BOOK REVIEW: LINNEBO, Ø., *Philosophy of Mathematics* (Princeton Universisty Press, 2017, 216, pages)

GIORGIO VENTURI[1]
https://orcid.org/0000-0001-9089-4340
[1]*University of Campinas*
Department of Philosophyv
Campinas, SP
Brazil
gio.venturi@gmail.com

FELIPE ALBARELLI[2]
https://orcid.org/0000-0003-3646-7553
[2]*University of Campinas*
Department of Philosophy
Campinas, SP
Brazil
felipesalbarelli@gmail.com

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Abstract: We review Linnebo's *Philosophy of Mathematics*, briefly describing the content of the book.

The present volume is a contribution to the book series *Princeton Foundations of Contemporary Philosophy*, and offers a self-contained presentation of fundamental topics of the contemporary philosophy of mathematics. Notwithstanding the intrinsic difficulties of the subject, the book is enjoyable and very well-written and Linnebo succeeds in giving a clear presentation of intricate debates and concepts. The focus of the text is on problems and consequently history is presented only to clear the origin of the various topics. For this reason this book perfectly complements the classic textbook by Shapiro [1], offering a more updated and problem driven introduction to the philosophy of mathematics.

The book is divided in twelve chapters, roughly grouped in two parts: the first seven chapters are devoted to the classic themes from which originate the contemporary debate, while the last five offer a selection of contemporary trends in the philosophy of mathematics. The content—and to some extent the structure—of the book reflects the division in sections of the evergreen anthology by Benacerraf and Putnam [2]: the presentation of the big three schools and the corresponding views on the foundations of mathematics, the ontological problem for mathematical objects, the epistemological problem for mathematical knowledge, and a discussion of the philosophy of set theory.

The book does not intend to be complete neither with respect to the contemporary debates, nor with respect to the history of the discipline. Nonetheless, Linnebo succeeds in giving a broad perspective which introduces some of the most interesting problems of the subject and that, therefore, can profitably be used as an introduction for both philosophers and mathematicians.

This is another merit of this book. It is explicitly structured to build a bridge between the two disciplines, being palatable for people with backgrounds from both philosophy and mathematics. Without diluting the beauty of the subject in shallow waters easy to explore, this book presents the classical ontological and epistemological problems in such a clear way to make them interesting and puzzling from both a philosophical and a mathematical perspective.

We therefore highly recommend this book as an introduction to the philosophy of mathematics; one that will surely convince the readers to investigate further this vast and intriguing domain. We conclude by surveying the content of each of the twelve chapters.

In Chapter One *Mathematics as Philosophical Challange*, Linnebo introduces the main themes of the book. Initially, the author presents the integration challenge: the attempt to reconcile on a philosophical ground the ontology of abstract mathematical objects with priori knowledge. Next, a section is dedicated to a brief presentation of Kant's philosophy of mathematics. Here the analytical/synthetic and the a priori/a posteriori distinction are presented. Finally, Linnebo ends the chapter describing the main philosophical positions with respect to mathematics, on the base of Kant's distinctions.

Chapter two *Frege's Logicism* is about Frege's philosophy of mathematics. Linnebo starts by outlining both Frege's logicism and his platonism about mathematical objects. Frege's argument for logicism is presented as an attempt to reduce sentences about arithmetic to sentences about cardinality, while the argument for platonism is introduced by considerations on the semantics of natural language. Subsequently, Frege's response to the integration challenge is presented. Towards the end, the author shows how Cesar's problem and Russell's Paradox undermine Frege's logicist project.

Chapter three *Formalism and Deductivism* addresses the two positions named in the title, explaining the motivations for each as well as the main problems that they face. Initially, game formalism is presented, this is the thesis according to which mathematics is the study of arbitrary formal systems, like a game of symbolic manipulation. Next, Linnebo presents term formalism, according to which mathematical objects are the symbols used by mathematicians. Finally, the chapter ends with a discussion on deductivism, the position that sees mathematics as the study of the the formal deductions from arbitrary axioms.

Chapter four *Hilbert's Program* is about Hilbert's formalistic project. This type of formalism differs from the previous ones in that it distinguishes between two types of mathematics: finitary, which deals only with arbitrarily large and finite collections, and infinitary, which deals with complete infinite collections. Linnebo argues that Hilbert assimilates finitary mathematics to term formalism, and infinitary mathematics to deductivism. Hilbert Program is presented as making use of finitary mathematics, which has few suppositions but limited applications, to justify the use of infinitary mathematics, that is epistemically problematic but without limits of application. The chapter ends with a discussion of how the discovery of Gödel's incompleteness theorems showed the impossibility of completing Hilbert Program.

Chapter five *Intuitionism* deals with the philosophical thesis bearing the same name, which stands out from those already presented by criticizing certain mathematical practices. Initially, Linnebo explains that intuitionists argue that proofs that use non-constructive methods implicitly presuppose platonism; thus the proposal to abandon principles like the law of the excluded middle. Next, the chapter discusses the intuitionist alternative ontology of mathematics in terms of mental constructions. A subsection is devoted to present intuitionist logic. The chapter ends by showing how the intuitionist view on infinity and proof can influence arithmetic and analysis.

Chapter six *Mathematics Empiricism* provides an initial section on Mill's philosophy of mathematics and then focuses on Quine's empiricism and the indispensability arguments. Initially, Linnebo presents Mill's argument in defense of the thesis that the principles of mathematics are empirical truths. Then, in discussing Quine, Linnebo presents the main arguments against the synthetic/analytical distinction and his holistic empiricism, together with the criticism that these positions received in the literature. The chapter ends with a presentation and discussion of Quine's famous indispensability argument, which defends mathematical platonism on the basis of the unavoidable use of mathematics in science.

Chapter seven *Nominalism* discusses the thesis that no abstract mathematical objects exist. Linnebo explains that the difficulty of explaining how we could have knowledge of abstract objects, given that they have no causal relationship, serves as a basis for nominalism. Then, Field's project to nominalize science is presented, which consists in showing that it is possible to do science with a language that is not committed to the existence of abstract entities. To this aim, Filed needs to reformulate scientific theories without impacting on their deductive capacities. Towards the end Linnebo also presents a form of nominalism that is not reconstructive.

Chapter Eight *Mathematical Intuition* is devoted to mathematical intuition. Here we face directly the difficult problem of the justification of mathematical knowledge. After having cleared that intuition is not an univocal notion and that intuitive knowledge may come in degrees, Linnebo proceeds in presenting the recent debate on the relevance of intuition in mathematics. We are briefly recalled that intuition is fundamental in the work of Hilbert, Russell, and Gödel and then the chapter goes on presenting recent defenses of mathematical intuition: the realist use that Maddy made of this notion, the position of Parsons which extends the line of Kant and Hilbert, and finally the phenomenological perspective of Føllesdal. The extent to which intuition can be used to justify higher mathematics is then referred to chapters ten and twelve.

Chapter Nine *Abstraction Reconsidered* goes back to the notion of abstraction, connecting Frege's work with the neo-logicist position inaugurated by Hale and Wright. Abstraction principles are presented as a form of access to abstract mathematical objects. The discussion starts from Russell's paradox and the consequent collapse of Frege's logicist program. After briefly presenting Russell and Whitehead proposal to avoid the paradoxes by means of type theory, Linnebo presents the neo-logicist prosopal: the attempt to derive arithmetic from safe abstraction principles. After a discussion of the logicality of Hume's Principle and after presenting the problem of distinguishing the good abstractions from the bad ones, the chapter ends with a discussion on a form of abstraction called dynamic, which better fits the iterative conception of sets.

Chapter Ten *The Iterative Conception of Sets* is meant to introduce the reader to the problems of the philosophy of set theory. After briefly recalling the definition of the cumulative hierarchy of the V^{α}, which form the intended interpretation of the axioms of set theory, Linnebo presents and discusses the axioms of ZFC. The chapter continues explaining the similarities between ZFC and type theory and then proceeds in addressing Boolos' stage theory—i.e. the philosophical counterpart of the formal presentation of a cumulative hierarchy—and the way in which this theory is used to justify a great amount of ZFC axioms. Linnebo then warns the reader from the difficulties of a literal understanding of the generative vocabulary and towards the end of the chapter he introduces the debate between actualists and the potentialists with respect to existence of the universe of all sets.

Chapter Eleven *Structuralism* presents the main tenants and problems of a view of mathematics centered around mathematical structures. Following the standard terminology, Linnebo divides structuralist positions between eliminative and noneliminative structuralism, according to the commitments to the existence or nonexistence of structures or patterns. By exemplifying these position Linnebo presents the case of arithmetic, discussing the origins of structuralist positions in the work of Dedekind. An interesting section then is devoted to the view according to which structures result from a process of abstraction. The chapter ends by outlining the connection between structuralism and the foundations of mathematics offered by category theory.

Chapter Twelve *The Quest for New Axioms* brings back the discussion to the philosophy of set theory. The starting point of the discussion is found in the widespread presence of independence in set theory, discovered after the proof of independence of the Continuum Hypothesis (CH) from ZFC. Linnebo then proceeds in discussing the main philosophical arguments offered to justify extensions of ZFC able to overcome independence: intrinsic reasons, connected to the use of intuition, and extrinsic reasons, based on the success of new set-theoretical principles. The chapter ends with a discussion on two common views on set theory: pluralism, that equally accepts different interpretations of ZFC, and a monist view, that has its roots on the quasicategoricity argument of Zermelo for second order ZFC.

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