

MICHAEL BEANEY, FREGE AND THE PARADOX OF ANALYSIS

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Abstract: *In his most recent book on Frege (1996), Michael Beaney defends the view that the sense/reference distinction introduced by Gottlob Frege was meant to solve the paradox of analysis. Upon considering many ways of interpreting the Fregean notions of sense and ways of splitting up the sense of a sentence (a thought), Beaney concludes that the Fregean distinctions cannot help to see a way out of this paradox. Here I will firstly discuss some of Beaney's attempts at making sense of the notion of a sentence's content. I will secondly disagree with Beaney's final proposal as to how to dissolve the paradox of analysis. And I will finally suggest that the solution of the paradox requires a notion of content which is essentially incomplete but completes itself partially as language evolves.*

Key-words: *Analysis; content; definitions; paradox; sense and reference; deduction; language; Frege; Beaney.*

1. PRELIMINARIES

A problem which philosophers have struggled with at least since Aristotle is that of providing an adequate explanation for the validity of our deductive practices. What makes it the case that the rules of classical logic, let us say, are the correct ones to apply? Why is it correct

to judge according to precisely these rules? One may say that this is so because they preserve truth. But there are many possible alternative answers to these questions.

Philosophers of a Leibnizian persuasion have usually felt that deduction does not extend our knowledge. By means of deductive reasoning we merely clarify or analyze knowledge that is already present in the reasoning's premises. This is hard to square with the fact that there are so many surprising proofs in mathematics and that was the reason why Kant preferred to say that the proofs of arithmetic and Euclidean geometry also extend our knowledge. He did not think the same of logic; according to Kant, logic consisted of analytic truths (i.e. those truths which are not knowledge-extending). Frege who was responsible for a remarkable extension of the Aristotelian paradigm in logic disagreed with Kant about the epistemological status of logic. For Frege, logical reasoning could also extend our knowledge. This means however that the fruitfulness or information-extending character of deductive reasoning must also be explained.

Those who required an account of both the validity and the fruitfulness of deduction often found it hard to reconcile these two apparently conflicting properties of deduction. This is because validity seems to demand of the conclusion that it does not add new information to that which is already present in its premises, whereas fruitfulness appears to require just the opposite. A comparison with induction might illustrate the point. Inductive reasoning is essentially fruitful but for that very reason it cannot be valid. It has been thought that with deduction the opposite occurs: because it is essentially valid, correct deductive reasoning cannot be fruitful.

Yet some philosophers have thought that, although hard to reconcile, validity and fruitfulness can be harmonized. This is the view of Michael Dummett who calls the task of finding an explanation which harmonizes these two properties of deduction the problem of

the justification of deduction.¹ Dummett also traces back to Frege a worry not just about the validity of deduction but also with its fruitfulness.²

A problem which relates to the justification of deduction is what has been called the paradox of analysis. Definitions are paradigmatic examples of conceptual analysis. Suppose that a concept is being analyzed by means of a definition. In order for the analysis to be correct something has to be preserved in the transition from the concept before analysis to the analyzed concept. Otherwise, how could we say that the analysis clarifies the unanalyzed concept rather than that it produces a new concept? It may be said by analogy with the preservation of truth for the case of valid deduction that what is preserved in the case of analysis is the extension of the concept to be analyzed. There are of course other properties of a concept whose preservation might be suggested as an explanation of the correctness of a concept's analysis; the concept's intension would be an example. We will come back to this idea in section 8.

Another feature of definitional analysis for which many philosophers require an explanation is fruitfulness. The idea behind this aspect of analysis is that there is at least a cognitive difference between the unanalyzed notion and the analyzed one; we know more about the concept under analysis after it has been subjected to analysis than we did before. Thus think of the Russellian definition of the number 3 as the class of all the three-membered classes. It is plausible to say that before Russell proposed this definition we did not know that being a class of classes was a feature of the number 3, as well as of the other natural numbers.

But if analytical definitions can be correct (that is, say, preserve a concept's extension) and fruitful, how can they possess both of these

¹ In Dummett (1973), for example.

² We will consider this aspect of Frege's philosophy in the following sections for the case of the paradox of analysis.

features simultaneously? For correctness demands, one might say, conservation of extension while fruitfulness is not incompatible with the non-conservation of the extension of the concept under analysis. For example, the Russellian or Fregean definitional analysis of the concept of natural number aims at preserving the extension of the old concept; if not how could such an analysis be deemed correct? On the other hand, the Russellian definition of number also aims at telling us of something we did not know before, namely that a number is a class of classes. The definition of number therefore excludes from the new concept's extension those things which are not classes of classes although they may be in the old concept's extension.

The paradoxical aspect of analysis boils down to the apparent impossibility of reconciling the demands of the two essential features of analysis: its correctness and its fruitfulness. Perhaps a good starting point for an attempt at dissolving the paradox would be the Fregean account of definitions. For Frege was the first philosopher to recognize the fruitfulness of analysis.

2. THE REFERENCE/SENSE DISTINCTION AND THE PARADOX OF ANALYSIS

Michael Beaney in his recent book *Frege: Making Sense* has maintained that Frege's aim in introducing the distinction between sense (*Sinn*) and reference (*Bedeutung*) was to provide a philosophical justification for the definitions of the natural numbers given in *Grundlagen der Arithmetik* (Frege (1884)) and the contextual definition of the extension of a concept present in the beginning of *Grundgesetze der Arithmetik*, volume 1 (Frege (1893)) (see, for example, Beaney (1996), p. 10). Justification is needed, Beaney insists, because Frege is, on the one hand, engaged in analyzing further the concept of a natural number (when he says that natural numbers are identical with the extensions of certain concepts) and, on the other hand, in maintaining the correctness of this analysis, that is, that the extension

corresponding to the analyzing notion is the same as the extension of the analyzed notion. The idea, as we have just supposed, is that the correctness of an analysis just requires that it (the analysans) preserves the extension or reference of the old notion while the fruitfulness of the analysis appears to be asking for some kind of cognitive difference between the new notion and the old one. Hence, a simultaneous explanation of the correctness and informativeness of analysis would demand a distinction like that between the reference and the sense of an expression.

The problem of justification would not arise if we saw Fregean definitions as merely stipulative since these definitions sacrifice fruitfulness for correctness; in Frege's terminology, their left- and right-hand sides possess both the same reference and the same sense. In the stipulative case, the definiendum is a completely new expression on which is bestowed the very same reference and sense of its definiens. It is true that within the system of *Grundgesetze* the number definitions are regarded as stipulations but it is also true that these definitions are seen by Frege as further clarifying the concept of natural number in *Grundlagen*. This amounts to the following: the definitional identities encountered in the process of looking for and hitting on the best notions for the analysis of a given concept *C* are analytic; the final presentation of the system including a purified version of *C* will contain a symbol which expresses the content of *C* and the definitions of this symbol will be stipulative.

But while there is hardly any doubt that Frege's contextual and explicit definitions should be seen as analytic rather than stipulative,³ it is not that easy to say how to cash out the notion of fruitfulness that

³ This is now Frege himself sees them in *Grundlagen* when he suggests that Kant's theory of definitions was poor in not recognizing the fruitfulness of certain definitions.

Frege takes to be one of the essential features of analytic definitions.⁴ On this of course depends his resolution of what Beaney calls the paradox of analysis which may be represented by the following question: how can an analytical definition be at the same time correct and informative? His huge effort to make sense of Frege's resolution of this paradox is perhaps Beaney's best contribution to recent Fregean scholarship. The detailed discussion of the topics related to Frege's struggle with the paradox occupies virtually all the book but most specifically its last four chapters. It is worth making a few comments about the steps needed for the dissolution of the paradox and the problems to be faced by the tools Frege uses (and Beaney tries to refine) in order to dissolve it.

The contextual definition of a natural number which Frege quotes from Hume is the following:

(HP)⁵ The number of F 's is identical with the number of G 's \equiv_{df} the concept F is equinumerous with the concept G .

The concept of equinumerosity is further explained in terms of a one-to-one correspondence between the F 's and the G 's. Another example of the analysis of a concept is Frege's explicit definition of a natural number:

(END)⁶ The number of F 's \equiv_{df} the extension of the concept *equinumerous to the concept* F .⁷

⁴ Of course there are analytic definitions which are not fruitful as there are some which are not correct. Frege maintains however that the most interesting cases of analytic definitions are both correct and fruitful. This is in Frege (1884) and in one of his posthumous writings called "Logic in Mathematics" (Frege (1914)).

⁵ For Hume's principle.

⁶ For explicit number definition.

But how can definitions like these be at the same time correct and yet tell us something new?⁸

3. CONTENT AND WAYS OF SPLITTING UP CONTENT

Frege suggests that the way to accommodate these two apparently conflicting features of analytic definitions is to hold, on the one hand, that the content of the left- and right-hand sides of the definition is the same and, on the other hand, that this same content is split up in different ways on each side.⁹

⁷ Frege (1884), § 68.

⁸ I am assuming of course that the reader is already familiar with Frege's attempts at defining a natural number in section 4 of *Grundlagen* (Frege (1884)). Hume's principle (HP) constitutes his second attempt and it is rejected because of the famous Julius Cesar problem (the problem of recognizing the symbol for a natural number when it is not of the form "the number of...". (Frege (1884), §§ 66-67) This is the reason why Frege settles for the explicit definition (END). Despite this, most of the discussion in this paper concentrates on HP. Firstly because, like the contextual definition of the direction of a straight line, HP is also an example of a fruitful definition that illustrates Frege's geometrical metaphor of the same propositional content being split up in different ways (see next footnote). The second reason for privileging HP is that after the discovery of Russell's contradiction and Frege's consequent rejection of Basic Law V of *Grundgesetze I* (Frege (1893)), people who still believe in the Fregean platonistic project had the idea of taking HP as the only non-logical axiom of Peano's arithmetic. For an excellent presentation of this neo-Fregean project, see Wright (1983).

⁹ Frege (1884), § 88. In this passage, Frege uses the geometrical metaphor of the same area being divided in their respective parts by different sets of lines. The common area is the representative of the same content of both sides of the definition and the different sets of lines correspond to different ways of analyzing that content. For example, the referential content of the sentence "hydrogen is lighter than carbon dioxide" may be analyzed as object (hydrogen) and one-place concept (x is lighter than carbon dioxide), as two objects (hydrogen and carbon dioxide) and two-place concept (x is lighter than y) or in various other ways.

But then how should the notions of content and way of splitting up content be spelled out? The *Begriffsschrift*'s notion of conceptual content may offer us a starting point for understanding this dichotomy. Two sentences have the same conceptual content according to the author of *Begriffsschrift*¹⁰ just in case they are logically equivalent (let us say that they have the same truth conditions). Following this line, the different ways of splitting up content would correspond to the different ways of carving up the truth conditions of a sentence. The notion of a way of carving up a sentential content is crucial to Frege's account of quantification and it should not be thought that it is reducible to a possible syntactical analysis of a sentence. For example, the analysis of "hydrogen is lighter than carbon dioxide" into the name 'hydrogen' and the predicate 'x is lighter than carbon dioxide' mirrors the analysis of the sentence's respective propositional content (its conceptual content) into an object (the chemical element hydrogen) and a concept of being lighter than carbon dioxide. Similarly, the alternative syntactic analyses correspond to other ways of splitting the saturated content of the sentence into unsaturated and saturated parts.

Suppose then that the left- and right-hand sides of HP have the same truth conditions. Despite this, the way these truth conditions are split up on either side is very distinct: the left-hand side asserts an identity between objects while the right-hand side affirms the relation of equinumerosity between two content. We need something to be preserved in the transition from the left- to the right-hand side of HP otherwise it could not be claimed that this analytical definition is correct. It may not be conceptual concept. But then what else would it be? The only candidate we have up to now given the Fregean dichotomy is conceptual content. But can we assume that the conceptual content of the two sides of HP is the same? The answer is

¹⁰ Frege (1879), section 1.

that we cannot, and more justification for this claim will be provided in the following sections.

But as far as Frege is concerned, it is fair to say that he realized that conceptual content is too strong to be what is needed for the preservation of the correctness of analytical definitions. A weaker notion like that of reference may be enough. Apart from that, a better explanation of the newness of some analytical definitions than that provided by the above-mentioned geometrical metaphor is also needed; the informativeness of analytical definitions requires something stronger than the idea of various ways of splitting up the same content. For given a certain content the amount of ways of carving it up must already be determined by the content in question; these possibilities of splitting-up are limited by Fregean linguistic and ontological categories (object, first-level concept, second-level concept, etc.). Fruitfulness seems to be asking for something like a new way of being presented with the same reference (a new sense). This is where the sense/reference distinction enters the scene in 1892 in place of the pre-“*Über Sinn und Bedeutung*”¹¹ dichotomy between conceptual content and ways of carving it up. Part of the story is already known. Frege rightly saw that his former *Begriffsschrift* notions of conceptual content and of different ways of splitting up conceptual content could not adequately explain the difference in cognitive value between analytic (uninformative) identity statements and true synthetic ones. Examples of these are respectively “the Morning Star is the Morning Star” and “the Morning Star is the Evening Star”.

The sense/reference distinction may work for the explicit number definition (END); Frege could maintain that the reference of either of its sides is the same while their corresponding senses differ.

¹¹ Frege (1892). Traces of the sense/reference distinction already appear in the earlier paper “Function and Concept” (Frege (1891)), as Beaney rightly points out.

The problem is that HP cannot be handled in a similar way; its correctness cannot be accounted for by saying that the truth-values of HP's right- and left-hand sides are identical. For if that were the case then any content that agreed in truth-value with HP's left-hand side would make this analysis of the concept of number correct.

4. SEMANTICAL AND EPISTEMOLOGICAL CONTENT

There must be a richer notion of content than that of reference that will succeed in explaining the correctness of analytical definitions of HP's kind. And something like Fregean senses would still be that in terms of which HP's newness is explained. The idea would then be to split the content of a sentence into the state of affairs referred to by the sentence or its truth-conditions (its semantical content)¹² and the thought expressed by it (its epistemological content).¹⁵ The distinction between the semantical and the epistemological content of a sentence is put forward by Beaney as a way of parsing the Fregean content/way of splitting up content dichotomy. This is how Beaney puts it:

As we saw in § 5.2, Frege held that the same 'content' can be 'split up' in different ways, a conception that was grounded in the idea of alternative function-argument analyses. This early distinction appeared to offer an attractive resolution of the paradox of analysis, 'content' being understood as 'logical content' and 'way of splitting up content' as 'cognitive content'. (Beaney (1996), pp. 226-7)

¹² This is partially Beaney's terminology, partially mine. He introduces the distinction between semantic and epistemological sense and identifies it with the distinction between logical and cognitive content (Beaney (1996), p. 227). I prefer to reserve the term 'sense' for epistemological content and speak of semantical content as corresponding to Beaney's logical content. The semantical content of a sentence would be the semantical equivalent of the syntactical notion of logical content. For Beaney, two sentences have the same logical content if they are provably materially equivalent (Beaney (1996), p. 226).

But, as Beaney himself recognizes, this new distinction has its own problems. The first is that the semantical content/epistemological content distinction departs a lot from Frege's cherished sense/reference distinction. For, as we saw, the semantical content of a sentence does not coincide with its reference. Neither does the distinction proposed by Beaney appear to coincide with the earlier Fregean distinction between content and ways of splitting up content. The problem now is with the coincidence between epistemological content and way of splitting up a given content. Does it sound plausible to say that the various different ways of carving up the content of a sentence correspond to different epistemological contents of it? The situation may be illustrated by the sentence "hydrogen is lighter than carbon dioxide". It would be extremely hard, I think, to defend the view that the two ways of partitioning the content of this sentence mentioned in footnote 11 correspond to two different epistemological contents of this sentence. These considerations show that Beaney's distinction is neither a good translation of Frege's distinction between content and way of splitting up content, nor of his later sense/reference dichotomy.

4.1 States of affairs and ways of splitting up states of affairs

But of course Beaney might say that the dichotomy he is putting forward is an improvement and a clarification of Frege's still imprecise sense/reference distinction; he could maintain that Frege should have held that the reference of a sentence is the state of affairs it represents. The following is the passage where Beaney mentions what would have been the natural candidate for the semantical content of a sentence:

There is undoubtedly a degree of unclarity in Frege's earlier notion of 'content' in both the *Begriffsschrift* and the *Grundlagen*, but in so far as a metaphysical gloss can be put on the notion, 'state of affairs' would probably be the best term to use in characterizing what Frege meant by the 'content' of a sentence. The 'natural answer' given at the end of the

last section to the question as to what propositions that have the same 'content' have in common is thus, arguably, indeed the answer that might be extracted from Frege's early philosophy. (Beaney (1996), p. 155)

It is undeniably true that Frege's argument for the identification of the reference of a sentence with its truth-value does not rule out any alternative candidate that satisfies Leibniz's principle concerning the identity of reference of the expressions which are intersubstitutable *salva veritate*. But even if one opted for states of affairs or facts as the referents of sentences (Beaney's suggestion) and so identified the semantical content of a sentence with the state of affairs it represents, it is highly doubtful that these notions (states of affairs or facts) could finally provide us with a respectable concept of content that would solve the paradox of analysis. A well-known difficulty with facts or states of affairs has been pointed out by Davidson in various places:¹³ if a sentence has a referent whatever it may be then all true sentences refer to the same thing. The background problem lies, as many commentators of Frege have noticed, in the association of sentences with proper names and with the consequent idea that sentences, like proper names, have a reference. The second problem with Beaney's distinction might be thus summarized: seeing semantical content as a clarification of Fregean reference and the semantical content of a sentence as the state of affairs it names leads to the absurdity that all true sentences name the same state of affairs.

Another obstacle to adopting states of affairs as the referents of sentences has to do with the ontological status of state of affairs. A

¹³ For example, in "Truth and Meaning" (Davidson (1967)) and in "True to the Facts" (Davidson (1969)). The argument Davidson employs to prove this has become known as the slingshot argument and Frege is said to have originated it. Actually in "*Über Sinn und Bedeutung*", Frege uses a similar argument to show that the thought expressed by a sentence cannot be identified with its reference.

thought might be that states of affairs resemble the shadowy entities that Wittgenstein in the *Tractatus*¹⁴ maintains propositions represent. Tractarian states of affairs are, however, not even part of extra-linguistic reality¹⁵ and would therefore be unsuitable for being the referents of sentences.

4.2 Truth-conditions and ways of splitting up them

But perhaps the solution to Beaney's troubles simply requires denying that states of affairs are the referents of sentences in favor of the thesis that the semantical content of a sentence consists of its truth-conditions; it looks as if the attempt to conceive the semantical content of a sentence in terms of its reference leads us into a dead end. This thesis finds favor, I take it, in Beaney's approach which seems to buy at least partially into the Tractarian thesis that propositions do not refer. For the early Wittgenstein, a proposition has a sense which is given by the conditions under which the proposition is true and those under which it is false. The new proposal would be that the semantical content of a sentence (its truth-conditions) can be split up in various different ways and these various splitting-ups of the sentence's semantical content would constitute its epistemological content.

The question – and this is our fourth problem with the semantical content/epistemological content distinction – is whether there is a criterion available for the identity of truth-conditions of two sentences which allows us to decide when they express the same semantical content. Beaney wants to assert that sentences like (1) “the Morning Star shines” and (2) “the Evening Star shines” have the same truth-

¹⁴ Wittgenstein (1922).

¹⁵ Reality contains, according to the author of the *Tractatus*, only objects and facts.

conditions relative to the same time of utterance.¹⁶ This seems plausible since both are true under the same condition that is, that the planet Venus is actually shining at the time of utterance. Yet it is also possible to defend the view for this example that the truth-conditions of the two sentences are different if one adopts a Russellian account of expressions like ‘the Morning Star’ and ‘the Evening Star’ and further contends that the truth of each of the two sentences requires the satisfaction of different predicates by Venus.

I take it that Beaney would favor the sameness of truth-conditions of these two sentences even if the analysis to be undertaken were the Russellian one. What matters for Beaney’s account of the semantical content of (1) and (2) is the co-extensionality of the predicates ‘ x is the Morning Star’ and ‘ x is the Evening Star’. Nonetheless, cases like the two sides of HP are much more difficult to accommodate within Beaney’s picture because they do not exhibit the same superficial syntax as the pair of sentences just mentioned. The trouble here is that for these more complex cases there is no clear criterion for the sameness of truth-conditions of a pair of sentences.

But let us suppose there was such a criterion and suppose further that under this criterion the two sides of HP possessed the same truth-conditions. Perhaps we could now talk about different analyses of a condition for the truth of a sentence in terms of different axioms for the reference of its constituent parts according to the privileged syntactical analysis of the sentence. Different reference axioms as well as different syntactical analyses would yield different ways of analyzing the same truth-condition. Maybe this is the case with HP. And maybe this is the best way of understanding the Fregean

¹⁶ See, for example, Beaney (1996), pp. 227-8 where he defends the sameness-of-truth-conditions view for the more controversial case of the pair of statements “Hydrogen is lighter than carbon dioxide” and “Carbon dioxide is heavier than hydrogen”.

notion of sense that agrees with his metaphor of the various ways of splitting up the content of a sentence. The semantical content of a sentence would be given by its truth-condition while its epistemological content would correspond to the specific way of partitioning that semantical content into constituent parts. Thus, HP's left- and right-hand sides would possess the same semantical content but different epistemological contents (one side expresses an identity statement between the numbers that belong respectively to two concepts and the other side a statement of the one-to-one correspondence between the extensions of the two concepts).

Various problems crop up here. One – our fifth problem with Beaney's distinction¹⁷ – is that epistemological content in the sense of a way of dividing up the semantical content of a complex expression cannot be what Frege had in mind when he introduced the notion of sense. For according to the above account the sentences "carbon dioxide is heavier than hydrogen" and "hydrogen is lighter than carbon dioxide" would express different epistemological contents.¹⁸ But according to Frege,¹⁹ two sentences possess the same sense if their cognitive value for a speaker is identical. That is: if it cannot be the case

¹⁷ This problem relates of course to the first problem discussed at the beginning of section 4.1.

¹⁸ Beaney maintains that the case of this pair of sentences is relevantly different from that of the two sides of HP since the syntaxes of the sentences of the former pair is the same while the syntaxes of the sentences of the latter pair differ. This, according to him, adds plausibility to the thesis that the sentences of the first pair possess the same sense, in the sense of being intensionally equivalent or intersubstitutable *salva veritate* in all intensional contexts, while the senses of the sentences of the second pair differ. But I do not see why one could not insist that both in the sentences of the first pair as in those of the second pair different concepts are being employed and hence different senses should be assigned to each member of the pairs. More on this below.

¹⁹ See, for example, Frege (1892).

that a speaker who believes that carbon dioxide is heavier than hydrogen does not also believe that hydrogen is lighter than carbon dioxide and it cannot be the case that he or she believes the second content without believing the first. We will come back to the criterion of sameness of sense in a moment. The important point here is that it can be argued that a speaker could not believe that A is lighter than B without at the same time believing that B is heavier than A ; somebody could not have learnt the grammar of the expression “ x is lighter than y ” without associating it tautologically with “ y is heavier than x ”. If this is so, then the notion of a way of splitting up semantical content cannot be employed to shed light on the Fregean notion of sense.

A further problem – the sixth – relates to the question of whether the different ways of splitting up a whole content (be it semantical or epistemological) into its constituent contents would not amount to different whole contents (despite what Frege’s metaphor suggests). If our starting point was the same as Frege’s – that is, the content of whole sentences – of course it would make sense to say that the same content, however it might be constituted, can be split up in many different ways. The opposite starting point – i.e., the contents of the parts of a sentence – would render nonsensical the talk of different contentual constituents producing the same whole content. But there is also room for a third starting point which consists in conceding the priority of the content of sentences over the content of their parts for the more basic sentences of the language (the sentences we learn first) while holding that for the other sentences (the sentences whose content we learn once we have grasped the content of their constituent parts) the content of their constitutors is prior to their respective sentence’s content. Within this third scenario it is possible to say that the identity of a new sentential content (for the latter sentences) is fully determined by the totality of its conceptual constituents (at the level of sense so as not conflate them with Fregean concepts).

The last scenario might suggest that the best manner of addressing the above question is from the point of view of a speaker's cognitive access to the content of the sentences of his or her idiolect. It is reasonable to say that a speaker grasps the thought expressed by a sentence when he or she understands the sentence. Now, does the understanding of sentences have priority over the understanding of its parts or is it the other way around? In other words: do we understand sentences in virtue of a prior understanding of its parts or do we grasp the content of its parts only in the context of an understanding of the whole sentence?

There is no need to decide the issue of where the priority lies in order to make sense of Beaney's proposal concerning the way to interpret Frege's dichotomy between a content and the different ways of decomposing that content. This is because he makes a new suggestion that may help to settle the question of whether or not two distinct ways of carving up a semantical content should constitute different epistemological contents.

5. THE THOUGHT EXPRESSED BY A SENTENCE AND THE THOUGHT AS EXPRESSED BY A SENTENCE

The idea seems to be that – and this is another suggestion by Beaney to dissolve the paradox of analysis – since the same thought can be expressed by different sentences, we must distinguish the thought expressed by a sentence from the thought *as* expressed by a sentence. Call the first content₁ and the second content₂. Beaney suggests the following illustration of this dichotomy: in order to understand the thought as expressed by the sentence “hydrogen is lighter than carbon dioxide” (i.e., its content₂) one must understand the expression “is lighter than”. Understanding this expression is, however, not necessary for grasping the thought expressed by this sentence (i.e., its content₁) because one may grasp this same thought by

understanding the sentence “carbon dioxide is heavier than hydrogen” (Beaney (1996), p. 239).

The new suggestion allows one to keep the Fregean thesis that a certain propositional content (content₁) may be analyzed in different ways (the thought as expressed by various different sentences) while at the same time making it sensible to say that the constituents of a propositional content as expressed by one sentence make up a different content (content₂) from the one yielded by the constituents of the same propositional content (content₁) as expressed by another sentence. According to Beaney:

The distinction just drawn here between the thought expressed by a sentence and the thought *as* expressed by the sentence may seem unnecessarily subtle, but it is essential to respect it if we are to preserve Frege’s fundamental conception that the same thought can be expressed in different linguistic forms. Indeed, it really amounts to no more than Frege’s original distinction between ‘content’ and ‘way of splitting up content’. (Beaney (1996), pp. 239-40)

Within this new proposal, HP’s left- and right-hand sides would exhibit the same content₁ but different contents₂. But what is the criterion for the sameness of content understood in this new light (content₁) if the same content can be expressed by more than one sentence? It may be said that such a criterion turns out to be the identity of truth-conditions. But then as we saw in section 4.2, no clear criterion for the sameness of truth-conditions of such complex cases as the two sides of HP is forthcoming. The same could be said of a criterion for the identity of content₁ that appealed to states of affairs; the question for a criterion like this would be: what makes it the case that either side of HP represent the same state of affairs? No sharp answer seems to be available²⁰ unless one said that was so by stipulation. But in that case one would have abandoned any attempt to

²⁰ See section 4.1.

solve the paradox of analysis.²¹ These remarks confirm Beaney's hunch that the solution of the paradox requires abandoning the notion of semantical content; as we will see in the next section his strategy will be to look for a sharper criterion for contents₁.

Moreover, if the proposal is correct, contents₂ must be as finely discriminated as there are sentences to express thoughts. Nevertheless, it would be plausible to maintain that according to the ordinary notion of content, contents themselves cannot be so finely discriminated. Suppose, for example, there are facts about the intertranslatability of words of a language into another; suppose that the English sentence "this dog barked all night long" translates perfectly well the Spanish sentence "este perro ladró toda la noche". Shall we therefore say that the two sentences express two different contents₂? We surely need a criterion for the identity of content₂ which does not subordinate a content₂ difference between the two sentences to some difference in their constituent expressions.

And finally if there are contents (contents₂) capable of being expressed by different sentences then it follows that a speaker may express the same content₂ without knowing that he or she is doing so. How could that be? Imagine, for instance, that there are two synonymous sentences²² in the same language and that a speaker can express them both. If nothing like full understanding of both sentences is required then there may be situations in which the speaker will assent to one of them and not to the other and vice versa; one would be forced to say that the two sentences did not have the same cognitive value for the speaker and hence not the same content₂. Here we have another motivation for demanding a clear criterion for the sameness of the thought as expressed by a sentence.

²¹ This should have been made clear in the discussion of section 2 about analysis and stipulation.

²² In the sense of expressing the same content₂.

6. A CRITERION FOR COGNITIVE EQUIVALENCE

An attempt to circumvent the above mentioned obstacles to the new proposal might be to appeal to the most promising principle of sameness of content (content₂ for the present case) put forward in Beaney's book.²³ The principle (SCE)²⁴ states that "two propositions *A* and *B* possess the same *sense* (express the same *thought*) if and only if anyone who understands both propositions at a given time can immediately recognize that *A* is true (or false) if they recognize *B* as true (or false), and vice versa" (Beaney (1996), p. 231). This certainly rules out the possibility that a speaker grasps the same thought expressed by two different sentences without knowing that the thought is the same. The reason is that a speaker who satisfies the right-hand side of SCE cannot fail to know that the content of sentence *A* is the same as the content of sentence *B*.

Beaney suggests, however, that the soundness of SCE is not beyond any reasonable doubt. The problem, he says, is that there would be a circularity if understanding a sentence were conceived as grasping the sense (content₂) of the sentence; and this is the way Frege conceives of it. The notion we are trying to clarify (that of the thought as expressed by a sentence [content₂]) is being presupposed in the right-hand side of SCE.

I do not see the problem though. The appearance of circularity disappears once one is sensitive to the following consideration. Let us say that SCE characterizes the notion of content₂ partially in terms of the concept of understanding. The thought is that the criterion for attributing understanding of a sentence (grasping its content) to

²³ It should be noticed that Beaney himself does not apply the following principle (SCE) specifically to contents₂. The proposal is mine since I am interested in seeing whether an interesting and sharp notion of content₂ can be isolated.

²⁴ For sense as cognitive equivalence.

someone is more basic than the criterion for sameness of content;²⁵ someone might say that there are observable signs of understanding of sentences although recognition of identity of content requires application of the criteria for understanding plus evidence that assenting to utterings of *A* can never be accompanied by dissenting from utterings of *B* and vice versa. Seen from this perspective SCE it involves no circularity.²⁶

Beane's problem, however, is not so much that of making sense of an identity of content₂; rather, what he needs is a criterion for sameness of content₁. For the aim is to solve the paradox of analysis and the strategy continues to be that of distinguishing two kinds of content (at present: content₁ and content₂) such that the correctness of an analytical definition is explained in terms of its definiens and definiendum possessing the same content₁ while its informativeness requires that their contents₂ be distinct.

A real test for SCE would then be to yield the identity of the senses (contents₁) of the two sides of HP that Beane would have liked (Beane (1996), pp. 231-2). The exceptional situation is easy to imagine: suppose someone could understand either of its sides although he or she did not know that they expressed the same content₁. Such an imagined situation can be made very plausible: think of a speaker who does not use Hume's principle for ascertaining that two sets have the same number; whenever faced with the problem of the identity of cardinality between two sets, he or she counts the objects belonging to both, say. This person may dissent from the claim that the set of

²⁵ More basic in the sense that in order to know whether someone understands a sentence we are not forced to use the criterion for sameness of content.

²⁶ The way to avoid the circularity is to take the notions of understanding and assenting to and dissenting from sentences as primitive. Once this is done, we can then try and extract a notion of sameness of content as characterized by SCE.

natural numbers is equinumerous with the set of even numbers while assenting to the sentence that says that there is a one-to-one correspondence between their elements. The reason it is plausible to think that the two sides of HP express different thoughts is that the conceptual resources of its two sides are very different. But then Beaney might respond that the difference in conceptual constitution is captured by the notion of the thought as expressed by a sentence (content₂), not by that of the thought expressed by a sentence (content₁).

Thus, the difference in conceptual constitution between the two sides of HP results from the different thoughts as expressed by either side since the thought expressed by the left-hand side of HP must be the same as the thought expressed by its right-hand side. The question of conceptual constitution should not arise for the thought expressed by either side of HP; one cannot say, for example, that the concept of a natural number must be grasped by anyone who is able to grasp the content of either side of HP since that content may be apprehended *via* the conceptual resources of HP's right-hand side. If the above suggestion is right, then there is no unique conceptual constitution of a propositional content although there is one for each propositional content as expressed by a sentence.

There is a difficulty with this suggestion though. It is connected with the claim that the concept of natural number is not essential for the apprehension of the thought expressed by either side of HP. If we add to this claim the hypothesis also defended by Beaney that the sides of HP are cognitively equivalent²⁷ (i.e., express the same thought [content₁]) then it might be argued that it is impossible to satisfy the requirements of claim and hypothesis. For the hypothesis can be interpreted as demanding that the concept of number be an essential conceptual constituent of the thought in question. This is because one

²⁷ According to SCE.

alternative for justifying the identity of content₁ of the two sides of HP is to appeal to Gareth Evans's generality constraint²⁸ with respect to the concepts of natural number and of one-to-one correspondence.

Also known in the literature as the systematicity of thought and language, the generality constraint says that the capacity to entertain certain contents or to use certain expressions is indissolubly connected with the capacity to entertain other contents or use other expressions. For example, if a subject has the capacity to believe that John is tall he must also have the capacity to believe that Smith is tall, that Susan is tall, and so on, as well as the capacity to believe that John is short, that John is fat, and so on. If human thought and language are systematic in this sense and if the pair of concepts *the number that belongs to a concept* and *one-to-one correspondence between the things that fall under two concepts* instantiate this type of systematicity (as the hypothesis mentioned above seems to imply), then possession of the concept of natural number must be essential for entertaining the thought expressed by either side of HP. QED.

7. THE HISTORICAL PERSPECTIVE

Beaney concludes his last chapter pessimistic about the prospects of dissolving the paradox within a Fregean framework. His final suggestion is that the dissolution of the paradox requires an historical perspective. The following passage summarizes his view nicely:

But if all this is right, then what it suggests is that the paradox of analysis cannot ultimately be 'solved' without taking an historical approach. For if what analysis involves is the crystallization of sense, then since this is a process that occurs over time, there is no ahistorically positioned answer as to whether it is both correct and informative. Before the theory is developed in which the analysis is offered, the analysis, if it is understood at all, will seem incorrect; and

²⁸ See, for example, Evans (1982), pp. 75, 100-5.

after it is developed, with the necessary transformation in our understanding effected, it will be correct but uninformative. To talk of 'correctness' is to make a move *within* a system; yet informativeness is located in the process of developing, learning and using a system. (Beaney (1996), p. 266)

The idea is that only by looking at the historical process of refinement of a concept (e.g. the concept of a natural number) can one account for the informativeness of analytical definitions; logical justification – interpreted as an ahistorical attempt to explain both the correctness and the fruitfulness of analytical definitions – cannot help us understand fruitfulness. But why should we be forced to interpret it this way? It is definitely true that for Frege concepts and sentential contents had no history; this is the reason why he felt forced to appeal to the distinct ways of carving up a sentential content so as to be able to explain the newness of analysis and of deductive proofs. Frege's efforts in this direction were fruitless, but that is no reason to be pessimistic about the possibility of a logical justification of analysis. What needs to be introduced into logical justification is a historical element; we must abandon the prejudice that this historical element is incompatible with logical justification. But more on this in the following section.

Nonetheless, conceding a historical element in the logical justification of analysis is one thing; it is quite another to reject logical justification altogether in favor of a completely historical perspective. The trouble with this perspective appears at the point where it presents an explanation of the correctness of analysis. We are told that the correctness of an analytical definition can only be assessed from within the system or theory of which the definition is a part; there is no external point of view from which to decide the question. This point of view can only be taken when it is a matter of evaluating the informativeness of such a definition. We are also told that correctness cannot be explained diachronically because a given analysis when seen

across time will always seem incorrect. This is because of the evolution or crystallization of the contents involved. It is only when we see an analytical definition as a move within its respective theory (when the theory itself is successful) that the definition can be said to be correct.

Still, one might insist that correctness also requires an external point of view, for otherwise there would be no way of knowing whether the new concept correctly analyses the old. If we cannot step out of the system (the theory) to which the new concept belongs, what guarantees that the system as a whole presents us with a correct analysis of the old concept? It is open to Beaney of course to say that the correctness of the theory containing the new concept stems from its success and acceptance. For example, the analysis of water as H_2O is correct because of its place in a widely accepted and largely successful²⁹ theory of the atomic structure of this substance. The following passage confirms the above:

In the end, what makes an analysis a good one is its success, as part of some overall theory, in convincing us that our ordinary discourse is indeed imprecise, and requires refinement for scientific purposes. (Beaney (1996), p. 263)

If this is so however then there will always be many possible analyses of a given concept as there could be many possible successful theories containing distinct and perhaps conflicting further analyses of a given concept. The intuition that there can only be one correct analysis of a concept cannot be refuted by Beaney's historical perspective.

8. INCOMPLETE CONCEPTS

So, how do we stand with respect to the problem of dissolving the paradox of analysis? A promising option, at this stage, would be

²⁹ Successful in the sense of its explanatory and predictive power.

perhaps to drop the hypothesis of preexisting content waiting to be analyzed in favor of that of constantly evolving concepts. These concepts will be fruitful or not depending on the success of the theory or theories they are embedded in. Beaney, as most of us nowadays, sides with the later Wittgenstein against Frege concerning the hypothesis that our concepts are essentially incomplete and in constant process of being partially completed. This hypothesis, however, implies that the notion of correctness in terms of sameness of old and new content must go. Of course the old and new content are different; the new concept extends the old concept in a new direction. Wittgenstein puts much emphasis on this point in his *Remarks on the Foundations of Mathematics*.³⁰ Consider, for example, the following passage:

But suppose I first introduce ' $p \vee q$ ' and ' $\sim p$ ' and use them to construct some tautologies – and then produce (say) the series $\sim p$, $\sim\sim p$, $\sim\sim\sim p$, etc. and introduce a notation like $\sim^1 p$, $\sim^2 p$,... $\sim^{10} p$... I should like to say: we should perhaps originally never have thought of the possibility of such a sequence and we have now introduced a new concept into our calculation. Here is a 'new aspect'. (Wittgenstein (1978), III-46, p. 178)

Wittgenstein sometimes expresses this idea by saying that the new concept adds a further determination to the old concept where there was none before. The following quote is explicit about the point:

The limit of the empirical – is *concept-formation*.

What is the transition that I make from "It will be like this" to "it *must* be like this"? I form a different concept. One involving something that was not there before. When I say: "If these derivations are the same, then it *must* be that..." I am making something into a criterion of identity. So I am recasting my concept of identity. (Wittgenstein (1978), IV-29, p. 237)³¹

³⁰ Wittgenstein (1978).

³¹ See also Wittgenstein (1978), III-46, p. 178; IV-30, pp. 238-9; V-9, p. 267; and V-40, pp. 294-5.

This can be best illustrated by Cantor's proof that the real numbers have a higher cardinality than the rational numbers. Following Wittgenstein's insight, one could say that Cantor's diagonal method added a further criterion for the application of the concept of a real number. Before his discovery of the method we did not know how to answer the question of whether the rational and the real numbers were equinumerous or not; Cantor taught us how to answer it.

But, one might ask: if the old and new concepts are not identical then how can it be claimed that the analysis of the first concept by the second is correct? A similar objection to Wittgenstein's claim about the evolving character of our concepts is put forward by Michael Dummett. Dummett objects that Wittgenstein's thesis that each mathematical proof provides a new criterion for the use of our concepts is incompatible with a satisfactory explanation of the correctness of our deductive practices.³² I have discussed this objection elsewhere.³³ What seems to me relevant for both discussions – that is, the one about the correctness of analysis and the other about the correctness of deduction – is Dummett's remark that a putative explanation of the correctness of deductive inference is incompatible with Wittgenstein's claim that the use of deduction may change our concepts.³⁴

³² The objection can be found in various of Dummett's texts. See, for example, Dummett (1959) and Dummett (1994).

³³ In Pinto (1998), sections 7.5.1, 7.5.3 and 7.5.4.

³⁴ Indeed Wittgenstein may be read even more radically as claiming that any new application of deduction changes our concepts. For example, when he claims that "the further expansion of an irrational number is a further expansion of mathematics." (Wittgenstein (1978), V-9, p. 267) I favor here the more cautious interpretation of his remarks suggested in the main text. But if we take this last quotation seriously then even a new use of an old concept (for example, that of addition) will have changed the concept. Such a thesis would be much harder to defend; this is why I decided to appeal to less controversial examples like those of Cantor's diagonal proof, the Continuum

According to Dummett, this is so when we interpret the latter claim as saying that deduction may change even the extensions of our concepts. For in this case it would be possible to find a counter-example to the new concept generated by a mathematical proof, say, which however falls under the extension of the old concept, or vice-versa. As an illustration of the situation Dummett has in mind, think of the geometrical theorem: the intersection between a cylinder and a plane is an ellipse. If the discovery of the theorem's proof had changed the extension of the concept of an ellipse, then it would be possible that a geometrical object were an ellipse, according to the new concept's criteria of use, although, when judged from the point of view of the criteria for the old concept, it would not be an ellipse.

If this is correct, then the same difficulty applies to the last proposal for explaining the correctness of analytical definitions. Here also it would be possible, Dummett might say, to find a counter-example to the pre-Fregean concept of a natural number that is a number according to the Fregean conception of number (for instance, extensions of concepts). But that possibility would only exist if the concepts in question had sharp boundaries. And, as we saw above, for constantly evolving concepts the question of whether a given object falls under a concept or not does not always have an answer.

Take, for instance, Dummett's geometrical example. The incomplete concepts proposal claims that the extension of the old concept of an ellipse is open with respect to the question of whether intersections of cylinders and planes belong to it or not; this is a question that the new concept given by the proof decides positively. The conflict between the old and the new concept does not exist because the modification in the extension of the concept of an ellipse

Hypothesis, the theorem about the ellipse and Euler's proof (the latter two below). The thought is that for these cases the thesis that a new use of deduction changes our concepts is much easier to defend.

effected by the proof corresponds to a decrease of the size of the area of uncertainty between the area including the objects which are ellipse and that including the objects which are not. The objects (O_1) that were ellipse according to the old concept continue to be so when the new concept is adopted (see figures 1 and 2). Likewise with the objects (O_3) that were not ellipses according the old concept.

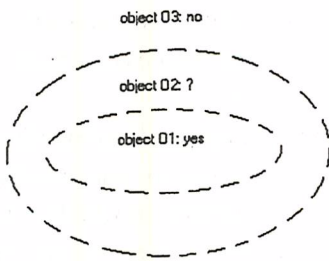


Fig. 1: old extension of the concept ellipse

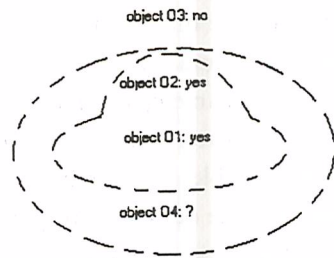


Fig. 2: new extension of the concept ellipse

Talk of an uncertainty area in the extension of a concept may give the impression that the lack of a decision procedure has to do only with our ignorance of the facts about our concepts. This is not what the incomplete concepts' proposal claims at all.³⁵ According to it, the facts, say, about the extension of the concept of an ellipse are constituted by the existence of identity criteria for the objects falling into it. Thus, imagine a group of people whose concept of an ellipse

³⁵ A reason for not calling Wittgenstein's account the vague concepts proposal was precisely that, contrary to other views of vagueness (for example, Williamson (1996)), Wittgenstein refuses to see vagueness as having anything to do with our ignorance (that is, to see it as epistemological in nature).

does not include the criterion that all intersections of cylinders and planes are ellipses. Suppose they never came across the proof mentioned by Dummett. For them, a plane curve is an ellipse if two symmetrical points on its main axis (its two foci) can be found such that the sum of the distance between the first focus and an arbitrary point on the curve (the supposed ellipse) and the distance between the second focus and the same arbitrary point remains the same for every point along the curve. It would be plausible to say that for this concept of an ellipse there is no fact about plane figures which are intersections of planes and cylinders belonging or not to its extension. Such a concept of an ellipse might be an ancestor of ours.

A better example is perhaps that of the concept of real number. It is possible to imagine a more primitive concept of real number which is like ours in all respects except that the question of whether the real numbers are equinumerous with the natural numbers cannot be decided by it. Likewise, it is also possible to imagine a more advanced concept of real number according to which the Continuum Hypothesis is decided positively or negatively. If the measuring rod is the more primitive concept of real number then there is no fact about whether real numbers have the same cardinality as the natural numbers. If the measuring rod is our concept of real number there is no fact about whether there is an intermediate cardinality between that of the natural numbers and that of the real numbers. This should suffice for motivating the incomplete concepts view that the facts about the constitution of the extension of a concept are given by the criteria for identifying objects as belonging to this extension.

It should also be noticed that Wittgenstein's proposal is not committed to the claim that whenever a new concept is produced from an old one (through analysis, proof or any other method) the extension of the new concept differs from that of the old one. There are also cases for which what distinguishes the old and the new concept is that the latter contains a further criterion for determining the same

extension that is not to be found in the former. Take as an example the famous proof by Leonhard Euler that there is no non-interrupted route through all the seven bridges in eighteenth century Königsberg.³⁶ Before the proof, there was already a criterion for deciding whether or not any continuous path through all the seven bridges which did not go through any of them twice contains only seven crossings. As there are only a finite number of such paths, it is just a matter of listing them all and checking that none of them satisfies the description of being a continuous path over all the bridges without repeating any of them. The new concept of an interrupted minimal path over all the Königsberg bridges has the same extension as the old one; they differ in that Euler's theorem is included as a criterion in the new but not in the old concept.

To summarize the Wittgensteinian proposal for solving the paradox of analysis. First, correctness is explained in terms of the lack of conflict between the respective extensions of the old and new concepts. That is, the extensions of the two concepts may differ; what cannot happen is that the criteria for one concept legislates in favor of including a given object in its extension while the other rules against that inclusion. If the new concept's criteria produce a positive or negative verdict and the old concept's criteria produces no verdict there is no problem because no contradiction is involved. Secondly, informativeness is accounted for in terms of the difference in content (or sense) and perhaps also in extension between the two concepts. According to Wittgenstein, analysis as well as deduction both contribute to the modification of our concepts.

But it might be objected that the new account of informativeness does not fare any better than the Fregean one since the former, like the latter, also appeals to a difference of content or sense

³⁶ See Terquen & Geronio (1851), pp. 106-19 for the French translation of Euler's paper (1736).

between the old and the new concept in order to make sense of the newness of analysis. Implicit in this objection is the idea that what caused the problem with the Fregean account (the notion of sense) is again being employed in the Wittgensteinian account. Hence, there is no explanatory gain in the transition from Frege to Wittgenstein. I think the objection is unfair. In my opinion, what made trouble for Frege was the idea that we need two notions of content: one for correctness (this kind of content does not change with the analysis of a concept) and the other for informativeness (analysis modifies this second kind of content). The difficulty lies in getting a kind of content that is rich enough to remain the same in analysis and yet another kind of content that will account for the newness of analysis. The lesson to be learnt from sections 3 to 7 is clear: splitting content in two does not help to solve the paradox of analysis.

Wittgenstein's account appeals to one single notion of content and analysis, as well as other types of new uses of language, may modify such a content. Content modification is great for fruitfulness; the trouble is that it seems incompatible with correctness. This is indeed the core of Dummett's above-mentioned objection. Moreover, remember that our task was to reconcile correctness with the fruitfulness of analysis and not to explain either of them separately. The Wittgensteinian strategy then boils down to this: concept modification resulting from analysis does not clash with correctness because the new concept extends the old concept in a new direction; a direction that was not written into the old concept in advance.

Finally, if the incomplete concepts proposal is right in claiming that the only way to account for both the correctness and fruitfulness of analysis is to view our concepts as permanently incomplete, then the idea that analysis must strictly preserve content and more particularly extension has proved untenable. But this does not mean that any modification in conceptual extension is allowed. A correct analytical definition must still avoid any conflict between the criteria for the old

and for the new concept within any range of objects for which they can both decide whether those objects belong or not to the extension of the concepts in question. Such a requirement of harmony is of course compatible with the thesis that analysis and deduction modify our concepts and is indeed what renders the latter thesis plausible, critics like Dummett notwithstanding.³⁷

BIBLIOGRAPHY

- BEANEY, M. (1996). *Frege: Making Sense* (London, Duckworth).
- CLARK, P. & HALE, B. (eds.) (1994). *Reading Putnam* (Oxford, Blackwell).
- DAVIDSON, D. (1984). *Inquiries into Truth and Interpretation* (Oxford, Oxford University Press).
- . (1967). "Truth and Meaning" in Davidson (1984).
- . (1969). "True to the Facts" in Davidson (1984).
- DUMMETT, M. (1959). "Wittgenstein's Philosophy of Mathematics" in Dummett (1978).
- . (1973). "The Justification of Deduction" in Dummett (1978).
- . (1978). *Truth and Other Enigmas* (London, Duckworth).
- . (1994). "Wittgenstein on Necessity: Some Reflections" in Clark, P. & Hale, B. (1994).
- EULER, L. (1736). "Solution d'un Problème appartenant à la Géometrie de Situation" in Terquem & Gerono (1851).
- EVANS, G. (1982). *The Varieties of Reference* (Oxford, Oxford University Press).

³⁷ I am enormously indebted to Marco Ruffino and also to an anonymous referee for their invaluable comments on an earlier version of this paper.

- FREGE, G. (1879). "Begriffsschrift, a formula language modelled upon that of arithmetic, for pure thought" in van Heijenoort (1967).
- . (1884). *Die Grundlagen der Arithmetik*. (Breslau, Köbner)
Translated as Frege (1950).
- . (1891). "Function and Concept" in Frege (1984).
- . (1892). "On Sense and Meaning" in Frege (1984).
- . (1893-1903). *Grundgesetze der Arithmetik*. (Hildesheim, Georg Olms Verlag).
- . (1914). "Logic in Mathematics" in Frege (1979).
- . (1950). *The Foundations of Arithmetic* transl. by J. L. Austin (Oxford, Blackwell).
- . (1979). *Posthumous Writings*. (Oxford, Basil Blackwell).
- . (1984). *Collected Papers on Mathematics, Logic, and Philosophy* (Oxford, Basil Blackwell).
- PINTO, S. (1998). *Wittgenstein, Meaning and Mathematics* Ph.D. dissertation (London University).
- TERQUEM & GERONO (1851). "Solution d'un Problème appartenant à la Géométrie de Situation, par Euler" in *Nouvelles Annales de Mathématiques* Première Serie (10): 106-19.
- VAN HEIJENOORT, J. (ed.) (1967). *From Frege to Gödel. A Source Book in Mathematical Logic, 1879-1931* (Cambridge, MA, Harvard University Press).
- WILLIAMSON, T. (1996). *Vagueness* (New York, Routledge).
- WITTGENSTEIN, L. (1922). *Tractatus Logico-Philosophicus* (London, Routledge)

- . (1978). *Remarks on the Foundations of Mathematics* ed. Anscombe, Rhees & Von Wright, 3rd edition (Oxford, Basil Blackwell) 1st edition 1956.
- WRIGHT, C. (1983). *Frege's Conception of Numbers as Objects*. (Aberdeen, Aberdeen University Press).

