach, for he has no idea about higher order vagueness, and about the implications that this could have for notions like validity and inference rules. But the main intuition behind supervaluationism is something that Frege could quite well have come to, had he worked out a little further his views on incomplete functions.

REFERENCES


