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HOW IS IT DETERMINED THAT THE TRUE IS NOT THE SAME AS THE FALSE?

OSWALDO CHATEAUBRIAND

Department of Philosophy
Catholic University of Rio de Janeiro
Rua Marquês de São Vicente, 225, Gávea
22453-900 Rio de Janeiro, RJ
BRAZIL

oswaldo@fil.puc-rio.br

Abstract: The question that I discuss in this paper is whether Frege has a criterion of identity for the objects the True and the False that he introduces as denotation of sentences. My answer is that he does not, either in general or within the system of *Basic Laws*.

Key-words: Frege. Truth-values. Identity criteria.

In section 62 of *The Foundations of Arithmetic* Frege formulates the principle:

- (1) If we are to use the symbol a to signify an object, we must have a criterion for deciding in all cases whether b is the same as a .¹

¹ The passage is the following (p. 73):

How, then, are numbers to be given to us, if we cannot have any ideas or intuitions of them? Since it is only in the context of a proposition

It follows from this that:

- (2) If we are to use 'the True' to signify an object, we must have a criterion for deciding in all cases whether b is the same as the True.
- (3) If we are to use 'the False' to signify an object, we must have a criterion for deciding in all cases whether b is the same as the False.

And since both 'the True' and 'the False' are used by Frege to designate objects it follows in particular that the criterion (or criteria) should decide whether the True is the same as the False.

The question that I want to discuss in this note is whether Frege does indeed have such criteria, either in general or within the formal system of *The Basic Laws of Arithmetic*. My view is that he does not have criteria (2)-(3) in general and that it is compatible with the system of *Basic Laws* that all singular terms designate a single object – that we might as well identify with the True. It follows that in *Basic Laws* Frege

that words have any meaning, our problem becomes this: To define the sense of a proposition in which a number word occurs. That, obviously, leaves us still a very wide choice. But we have already settled that number words are to be understood as standing for self-subsistent objects. And that is enough to give us a class of propositions which must have a sense, namely those which express our recognition of a number as the same again. If we are to use the symbol a to signify an object, we must have a criterion for deciding in all cases whether b is the same as a , even if it is not always in our power to apply this criterion.

cannot distinguish assertion from negation. Moreover, all of this is independent of the axioms that Frege chose for his system.²

1

In the various texts in which Frege introduces truth-values, they are introduced as objects designated by sentences.³ A question we can raise, therefore, is when two sentences designate the same truth-value. A possible answer is:

- (4) Two sentences S and S' designate the same truth-value if and only if S and S' are materially equivalent.⁴

² These arguments are formulated in pp. 274-80 of my 2001, but I think that for purposes of discussion it may be worth presenting them again in this more self-contained form. For a recent analysis and survey of the issues relating to section 10 of *Basic Laws*, see Ruffino 2002.

³ See “Function and Concept” pp. 13-4, “On Sense and Reference” pp. 33-4, *Basic Laws*, pp. 6-7.

⁴ In this formulation the variables ‘ S ’ and ‘ S' ’ range over sentences, but we can also formulate (4) as

$$(4^*) S = S' \text{ if, and only if, } 'S' \text{ is materially equivalent to } 'S',$$

where the variables are substitutable by sentences and the quotes are interpreted as quasi-quotes. For, as Frege says in “Function and Concept” (p. 14):

Thus, just as we write:

$$'2^4 = 4.4'$$

we may also write with equal justification

$$'(2^4 = 4^2) = (4.4. = 4^2)'$$

and ‘ $(2^2 = 4) = (2 > 1)$ ’.

This is not very informative, because material equivalence is defined in terms of truth-value (whether objectified or not), but if we presuppose the notions of truth and falsity, then we could use (4) as a criterion of identity for truth-values. Nevertheless, (4) suffers from the same deficiency that Frege emphasizes in connection with his criteria of identity for numbers in *Foundations* and for courses-of-values in *Basic Laws*; namely, that they only work for a limited range of designating expressions. For, as Frege points out in *Foundations*, the criterion

- (5) The number which belongs to the concept F is the same as the number which belongs to the concept G if and only if there is a one-to-one correlation between F and G ,

does not meet the conditions stipulated in (1) because it will only settle the question of identity for designating expressions of the form ‘the number which belongs to the concept ‘ Z ’.⁵

We could also formulate (4') with a biconditional as

$$(4'') S = S' \text{ if, and only if, } S \text{ if and only if } S',$$

but if we analyze the biconditional in the right hand side in terms of denotation of truth-values, then the criterion is clearly circular.

⁵ This is the so-called Julius Caesar problem. In the case of truth-values the problem is what determines that Julius Caesar, or England, is not a truth-value (pp. 68, 78). In his recapitulation Frege says (p. 117):

A recognition statement must always have a sense. But now if we treat the possibility of correlating one to one the objects falling under the concept F with the objects falling under the concept G as an identity, by putting for it: “the Number which belongs to the concept F is identical with the Number which belongs to the concept G ”, thus introducing the expression “the Number which belongs to the concept F ”, this gives us a sense for the identity only if both sides of it are of the form just mentioned. A definition like this is not enough to enable us to decide whether an identity is true or false if only one side of it is of this form.

Frege's solution to this problem in *Foundations* is to bring in the extensions of concepts as logical objects and to define

- (6) The number which belongs to the concept F is the extension of the concept under which fall all concepts that can be put into one-to-one correlation with F .

This solution is quite general and could be applied to the truth-values as well. We could define

- (7) The truth-value of a sentence S is the extension of the concept under which fall all sentences materially equivalent to S .

Unfortunately, however, as Frege acknowledges in *Basic Laws*, the natural criterion of identity for extensions

- (8) The extension of the concept F is the same as the extension of the concept G if and only if all objects which fall under F fall under G and vice-versa,

is open to the same objections as (5) – i.e., (8) cannot be applied to designating expressions that are not of the form ‘the extension of the concept Z ’.⁶ Obviously, if all objects could be presented as the

⁶ Frege says (p. 16):

Although we have laid it down that the combination of signs “ $\epsilon\Phi(\epsilon) = \alpha\Psi(\alpha)$ ” has the same denotation as “ $\text{---}\alpha\text{---}\Phi(\alpha) = \Psi(\alpha)$ ”, this by no means fixes completely the denotation of a name like “ $\epsilon\Phi(\epsilon)$ ”. We have

extension of some concept, then there would be no problem, but Frege himself argues that this cannot be done in general.⁷ So what he tries to do in *Basic Laws* is to identify the truth-values with some specific extensions and to make sure that he can always settle identity questions within the system by working exclusively with extensions. To achieve this he views the system of *Basic Laws* as a sort of “constructive” system in which objects and functions are introduced step by step in a dependent way.

2

Frege starts setting up the system of *Basic Laws* introducing the truth-values as objects and using them to define the horizontal function, the negation function, the identity function and the generality function for functions of objects. He then introduces the courses-of-values as objects subject to the condition that

$$(9) \dot{\epsilon}\Phi(\epsilon) = \dot{\alpha}\Psi(\alpha)$$

has the same denotation as

$$(10) \text{---} \underline{\alpha} \text{---} \Phi(\alpha) = \Psi(\alpha).$$

only a means of always recognizing a course-of-values if it is designated by a name like “ $\dot{\epsilon}\Phi(\epsilon)$ ”, by which it is already recognizable as a course-of-values. But we can neither decide, so far, whether an object is a course-of-values that is not given us as such, and to what function it may correspond, nor decide in general whether a given course-of-values has a given property unless we know that this property is connected with a property of the corresponding function.

⁷ *Basic Laws*, p. 18, n. 17.

But this, Frege argues, does not fix the denotation of “ $\hat{\epsilon}\Phi(\epsilon)$ ”:⁸

If we assume that

$$X(\xi)$$

is a function that never takes on the same value for different arguments, then for objects whose names are of the form

$$“X(\hat{\epsilon}\Phi(\epsilon))”$$

just the same distinguishing mark for recognition holds, as for object signs for which are of the form “ $\hat{\epsilon}\Phi(\epsilon)$ ”. To wit,

$$“X(\hat{\epsilon}\Phi(\epsilon)) = X(\hat{\alpha}\Psi(\alpha))”$$

then also has the same denotation as [(10)]. From this it follows that by identifying the denotation of [(9)] with that of [(10)], we have by no means fully determined the denotation of a name like “ $\hat{\epsilon}\Phi(\epsilon)$ ” – at least if there does exist such a function $X(\xi)$ whose value for a course-of-values is not always the same as the course of values itself. How may this indefiniteness be overcome? By its being determined for every function when it is introduced what values it takes for courses-of-values as arguments, just as for all other arguments. Let us do this for the functions introduced up to this point. There are the following:

$$\xi = \zeta, \text{ — } \xi, \text{ — } \neg \xi.$$

Frege goes on to argue that the negation function can be left “out of account, since it can be considered always to take a truth-value as argument”, that the horizontal function can be reduced to the identity function as the function $\xi = (\xi = \xi)$,⁹ and that since the only objects

⁸ Continuing the remarks quoted in note 6. This argument of Frege’s is quite general and would apply to any of the definitions by abstraction that Frege considers (numbers, directions, etc.).

⁹ But to reduce the horizontal function to the identity function in this way we must make sense of the identity of ξ to $\xi = \xi$, and to make sense of this we need truth-values as denotation of sentences.

other than courses-of-value introduced so far are the truth-values, the question of the criterion of identity for courses-of-value reduces to the question of whether the truth-values are courses-of-value or not. He argues further that this also cannot be settled by the stipulation that (9) is to have the same denotation as (10), but that it is consistent with this stipulation that “it is always possible to stipulate that an arbitrary course-of-values is to be the True and another the False”. He therefore stipulates that the True is

$$(11) \hat{\epsilon}(\text{---}\epsilon),$$

and the False is

$$(12) \hat{\epsilon}(\epsilon = (\text{---}a \text{---} a = a)).$$

The problem, however, is that Frege’s system cannot guarantee that the True is different from the False.

If the True is the same as the False, then every function of objects that Frege has introduced so far has the True as value for any object as argument, and the generality function has the True as value for any such function as argument. Thus, unless Frege can distinguish the True from the False, the system does not even get off the ground. But this cannot be done *within* the system, because within the system the identity function denotes the True if the arguments are the same and the false if they are not the same. To postulate in the system that

$$(13) \text{---} \text{---} \text{ the True} = \text{the False}$$

does not help, because if the True *is* the False, then both

$$(14) \text{---} \text{---} \text{ the True} = \text{the False}$$

and

$$(15) \text{---} \text{---} \text{ the True} = \text{the False}$$

denote the True. And to identify the True with (11) and the False with (12) does not help either, because (10) cannot determine whether *any* courses-of-values are the same or different unless *something* determines whether the True and the False, upon which are based the logical functions involved in (10) are the same or different.

The problem is that Frege's system is trivially consistent (or inconsistent) in the sense that there is an interpretation of the system that consists of only one object and such that every function is essentially an identity function. This consistency (or inconsistency) of Frege's system is completely independent of the axioms and principles of inference that he postulates. What Russell's paradox shows is that the trivial interpretation is the *only* interpretation of Frege's axioms and principles of inference. With other axioms there may be non-trivial interpretations, but what my argument shows is that no choice of axioms can eliminate the trivial interpretation. It is in this sense that we may say that the trivial interpretation actually reveals an "inconsistency" in the system. Another way of putting the point is to say that Frege's system cannot guarantee the distinction between assertion and negation (because it cannot guarantee that the True is different from the False). This is clear from the definition of the negation function in terms of truth-values.¹⁰

¹⁰ Frege introduces the negation function as follows (p. 10):

We need no special sign to declare a truth-value to be the False, so long as we possess a sign by which either truth-value is changed into the other; it is also indispensable on other grounds. I now stipulate:
The value of the function

$$\neg \xi$$

shall be the False for every argument for which the function

$$\xi$$

is the True; and shall be the True for all other arguments.

3

It seems to me that the culprit for this situation is the idea that one can do the whole logic denotationally in agreement with principle (1) – which is introduced by Frege as a version of the context principle. The notions of truth, falsity and negation cannot be treated in this cavalier manner. It is true that given two distinct objects one could stipulate that all true propositions denote one and that all false propositions denote the other, but this will presuppose as given the objects and a notion of distinctness that cannot be explained in turn in terms of denotation of the objects.¹¹

It is quite obvious that Frege assumes that the two truth-values are distinct objects, and one could argue that this is a presupposition of the system that is guaranteed outside the system. I think, however, that as long as one adheres consistently to principle (1) it cannot be guaranteed even outside the system. For, if not in term of extensions, then how is the criterion of identity that justifies the introduction of truth-values as logical objects to be formulated? And how is the criterion of identity that justifies the introduction of extensions (or courses-of-values) as logical objects to be formulated? It is an old insight of Aristotle's that not everything can be proved and that not everything can be defined. Similarly, not everything can be introduced in accordance with principle (1).¹²

¹¹ I am not arguing here that one cannot give a reasonable account of the True and the False (see my 2001, p. 416 ff.), but only that one cannot do the logic in this purely denotational way.

¹² It is interesting that by adhering consistently to the principle "No entity without identity" Quine is led in his 1976 to an ontological conclusion that is close in spirit to Frege's in *Basic Laws* – namely, that the entire ontology of science consists exclusively of pure sets – and for which one can raise equally vexing problems (see my 2001, p. 361f.).

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