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Review of FERREIRÓS, J; LASSALLE CASANAVE, A. El árbol de los números. Editorial Universidad de Sevilla: Sevilla, 2015

Bruno Mendonça

Universidade Estadual de Campinas
Departamento de Filosofia
Rua Cora Coralina, 100, Campinas, SP, Brasil.
bruno.ramos.mendonca@gmail.com

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Abstract: We review Ferreirós and Lassalle Casanave's recently publishedbook "*El árbol de los números*". The book is a result of the Brazilian-Spanish conference "*Sobre la elucidación del concepto de número: cognición, lógica y práctica matemática*" hosted in Sevilla in 2013, and collects new papers on History and Philosophy of Mathematics as well as Mathematical Practice. These papers present results of investigations in Cognitive Sciences, Logic and Epistemology of mathematical certainty. In this review we present a general overview of the papers' contents, and advance a critical analysis of them.

Keywords: Philosophy of Mathematics; Mathematical practice; Logic; Mathematical cognition; Mathematical certainty.

The result of the Brazilian-Spanish meeting "*Sobre la elucidación del concepto de número: cognición, lógica y práctica matemática*" hosted in Sevilla in 2013, this book presents new results in Philosophy of Mathematics due to researchers from Brazil, France, Italy and Spain. As the book title suggests, the articles share a common anti-foundationalist standpoint; that is, they are more concerned with the cognitive aspects of mathematical practice than with traditional questions about the logical and epistemological bases of the discipline (even though, as the organizers make clear in the introduction, the motivating questions are just the classical ones in which philosophers have been interested since antiquity).

The book is divided into three parts. The first one is composed by three papers on cognitive studies about mathematical cognition, by Valeria Giardino, Tatiana Arrigoni & Bruno Caprile, and Hourya Benis Sinaceur.

Giardino's paper, "*¿Dónde situar los fundamentos cognitivos de las matemáticas?*", presents three experimental scenarios in order to test a hypothesis about the development of mathematical (especially arithmetical and geometrical) cognition: the first and second scenarios concern non-human and infant mathematical cognition, respectively; the third one concerns mathematical cognition in populations with a poor linguistic system. Giardino's hypothesis is that mathematical cognition, both in human as well as in non-human animals, starts as an evolutionarily advantageous form of perceptual cognition, namely, as perceptual recognition of numerosity and geometrical forms, but evolves in human beings so as to be able to process more abstract patterns through the development of skills related to counting, using maps and manipulating symbols. Further, Giardino seeks to point out the roles of both biological and cultural factors in the development of mathematical cognition.

Arrigoni & Caprile's "*La cognición de los enteros: una nueva propuesta*" investigates the cognitive processes that occur in the formation of a theory of integers. After presenting a revision of the literature in cognitive sciences, the authors argue that preschool children's theory of integers lacks some characteristics of the integers, especially infinitude. Further, they claim it is only when that theory is enriched with the missing properties that a theory of integers as abstract objects arises.

In "*Filosofía de la biopsicología del número*", Benis Sinaceur compares properly numerical cognitive attitudes with more general conceptual attitudes. Firstly, through an analysis of brain mapping results, Benis Sinaceur notes that, unlike other senses, perceptual recognition of numerosity is not related to a specialized part of the brain. Further, by reflecting on the different mental operations for processsing quantity and number, the author proposes a characterization of the different mental operations that are involved in perceptual recognition of numerosity as well as in numerical cognition.

All of these works present the reader with an excellent overview of recent results regarding mathematical cognition from the point of view of cognitive science (and also, neurophysiology), and also advance interesting theses about the processes involved in the formulation of mathematical knowledge by human beings.

The second part of the book, on logic, starts with Oswaldo Chateaubriand's "*Números como propiedades de segundo orden*" followed by Frank Thomas Sautter's "*Relaciones euclidianas de equinumerosidad".* These papers both propose alternatives to traditional perspectives on mathematical ontology and set theory. Chateaubriand argues against an ontological theory of numbers as objects in favour of a theory of numbers as higher-order properties. The latter theory faces at least two objections. Firstly, it implies assuming an ambiguous characterization of number (since there are infinite notions of number, each one for each rank in the infinite hierarchy of higher-order properties being assumed). Secondly, following Quine, it is possible to claim that, since properties do not have clear identity criteria, they do not fulfill a good ontological basis for numbers. Chateaubriand anticipates and replies both objections, in the case of the former objection by biting the bullet, and in the case of the second objection by claiming that objects do not satisfy a clear criterion of identification either.

By his turn, in a very careful investigation, Sautter explores some alternatives to Cantor's definition of equinumerosity that, contrary to the Cantorian notion, respect the principle that the whole is greater than any of its proper parts (hereafter, Euclidean principle). The first alternative considered by the author, which he calls 'total equinumerosity', says that equinumerosity holds between two sets, *A* and *B,* if and only if all injective functions between *A* and *B* are bijective and there is an injection between *A* and *B*. This is a very restrictive alternative since it only identifies equinumerosity of finite sets. Sautter's second alternative, called 'compositional equinumerosity', is obtained by recognizing equinumerosity of any two sets *A* and *B* for which there are subsets *C* and *D* of *A* and *B*, respectively, such that *A-C=B-D* and *C* is totally equinumerous to *D*. This second alternative may be regarded as more permissible than the first. Sautter's third alternative, inspired by Bolzano, says that two sets, *A* and *B,* are equinumerous if and only if either they are Cantorian equinumerous and there are no subsets of *A* (respectively, *B*) which are compositionally equinumerous with *B* (respectively, *A*) or they are compositionally equinumerous between themselves. The fourth and last alternative considered by the author is obtained by relaxing the requirement of compositional equinumerosity that two equinumerous sets have finite equinumerous differences. Sautter shows then that the first three alternative notions of equinumerosity satisfy the Euclidean principle, either in a weak or in a stronger sense. The latter alternative, on the other hand, does not satisfy the Euclidean principle at all. Sautter's study is of interest for any philosopher motivated by the type of question raised, for instance, by Tiles (1989) on the soundness of Cantor's cardinal arithmetic of infinite sets, as well as on the philosophical meaning of the mathematical questions posed by that work.

The second part of the book continues with Sérgio Schultz's "*Gödel versus Hilbert y su concepción simbólica de conocimiento*" followed by Concha Martínez Vidal's "*A vueltas con la intuición en el conocimiento matemático*". Schultz presents some historical evidence for the thesis that Gödel's apparent anti-Platonism in the end of the 1920's is due to a subscription to methodological as well as conceptual aspects of Hilbert's program, more specifically the idea that mathematical truth is accessed via symbolic knowledge only. The author argues that it is just when such epistemological stance is questioned in the face of Gödel's second incompleteness theorem that Gödel adopts a Platonist conception about the epistemology of Mathematics.

Vidal's paper advances an analysis of the concept of mathematical intuition in Gödel, Parsons and Bealer. Vidal wants to show how these authors understand intuition as a source of beliefs, and also whether intuition can provide any epistemic grounding. Further, Vidal considers whether, for these authors, intuition generates *a priori* knowledge. Vidal shows then that the first two authors understand intuition as a cognitive activity analogous to perception, but characterized in slightly different terms. For Gödelintuition is primarily a cognitive attitude towards propositions, and only derivatively towards objects, whereas Parsons takes the opposite view. Vidal argues that both characterizations face problems for which they do not provide a sufficient response: Gödel's proposal does not characterize adequately the relationship between primary and derivative forms of intuition, whereas Parsons does not provide a good explanation of the way in which intuition bases belief about infinitary properties. According to Vidal, Bealer, by his turn, characterizes intuition as an irreducible form of propositional attitude, associated with the possession of concepts. Vidal argues then that Bealer's thesis cannot characterize the way in which intuition can be accounted as an epistemological grounding. Vidal's conclusion is that none of these proposals present a sufficient account of the notion of mathematical intuition. Vidal's study is very useful to any reader wishing for a complete overview of the literature, but she does not proceed to a more popositive thesis on the subject, nor does she consider experimental data provided by cognitive science, differently from the methodological attitude of the works composing the first part of the book that deal very strongly with results coming from cognitive science as well as related areas.

The third and final part of the book, on mathematical practice, collects José Ferreirós' "*Sobre la certeza de la aritmética*", Abel L. Casanave's "*Conocimiento simbólico y aritmética en Hilbert*" and, lastly, José M. Sagüillo's "*Números y proposiciones en las formalizaciones de la aritmética de Peano, Gödel y Whitehead-Russell*".

Ferreirós argues that Peano Arithmetic (hereafter PA) is sound with respect to the collection of practices involved in the act of counting. This provides epistemological certainty for the basic Arithmetic which is axiomatized by that system. Further, Ferreirós claims that such certainty does not propagate for what he calls arithmetical number theory, which concerns the more advanced properties of Arithmetic (Ferreirós advocates such a dichotomy by considering the existence of advanced arithmetical theorems that *up to now have not been* proved using only resources of basic Arithmetic, e.g., Fermat's last theorem). Arithmetical number theory is based not in certainty but in quasi-empirical hypothesis, according to Ferreirós. The author focuses on arguing that the practice of counting verifies the induction axiom. This is the more problematic case for two reasons: firstly, the induction axiom allows us to deduce knowledge about infinitary properties from analysis of finitary cases only; secondly, the induction axiom in its original formulation is very liberal and suffers from impredicativity. On the second question, Ferreirós replies that the induction axiom allowed by the practice of counting is even more liberal than that formulated in PA. So, even if the various possible restrictions in the formulation of induction axiom can be an interesting theme of logical study (specially, for investigations in reverse Mathematics), they are not relevant for the question posed by Ferreirós. On the first question, Ferreirós claims that the induction axiom is verified through the very complex cognitive capacity of human beings of passing from the counting of concrete numbers to the imagination of an arbitrary possible number. Finally, the author considers whether the existence of non-standard models of PA is an objection to his claim. Calling attention to the fact that model theory is not grounded in basic Arithmetic (it is based in set theory), Ferreirós claims that the existence of non-standard models of PA is a result of arithmetical number theory and so does not enjoy nor conflicts with the certainty of basic Arithmetic (from the final comment it is possible to infer Ferreirós' negative opinion about the various enterprises on logical foundations of Mathematics, which is explicitly expressed by him at the conclusion of the paper).

Lassalle Casanave's entry continues the debate about certainty in Arithmetic in a more historical perspective, providing a philosophical analysis of Hilbert's dichotomies between notions that are real or ideal, finite or transfinite, intuitive or formal, concrete or abstract, contentful or without content, unproblematic or problematic. Firstly, Lassalle Casanave presents historiographical evidence that Hilbert's dichotomies cannot be interpreted by reducing the first and second elements of each pair to meaningful and meaningless notions, respectively. Hence, the author argues, Hilbert's dichotomies need to be understood in light of the tradition of symbolic knowledge that started with Leibniz, according to which mathematical knowledge can only be gained via symbolic manipulation. Lassalle Casanave claims that, for Hilbert, real, finite, intuitive etc. mathematical notions are the semantic counterparts of symbolic knowledge as surrogative knowledge. On the other hand, ideal, infinite, formal etc. mathematical notions are what are *exhibited* (a technical notion owing to the tradition of symbolic knowledge) in symbolic knowledge as a non-surrogative form of knowledge. It is possible to notice a disagreement between Ferreirós' and Lassalle Casanave's stances on the subject of certainty in Arithmetic; namely that by subscribing Hilbert's dichotomies, Lassalle Casanave limits certainty to Arithmetic's finitary part, whereas Ferreirós attributes certainty to the induction axiom as well.

Finally, Sagüillo's paper proceeds to a compare analyses of Gödel's, Peano's and Whitehead & Russell's formalizations of Arithmetic, both in their technical aspects as well as in their philosophical assumptions. More specifically, Sagüillo claims these works give different definitions of both the domain of the discipline (i.e., the ontological commitments taken by arithmeticians) as well as the universe of discourse of the formalization (i.e., the ontological commitments associated with the given formal system). So, whereas Gödel's system only quantifies over numerical variables, Peano's system quantifies over individuasl in general and relativizes its arithmetical axioms to numbers. Further, Sagüillo argues that, Whitehead & Russell, by their turn, consider higher-order sets as composing the domain of Arithmetic and individuals in general as composing the universe of discourse of their formal system. Sagüillo sees in the variations between these treatments different ways of making a correspondence between arithmetical propositions and formulas of a formalized system of Arithmetic.

In general, the book advances very engaging arguments regarding the History and Philosophy of Mathematics as well as on mathematical cognition. In this sense, the works there collected strongly acknowledge the importance of empirical and historical studies for the investigation on Philosophy of Mathematics. Thus, these works are relevant to anyone that subscribes to a more naturalized approach to the subject. Furthermore, as a reflex of the peculiar anti-foundationalist standpoint there presented, the works relativize the philosophical importance of logical studies in foundations of Mathematics, even though they do not reject completely the utility of this type of enterprise.

References

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