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LOGICISM: FREGEAN AND NEO-FREGEAN*

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In this paper I first reconstruct the main steps of the development of Frege's logicism; along the way, I point out possible solutions of a number of interpretative puzzles about Frege's procedure raised in the scholarly literature. Next, I discuss some problems faced by the contemporary reconstruction of Frege's arithmetic embraced by, among others, Crispin Wright. Finally, I point out what seems to me to be missing from the contemporary discussion about Frege's rejection of Hume's Law as basic law of arithmetic.

INTRODUCTION

The work of Gottlob Frege has exerted an enormous and widespread influence in modern analytic philosophy, especially among philosophers interested in language, semantics, and epistemology. Until recently, less attention had been given

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to Frege's philosophy of mathematics, despite the fact that Frege's lifelong project, to which all other branches of his philosophy were subordinated, was to find the correct foundation of arithmetic.

The neglect which the mathematical part of Frege's philosophy has received is partly due to the well known fact that his formal system is inconsistent, and partly to the inconvenience of reading and understanding his logical notation. Apparently, the opinion underlying much of the philosophical research on Frege is the following: his philosophy of arithmetic is useless due to the inconsistency; only his early work on pure logic, as well as his insights concerning language (sense and reference) deserves attention.

Some major exceptions to this received opinion are Crispin Wright's *Frege's Conception of Numbers as Objects* (1983) and Dummett's *Frege: Philosophy of Mathematics* (1991). Despite my disagreement on many fundamental points with Wright and Dummett, I think that they succeeded in calling attention to the great sophistication and philosophical value of Frege's original project, as well as in bringing Frege's reflections about the foundations of arithmetic to the center of the modern debate on the philosophy of mathematics. My paper is meant as a contribution in this same direction. In my opinion, some of the most powerful insights that Frege had, such as his views on the ontology of mathematics and of logic, or the distinction between concept and object, or even the sense-reference distinction, are by-products of his efforts to create an adequate formalism in which the reduction of the basic laws of arithmetic to the axioms of logic could be carried out. So, I believe, a better understanding of the development of Frege's philosophy of arithmetic will certainly contribute to a better understanding of the genesis and nature of these other insights.

In this paper, I shall first present some of the fundamental steps of the evolution of Frege's logicism in a systematic way. I shall also suggest a solution to an interpretative problem concerning the so-called context principle and Frege's apparent formulation of a contextual definition in *GLA*.[1] Next, I will present and discuss a current issue in the literature about Frege. Finally, I shall indicate what is, in my view, the main defect of the neo-Fregean project as it has been formulated by Wright (and embraced by other philosophers).

1. FREGE'S LOGICIST PROJECT

From the beginning of his philosophical career Frege had in mind the task of providing a deep analysis of the fundamental concepts of arithmetic. Frege's logical notation, the *Begriffsschrift*, presented in a short book published in 1879, was meant to be an adequate tool for the rigorous analysis of (primarily) arithmetical concepts, a device which could show precisely how and when (if at all) elements deriving from empirical or pure intuition are essential to arithmetical concepts and, consequently, how and when (if at all) these elements are required in the derivation of arithmetical statements. Frege believed that a careful analysis of the fundamental arithmetical concepts would show that no element extraneous to pure logic was involved in the basis of arithmetic. As he states in the preface:

> We divide, accordingly, all truths in need of a justification into two kinds, such that the proof of the first kind is purely logical, and the proof of the other kind of truths needs the

[1] See bibliography for the abbreviations of Frege's works used in this paper.

> support of facts of experience[...] Since I posed to myself the question to which one of these two categories arithmetical judgments belong, I had first to check how far one can go in arithmetic using inferences, having as support only the laws of thought, which are above all particulars. The path here was the following: I first tried to reduce the concept of ordering in a sequence to that of *logical* implication [*logische Folge*], and from this I tried to progress to numbers. (*BS* iii-iv)[2]

This passage expresses the question motivating the whole of Frege's philosophical work. The last sentence describes the program that, so he thought, could answer the question. However, only the first part of the program – the explanation of ordering in a sequence in terms of logical implication – is developed in the *Begriffsschrift*. No indication is given in this book about the part of the program vaguely announced in the last part of the last sentence. That is, no indication is given in *BS* about how to progress from the notion of ordering in a sequence to the notion of numbers.

There is no indication that by 1879 Frege already had in mind a definite idea about the ontological status of numbers. It was not until five years later, in *GLA*, that Frege claimed that numbers are objects. His position in *GLA* was the product of a sophisticated philosophical analysis, the main steps of which I shall briefly review in the next section.

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[2] My translation. Bynum translates "*logische Folge*" as "logical ordering" in Frege (1972). It seems, however, that what Frege is pursuing in part III of *BS* is a definition of the notion of order that employs only the symbols for logical implication (which is actually material implication) and for first and second-order quantification. This is the reason why, in my view, "logical implication" is a better translation.

2. THE DEFINITIONS OF NUMBERS IN *GRUNDLAGEN DER ARITHMETIK*

2.1 NUMBERS AS SECOND-ORDER CONCEPTS

It is clear that up to *GLA* § 44, Frege's discussion is mainly negative in purpose: the main point of his arguments is to refute some rival conceptions of arithmetic, especially the views of Kant, Mill and several versions of psychologism. In *GLA* § 46, Frege establishes the first fundamental result of his constructive analysis. This result is based on the claim that, given a certain aggregate of objects, we can attribute different numbers to the same aggregate, depending on the way this aggregate is described. For example, we can say of the same group of trees 'here is one forest' and 'here are 600 trees'. What changes from one statement to the other one is not the aggregate of things itself but the concept under which this aggregate is considered. So, Frege concludes, a statement of number (*Zahlangabe*) is a statement about a concept (*GLA* § 46).

If statements of number are always about concepts, it seems natural to adopt the position on the ontological status of numbers according to which they are second-order concepts. That is to say, the suggestion is that a statement of number says that a second-order property holds of a concept. In *GLA* § 53 Frege explains the suggestion with an analogy between a statement of number and existential statements: an assertion that *F* exists is nothing but a denial that the number zero belongs to the concept *F*. This is indeed the analogy that Frege uses in *GLA* § 55 when he proposes a first set of definitions of numbers. Here numerals figure as part of existential quantifiers:

$$(\exists_0 x)Fx \leftrightarrow (\forall x)\neg Fx$$

$$(\exists_1 x)Fx \leftrightarrow (\neg(\forall x)\neg Fx \wedge (\forall x)(\forall y)((Fx \wedge Fy)\rightarrow x=y)$$

$$(\exists_{n+1} x)Fx \leftrightarrow (\exists x)(Fx \wedge (\exists_n y)(Fy \wedge y\neq x))$$

(where '$(\exists_n x)Fx$' abbreviates 'the number n belongs to the concept F'). The first sentence is supposed to define the number zero, the second defines the number one, and the third the successor of a number. Despite the intuitiveness of this set of definitions, Frege rejects it in the following famous passage of *GLA* § 56:

> [...] we can never – to take a crude example – by means of our definition, decide whether the number Julius Caesar belongs to a concept, whether this famous conqueror of Gaul is a number or not. Moreover, we cannot show by means of our attempted explanation that $a = b$ must be the case if the number a belongs to the concept F, and the number b belongs to the same concept. In this way the expression "the number that belongs to the concept F" would not be justifiable, and hence it would be impossible to prove a statement of number because we would be unable to achieve a determinate number. It is only appearance that we have defined the 0, the 1; actually, we only fixed the sense of the phrases "the number 0 belongs to" and "the number 1 belongs to"; but we are not allowed to distinguish here the 0, the 1 as self-subsisting objects that can be recognized as the same again.

The first part of the quotation expresses Frege's dissatisfaction with the fact that genuine numerical equations cannot be derived from these definitions. This is so because numerical equations treat numbers as objects – since identity is a first-order relation – while the above definitions treat them as second-order concepts. This reason, however, is not decisive. Numerical identities could be interpreted as an "improper" way of expressing the fact that we recognize two numerical quantifiers as being "the same". But even if this reinterpretation of numerical identities were available, it is not likely that Frege would have accepted it. The last sentence of the quotation expresses the belief (which he never abandoned) that numbers are objects, and the greatest sin of the proposed definition is to fail to treat them as such. That is to say, even if the proposed definitions worked with arithmetical identity-statements reinterpreted in some appropriate way, in this reconstruction the *real* nature of numbers would be disguised.

What exactly the basis is for Frege's assumption that numbers are objects is a very delicate question, and it certainly deserves a much deeper answer than the one I shall outline here. One of the reasons is explicit in Frege's writings. On several occasions he offers as evidence for this ontological thesis some grammatical features of ordinary arithmetical statements. One of these features that seems to have deeply influenced Frege is the fact that numerals do not admit of a plural form (*GLA* § 38). Another related grammatical feature is that they are often preceded by the definite article (*GLA* § 38, § 57, § 68 n. 1), but never by the indefinite article. These two facts together contrast numerals with conceptual terms. For conceptual words – and only conceptual words – admit of pluralization (*GLA* § 38), and the possibility of the use of an indefinite article with a word indicates that it is meant to be conceptual (*GLA* § 51). The definite article, in contrast to the indefinite, indicates that we are using the word to refer to one single thing, and not to a multiplicity of things (*KS* 127; *WB* 195; *GGA* II § 100). On several occasions, Frege expresses the view that the presence of the definite article in an expression is justified if and only if the expression denotes one and only one object (*e. g.* in *GLA* § 23, § 74, n.1, § 76, § 97). Frege seems to never have abandoned the idea that the presence of the definite article indicates that the term in question is a proper name – and hence, if it refers at all, it refers to an object.[3] This principle survives until his very last writings (it appears, *e. g.*, in *NS* 274, 288), even after he gave up the idea that numbers are logical in nature.

Another grammatical feature that strongly influenced Frege was the resemblance that statements like 'two is a prime number' seem to bear to statements like 'Mars is a planet'. The first seems to resemble the second in asserting that a certain object falls under a certain concept (*NS* 282, 284; *WB* 271). A related consideration is that in arithmetic numerals are stan-

[3] It has been the source of some perplexity among scholars that the same Frege who so insistently raised doubts about the reliability of ordinary language as a guide in philosophical thinking took so seriously the presence of the definite article as a sign that some expressions (*e. g.*, numerals or names of extensions), are singular terms and hence refer to objects. I shall not discuss here in detail this apparent tension in Frege's philosophy, but it seems to me that it has been exaggerated by most Frege scholars. It is not quite true that Frege blindly took the presence of the definite article as a sure sign that certain terms are referential and function as proper names. Rather, Frege shows on a number of occasions an acute consciousness that assumptions regarding referentiality (even about the most primitive terms in logic) are essentially hypothetical, and, like any other hypothesis, they should by tested. In 1893, for example, he concedes that sometimes, although we use some terms believing that they are referential, they may lack a reference (*GGA* xxi). Unfortunately, this turned out to be true of some of the primitive names in his own system, as Russell's paradox of 1902 showed. Many years later, in 1924-25, Frege writes that it is quite difficult – and frequently impossible – to prove the logical harmlessness (*i. e.*, referentiality) of all terms employed in science (*NS* 289). Hence, the referentiality of many of these terms always remains a hypothesis. This fits nicely, I think, Frege's realist perspective on the nature and existence of mathematical entities.

dardly used in identity statements (*GLA* § 57). Now, Frege's analysis of identity underwent some changes between 1879 and 1893, but this much was clear to him from the beginning: whatever occurs on either side of the identity symbol must be a proper name. So numerals must be proper names.

There is a different reason for regarding numbers as objects instead of concepts.[4] There is no textual evidence that Frege thought about this particular reason, but it coheres with everything else that he says. It is natural to expect from a formal theory of the natural numbers that it can prove that every number has a successor. Now the definition of successor of *GLA* § 76 is:

> "there is a concept *F* and an object *x* falling under it such that the number that belongs to the concept *F* is *n* and the number that belongs to the concept 'falling under *F* but different from *x*' is *m*"
>
> means the same as
>
> "*n* follows immediately after *m* in the series of natural numbers".

If it must be a consequence of Frege's theory that any *n* has a successor, then there must be a concept *F* and an object *x* falling under it, such that there are $n+1$ objects falling under *F*, and *n* is the number of objects falling under *F* that are not *x*. That is to say, the theory, if it is to show that *n* has a successor, must require the existence of at least $n+1$ objects. In particular, if the theory is to show the infinity of the series of finite natural numbers, it follows that the theory will require the exis-

[4] As far as I know, Dummett was the first to point out this implicit reason (Dummett (1991), p. 132).

tence of infinitely many objects. Now, if numbers are not themselves objects (as in the definition of *GLA* § 55), the theory will depend, for its soundness, on the existence of an infinity of (non-logical) objects in the universe. This seems to be at odds with Frege's project of showing that all laws of arithmetic (including the infinity of the series of natural numbers) rest purely on a logical basis. It is true that the definition quoted above occurs in the text after Frege had already decided that numbers are objects, but the definition itself does not assume anything about the categorical status of numbers. Indeed, an entirely analogous explanation of successor is part of the definition of numbers as second-order concepts already in *GLA* § 55. It seems plausible to assume that Frege already had a picture about what the concept of successor in the number-series must be like, and at a certain point he saw that this requires the existence of infinitely many objects – which were the numbers themselves. No extra-logical existential assumptions were necessary.

2.2 HUME'S PRINCIPLE AND THE JULIUS CAESAR PROBLEM

After rejecting the possibility of defining numbers as anything but objects, Frege takes a different line of approach and proposes a second candidate for the definition of numbers in *GLA* §§ 63-5. This definition results from an analogy with the geometrical definition of direction, which does not say directly what directions are, but rather fixes the truth-conditions for two lines having the same direction. The definition of direction is:

the direction of a = the direction of b iff a and b are parallel

in which the identity of the left-hand side is explained by the right-hand side, which has epistemic priority. It says that two lines have the same direction if the relation of parallelism holds between the two. By analogy with this definition, Frege proposes, for any two concepts *F* and *G*:

the number of Fs = the number of Gs iff F and G are equinumerous.

The definition of the identity of numbers here is in terms of the relation of equinumerosity, and the explanation of the latter does not use anything extraneous to logic: *F* and *G* are equinumerous if and only if there is a 1-1 correlation between the objects falling under *F* and the objects falling under *G* (*GLA* § 72). In this definition, Frege is following what appears to be the statement of his general strategy at the beginning of *GLA* § 62:

> To obtain the concept of number, we must fix the sense of a numerical identity.

That is to say, Frege tries to fix the reference of numerical terms not by directly stating what they refer to, but rather by explaining identities between numerical terms in terms of statements where numerical terms do not occur. This is indeed crucial to his anti-Kantian conception of numbers, since by means of this definition it should become clear how we have cognitive access to numbers without having any representation of them: our cognitive access to natural numbers is *via* our access to concepts combined with our basic capability of recognizing whether or not an equivalence relation (in this particular case, a 1-1 correlation) holds between these concepts.[5]

[5] Bob Hale (1987) extensively discusses and generalizes this strategy as a possible response to the causalist challenge formulated by Benacerraf (among others) to Platonism (Benacerraf (1973)).

Frege's second definition, which I shall abbreviate as:

$$Nx{:}Fx = Nx{:}Gx \leftrightarrow F \text{ and } G \text{ are equinumerous}$$

has been frequently referred to in the literature as "Hume's principle", in accordance with Frege's attribution of the original form of this principle to Hume.[6] Despite its ingenuity, this definition is also rejected by Frege in the following famous passage:

> In the proposition "the direction of *a* is identical with the direction of *b*" the direction of *a* appears as an object, and we do not have in our definition a means of recognizing this object as the same again, if it should happen to crop up in some other guise, say as the direction of *b*. But this means does not provide for all cases. One cannot, for instance, decide by means of it whether England is the same as the direction of the Earth's axis – if I may be forgiven an example which looks nonsensical (*GLA* § 66).

We do know that the object that is meant by 'the direction of the Earth's axis' is not the same as England: one is an object of geometry, and the other is a concrete object (if we are referring to the territory, and not to something more abstract like the nation, or the state, etc.). However, Frege thinks, the definition itself plays no role in the discovery or formulation of this inequality. Indeed, this definition can only determine the truth-value of '$a = b$' when both 'a' and 'b' are of the form 'the

[6] George Boolos is the main author responsible, I believe, for the popularization of this designation for Frege's second definition. Boolos does not mean, however, that this principle is really to be found in Hume in the same way that Frege understood it. "Hume's principle" is employed by Boolos (and others) just as a name, with no historical commitment. I shall follow Boolos here.

direction of *c*'. This limitation is regarded by Frege as a deficiency:

> Obviously no one is going to confuse England with the direction of the Earth's axis; but that is not due to our definition of direction (*ibid.*).

This problem is known in the Fregean literature as the Julius Caesar problem, the problem of whether the definition of numbers is able to differentiate between numbers and objects of some other sort.[7] As Frege remarks in *GLA* § 66, it is of no help to say that 'the number of $F = b$' is false if b is not of (or cannot be translated into) the form 'the number of G', for this would presuppose that we are already in possession of the concept of number, which is not yet the case. And, he adds in *GLA* § 67, it is also of no help to say that q is a number if it can somehow be introduced by the definition in question. The reason why this is so, according to Frege, is that this solution presupposes that the way an object is introduced in a theory can be seen as a property of that object. But, he claims, a definition can never determine a property of an object: it can only fix the reference of names.

[7] This designation comes from the complaint in *GLA* § 56 (which I quoted before) that we could not know, by means of the previous definition, whether Julius Caesar is a number or not. Some authors have a stronger formulation of the Julius Caesar problem. (cf. Wright (1983), p. 108; Demopoulos (1995), p. 8). It points, they claim, to the requirement that the inequality between numbers and *any* other kind of object should be a consequence *only* of second-order logic, Hume's principle, and definitions. As I will mention in my closing remarks, it is dubious that this stronger version of the problem is what Frege used in *GLA* to criticize Hume's principle as a definition.

2.3 HUME'S PRINCIPLE AND THE CONTEXT PRINCIPLE: AN INTERPRETATIVE PUZZLE

There has been some debate in the literature concerning the role of the so-called context principle in the formulation of Frege's second definition in *GLA* §§ 62-5. The principle was first stated in the introduction of *GLA*:

> we should ask for the reference of words in the context of a proposition, but never in isolation (*GLA* x).

And in *GLA* § 62, when Frege outlines the project of the second definition, he asks:

> How can then a number be given, if we cannot have any representation or intuition of it? It is only in the context of a proposition that words have reference.

It seems clear that the second sentence of the quotation is meant as an application of the context principle as a guide in Frege's pursuit of a correct definition of number. This gives rise to a puzzle about Frege's real intention in applying the principle at this particular point of *GLA*. Some scholars tend to see the principle at this point as a *prima facie* justification of contextual definitions.[8] The problem for this interpretation is that not only Frege rejected the particular contextual definition provided by Hume's principle, but he also adopted instead a definition that is not contextual but direct. Frege's actual procedure seems to ignore the context principle, at least if this is supposed to be a general defense of contextual definitions.

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[8] Cf. Dummett (1991), pp. 125, 180, 200; also Hacker (1979), p. 222.

Other scholars have denied that the context principle is meant as a justification of contextual definitions.[9] But these scholars have failed to explain why, if this is the case, one of the few explicit appearances of the principle in the text of *GLA* is exactly at the point where Frege is elaborating a contextual definition for numbers.

The correct perspective, in my opinion, is the following. The statement of the context principle at the beginning of *GLA* § 62 suggests that a statement of the form

$$Nx{:}Fx = Nx{:}Gx \leftrightarrow F \textit{ and } G \textit{ are equinumerous}$$

is essential to explain how we apprehend numbers as logical objects. The question of whether or not this statement should count as a *definition* is, however, a *further* question. Hume's principle is indeed what Frege is aiming at, although he may recognize that this statement would not work as a definition (due, as we saw, to some residual indeterminacy). This does not imply that Hume's principle is not *the* crucial statement that should be established (possibly as a theorem?). Indeed, Hume's principle is the first important result that Frege extracts from his explicit definition of numbers as extensions. And its proof is the only place in *GLA* where Frege makes an essential (*i. e.*, non-eliminable) use of the definition. Every other fundamental result of the theory of numbers that Frege arrives at in *GLA* is derived from Hume's principle and some other auxiliary definitions. The occurrence of the context principle can be then seen as a defense of the centrality of the Hume's principle to Frege's overall project, but not necessarily of its centrality *qua* definition.

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[9] Cf. Wright (1983), p. 9.

2.4 NUMBERS AS EXTENSIONS

In *GLA* § 68 Frege takes a different line of approach, and outlines his third and final definition of number. His proposal is a direct definition. Instead of laying down the truth-conditions of sentences in which 'the number that belongs to the concept *F*' occurs, he now says directly what the term refers to: the number that belongs to the concept *F* is the extension of the concept denoted by 'equinumerous with *F*'. Since extensions of concepts are objects for Frege, this definition respects the objectual nature of numbers.

In *GLA* §§ 71-83 Frege briefly outlines how the other fundamental concepts of arithmetic are to be developed, taking numbers as extensions. In *GLA* § 72 equinumerosity between two concepts *F* and *G* is defined in terms of the existence of a 1-1 correlation between them. Also in *GLA* § 72 the concept of number is defined in the following way: *n* is a number if and only if there is a concept such that *n* is the number that belongs to that concept. In *GLA* § 73 Frege outlines the proof of Hume's principle (this time not as a definition, but as a theorem). In this proof, an implicit use is made of the criterion of identity for extensions (which will later be formulated as Axiom V of his *GGA*). In *GLA* § 74 the number 0 is defined as the number that belongs to the concept denoted by 'distinct from itself'. In *GLA* § 76 the successor-relation is defined (in the passage quoted above on p. 00). The number 1 is defined in *GLA* § 77 as the extension of the concept denoted by 'identical to 0'. Since the number that belongs to the concept denoted by 'identical to 0 but not identical to 0' is 0, it follows by the definition of the successor-relation that 1 follows 0 in the series of numbers.

It is important to stress here the feature of Frege's definitions which was mentioned earlier, that the existence of an infinity of numbers is independent of the existence of any other objects besides numbers themselves. That is to say, if there are 0 objects, then necessarily the number 0 exists. If 0 exists, then necessarily the number 1 exists. If 1 exists, then necessarily the number 2 exists, and so on. Frege's way of generating infinity may instructively be compared with Russell's, which famously needed to postulate an axiom of infinity. Frege's way of generating infinity contrasts also with the foundation that Dedekind provided for arithmetic in *Was sind und was sollen die Zahlen*. Dedekind needed a proof of the claim (Theorem 66) that there is at least one simply infinite system. The proof is given by showing that there is a 1-1 function mapping the totality S of all things that can be "objects of my thought" into S itself. This function is such that, if *s* is a possible object of my thought, then the object associated with it is the thought that *s* can be an object of my thought. There is one object ("*mein eigenes Ich*") which is in the domain but not in the image of the function. It follows that the system is infinite by Dedekind's definition of infinity. What is crucial here is that Dedekind, like Russell, needs an infinity of non-logical objects in order to obtain the infinity of numbers, while Frege does not appeal to the existence of anything besides that of logical objects themselves.

Certainly the most important result is the one that Frege enunciates in *GLA* § 79 and for which he outlines a proof in *GLA* §§ 79-83, that every finite number has a successor. Together with the claim that no finite number follows itself in the series of natural numbers beginning with 0, this implies the infinity of the number-series. The proof proceeds by showing that for every number *n*, there is a concept *F* and an object *a* such that n is the number of *Fs* that are not a. Frege remarks that this concept can be the one denoted by 'member of the series of numbers ending with n' (i.e., $x \leq n$), and the proof that this is so – the details of which I will skip here – uses the definitions of 'y follows x in the series generated by f' and 'y belongs to the series starting with x' which were already shown to be definable in purely logical terms in *BS* § 26 and *BS* § 29, respectively.

In these few sections of *GLA*, Frege outlines the basic proof of all fundamental theorems of arithmetic. In particular, each of Peano's axioms has an equivalent among Frege's fundamental theorems. The first axiom (zero is a number) follows trivially from the definition of 0. The second axiom (every number has a successor) is the result the proof of which is outlined in *GLA* §§ 79-83. The third axiom (the successor-relation is one-one) is indicated in *GLA* § 78 (theorem 5) as a conclusion that easily follows from the definitions. The fourth axiom (0 has no antecedent) is also one of the consequences listed in *GLA* § 78 (theorem 6). And the fifth axiom (induction), follows from the definition of finite number in *GLA* § 83 as all (and only) objects that belong to the series generated by the relation denoted by 'n follows immediately after m in the sequence of natural numbers beginning with 0' and from Frege's definition of 'y belongs to the series beginning with x'.

In summary: in *GLA*, numbers are a particular kind of object, viz., extensions of concepts. Frege claims that, from this basic assumption, and by means of appropriately chosen definitions, we are able to derive virtually all of arithmetic, including propositions about infinite cardinals. But no further explanation is given in *GLA* about the notion of extension itself.

3. THE INCONSISTENCY OF FREGE'S AXIOMATIC SYSTEM

Frege's full development of his logicist program appeared nine years after *GLA* in his most impressive work, the *Grundgesetze der Arithmetik* (the first volume from 1893, and the second from 1903). Here the product of the sharp philosophical analysis of *GLA* is combined with the axiomatization and formal rigor of the system of *BS* (to which some new axioms and other technical devices were added). There are initially in *GGA* two kinds of objects in the ontological basis of Frege's arithmetic: truth-values and value-ranges. The notion of value-ranges is a generalization, for arbitrary functions, of the notion of extensions of concepts. In *GGA* § 10, however, Frege establishes that truth-values are extensions;[10] it follows that there is, after *GGA* § 10, really only one kind of object, value-ranges, in the ontological basis of his arithmetic.

Two axioms are responsible for the introduction[11] of objects in Frege's formal system. One of them is Axiom IV:

$$a = b \vee a = \neg b$$

[10] Some scholars (most notably Moore and Rein) have interpreted Frege's procedure of identifying truth-values with their corresponding singleton sets as ultimately inconsistent with his realism about logical objects (Moore and Rein (1986)). However, as Tyler Burge argued (1986), Frege's procedure in *GGA* § 10 should not be understood as an arbitrary *stipulation*, but rather as a *recognition* that truth-values as objects have the corresponding property. This stems, Burge argues, from Frege's broader views on logic and truth.

[11] "Introduction" should not be understood here in a constructivist sense, since in Frege's realist perspective, we cannot create logical objects, but only recognize their existence and properties.

where *a* and *b* are any sentences. The axiom says that, for any two sentences *a* and *b*, either *a* is materially equivalent to *b*, or *a* is materially equivalent to the negation of *b* (the symbol of identity is interpreted by Frege as material equivalence when applied to sentences). Since it treats material equivalence as a particular case of identity, Axiom IV amounts to a formal recognition that the references of sentences – truth-values, according to Frege – are objects, and that there are only two of them, namely, the true and the false. The other is Frege's infamous Axiom V:

$$\grave{\varepsilon}f(\varepsilon) = \grave{\alpha}g(\alpha) = (\forall x)(f(x) = g(x))$$

where '$\grave{\varepsilon}f(\varepsilon)$' is the symbol for the value-range of the function $f(x)$. Frege's Axiom V captures and generalizes the intuitive claim that two concepts have the same extension if and only if they have the same value for all objects as arguments.

Apart from these two axioms and Axiom VI (which introduces the definite description operator), the rest of Frege's axiomatic system corresponds to ordinary second-order logic. The main definitions and proofs essentially formalize the definitions and proofs outlined in *GLA*. In *GGA* § 31 Frege tries to provide a proof that all names of his system – proper names and names of functions – have references. This proof, had it been sound, would have guaranteed the consistency of Frege's system. But, as we know, Frege's system was inconsistent, as the discovery of Russell's paradox in 1902 showed. Frege immediately acknowledged the paradox in a postscript added to *GGA* II (1903). The paradox can be derived within his system roughly in the following way: let ∇ be the class of classes that do not belong to themselves, *i. e.*,

$$\nabla = \dot{\varepsilon}\ [(\exists\Phi)(\varepsilon = \dot{\alpha}\Phi(\alpha) \wedge \neg\Phi(\varepsilon))]$$

where 'Φ' is a second-order variable. We can abbreviate the name of the function that follows '$\dot{\varepsilon}$' in the above expression by '$F(x)$', so that

$$\nabla = \dot{\varepsilon}\ F(\varepsilon).$$

From Axiom V it follows

$$(\dot{\varepsilon}f(\varepsilon) = \dot{\varepsilon}\ F(\varepsilon)) \rightarrow (f(\nabla) \leftrightarrow F(\nabla))$$

for any $f(x)$. By the definition of ∇,

$$(\dot{\varepsilon}f(\varepsilon) = \nabla) \rightarrow (f(\nabla) \leftrightarrow F(\nabla)).$$

By propositional logic,

$$F(\nabla) \rightarrow ((\dot{\varepsilon}f(\varepsilon) = \nabla) \rightarrow f(\nabla))$$

and by universal generalization

$$F(\nabla) \rightarrow (\forall\Phi)[(\dot{\varepsilon}\Phi(\varepsilon) = \nabla) \rightarrow \Phi(\nabla)].$$

Again, by predicate logic,

$$F(\nabla) \rightarrow \neg(\exists\Phi)[(\dot{\varepsilon}\ \Phi(\varepsilon) = \nabla) \wedge \neg\Phi(\nabla)]$$

and by the definition of '$F(x)$',

(I) $$F(\nabla) \rightarrow \neg F(\nabla).$$

On the other hand, by the definition of ∇,

$$\neg F(\nabla) \rightarrow (\forall \Phi)[(\nabla = \dot{\varepsilon}\, \Phi(\varepsilon)) \rightarrow \Phi(\nabla)]$$

and by universal instantiation,

$$\neg F(\nabla) \rightarrow [(\nabla = \dot{\varepsilon}\, F(\varepsilon)) \rightarrow F(\nabla)]$$

but by definition $\nabla = \varepsilon\, F(\varepsilon)$; hence

(II) $$\neg F(\nabla) \rightarrow F(\nabla)$$

I and II above together imply

$$F(\nabla) \leftrightarrow \neg F(\nabla)$$

hence the contradiction. The only ingredient in this derivation besides ordinary second-order logic is Axiom V. Therefore, Frege concludes in *GGA* II 257, it must be this axiom that is responsible for the contradiction.[12]

In the same postscript, Frege proposes a weakened version of Axiom V which could, he believes, avoid the contradiction. The modified version of Axiom V (which he calls Axiom V' in *GGA* II 262) is

$$(\dot{\varepsilon} f(\varepsilon) = \dot{\alpha} g(\alpha)) = (\forall x)((x \neq \dot{\varepsilon}\, f(\varepsilon) \wedge x \neq \dot{\alpha} g(\alpha)) \rightarrow (f(x) = g(x)))$$

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[12] Although Frege's system as a whole is inconsistent, Terence Parsons showed that the first-order fragment (*i.e.*, Axiom V plus only first-order predicate calculus) of Frege's system is consistent (Parsons (1987)). Dummett advocates the view that the main reason for the inconsistency of Frege's system is not Axiom V *per se*, but Frege's incautious way of introducing second-order quantification in *GGA* (Dummett (1991), pp. 217-219).

It says that the extension (value-range) of two concepts (functions) $f(x)$ and $g(x)$ coincide iff the value of $f(x)$ and of $g(x)$ is the same for all objects that are not the extensions (value-ranges) of $f(x)$ or $g(x)$. In the postscript, Frege shows that the Russellian paradox cannot be derived from Axiom V', but he does not try to provide anything like a proof of consistency. It turns out that the system resulting from the combination of second-order logic and Axiom V' is also inconsistent, as Quine showed many years later (1968).[13]

After 1903, Frege shows an increasing skepticism towards his own logicism and towards the notion of logical object. His skepticism culminates in 1924 with the complete rejection of the notion of class and of his logicist program altogether (*NS* 288-9, 296-7).

4. HUME'S PRINCIPLE AND NEO-FREGEANISM

In recent years, some philosophers motivated by some technical results have been seriously reconsidering Frege's original idea. It is commonly supposed that Frege introduced the notion of value-range (and with it the inconsistent Axiom V) in his system in order to solve the Julius Caesar problem. However, not only did the introduction of value-ranges *via* Axiom V not provide a solution for the problem (for it cannot say anything informative about the identity between Julius Caesar and any extension), but it also transformed Frege's system into an inconsistent one. On the other hand, although Frege rejects Hume's principle as a definition in *GLA* §§ 66-7, he still seems to regard it as fundamental to his explanation of the no-

[13] Quine (1955) refers to earlier derivations of the same paradox from Axiom V' by other logicians.

tion of number. For, as I said before, immediately after Frege establishes the definition of numbers as extensions in *GLA* § 68, the first important result that he derives from it is Hume's principle (*GLA* § 73). This is indeed the *only* place where Frege makes a fundamental (*i. e.*, ineliminable) use of his definition of numbers as extensions (and, implicitly, of Axiom V). From this point on, the outlined proofs of all relevant results depend only on Hume's principle.

It was first noticed by Charles Parsons that the axioms of Peano's arithmetic can be derived without any assumption concerning sets or extensions, but by simply taking Hume's principle as an axiom.[14] Later, following the sketches that Frege himself presents in *GLA* §§ 71-83, Crispin Wright developed, in section xix of *Frege's Conception of Numbers as Objects*, proofs for each of Peano's five axioms. Wright's proofs assume only second-order logic and Hume's principle. Moreover, Wright argued that the analogue of Russell's antinomy cannot be derived from Hume's principle ((1983), pp. 155-7), concluding that "there are grounds, if not for optimism, at least for cautious confidence that a system of the requisite sort is capable of consistent formulation" (*ibid.*, p. 155).

Some years later George Boolos (1987) showed that the theory consisting of second-order logic supplemented with Hume's principle is equiconsistent with Peano's arithmetic; that is to say, this theory is consistent if and only if Peano's second-order arithmetic is consistent. Consequently, if we abandon Frege's definition of numbers in terms of extensions and accept Hume's principle as the basic explanation of the notion of number, we no longer need Axiom V to fix the truth-conditions of identities between extensions, and hence the

[14] Parsons (1965), p. 164.

source of inconsistency is eliminated from Frege's system.[15] Or, better said, we get a system that is at least as consistent as Peano's arithmetic. (As we know, by Gödel's second incompleteness theorem the consistency of Peano's arithmetic cannot be proven in Peano's arithmetic itself if it is in fact consistent.) If extension (value-ranges), is a problematic notion, it does not follow that number must also be a problematic notion just because Frege defined it (wrongly, according to Wright) in terms of extensions.

An interesting question is whether the same reconstruction could be carried out for the proofs developed in *GGA*. In an elegant article, Richard Heck Jr. (1993) argues that the answer is affirmative. It indeed requires only a minor (from a technical perspective) modification of Frege's actual procedure. Carefully examining the development of Frege's definitions and proofs, Heck argues that:

> [...] with the exception of the use in the proof of Hume's principle itself, *all* uses of value-ranges in Frege's proof of the basic laws of arithmetic can be eliminated in a uniform manner. Moreover, with that exception, Frege uses value-ranges merely for convenience ((1993), p. 581).

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15 Boolos did not believe, however, that Hume's principle is a logical law, since it can only be true if infinitely many objects exist ((1990), p. 249). Indeed, he suggests that it is probably false, for in the same way that it proves the existence of the number corresponding to 'not identical to itself', it can probably prove the existence of the number corresponding to 'identical to itself' – *i. e.*, the number of all that there is – which is a result worthy of suspicion ((1990), p. 251). Consequently, although Boolos believes that Frege's construction can be saved, he does not regard this fact as supporting the logicist thesis.

In our case, the number 3 can legitimately be regarded as an instance of the concept of Roman emperor if the criterion of identity for numbers (equinumerosity) can plausibly be used as the criterion of identity for some Roman emperors. Since this seems to be absurd, it follows by Wright's N^d that the number 3 does not instantiate the concept of Roman emperor.

It should be noticed that the formulation of Wright's solution is sufficiently cautious to leave room for the possibility that numbers do fall under some other sortal concepts such as, *e. g.*, extension. For – and this seems to be Wright's idea – there may be some extensions for which equinumerosity could be a sufficient condition of identity.

It is not entirely clear what the status of Wright's N^d thesis above is. But there is one reason for suspicion about it: it is not clear whether the test recommended by N^d can be applied independently of our having already at our disposal a solution for the Julius Caesar problem. In other words, how can we be certain that, in the case of the Roman emperors, there are no two instances of this sortal which could involve equinumerosity as a criterion of identity? It seems to be already presupposed that we know that no numbers can be identical with Julius Caesar or any other Roman emperor. Hence, it seems, it is not clear that Wright's solution does not involve some circularity.

6. THE LOGICALITY OF HUME'S PRINCIPLE

A crucial question for a reconstruction of logicism along the lines that Wright suggests is to what extent it can still be seen as logicism. More precisely, how confident can we be that Hume's principle is a logical law? It implies, like Axiom V, the existence of objects. Indeed, it implies the existence of infinitely many objects, since the infinity of the number series is, source of inconsistency is eliminated from Frege's system.[15] Or, better said, we get a system that is at least as consistent as Peano's arithmetic. (As we know, by Gödel's second incompleteness theorem the consistency of Peano's arithmetic cannot be proven in Peano's arithmetic itself if it is in fact consistent.) If extension (value-ranges), is a problematic notion, it does not follow that number must also be a problematic notion just because Frege defined it (wrongly, according to Wright) in terms of extensions.

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Moreover, Heck identifies four theorems that Frege proves in *GGA* that are quite close to Peano's axioms (*ibid.*, 598), and also shows that Frege was aware of the fact that these theorems taken together determine a structure that is isomorphic to the natural numbers.

What I have just presented is a summary of some technical aspects of Frege's system that motivate the neo-Fregean project along the lines put forward by Wright. There are, nevertheless, some problems that the project faces, which I shall discuss in what follows.

Axiom V and Hume's principle are both abstraction principles, *i. e.*, both have the same general form

$$\eta x = \eta y \leftrightarrow \Omega(x,y)$$

where x and y are entities of a given primitive domain over which we abstract. η is an abstraction operator; it associates with each entity of the primitive domain an entity of a new kind (the abstracts). Ω is an equivalence relation defined over the entities of the basic domain. Two entities of the basic domain then have the same abstract associated with them if and only if they are in the relation Ω. η can be defined over objects (in which case the principle is a first-order abstraction principle), as is the case of the direction operator defined over lines. It can also be defined over concepts (in which case the principle is a second-order abstraction principle), as are the value-range operator and the numerical operator.

Although the two abstraction principles that we are considering are structurally similar, Axiom V is inconsistent while Hume's principle is consistent (a model for the latter can easily be constructed, as George Boolos has shown).[16] Now given

[16] Boolos (1987), p. 7.

that Axiom V is an instance of the contextual strategy for the introduction and explanation of abstract entities, the strategy motivated by the context principle, does the inconsistency of this abstraction principle compromise the whole strategy? That is to say, should we reject contextual definitions altogether because some of them are inconsistent? If the answer to this question is no, then the further question is whether we have at our disposal criteria for distinguishing, among the many abstraction principles, those that are acceptable from those that are not.

The two questions above constitute the main challenge posed by Dummett to Wright's project of reconstructing Frege's logicism by taking Hume's principle as the basic explanation of number. Given the fact that some contextual definitions are inconsistent, Dummett comments:

> In this case, we therefore have three options: to reject the context principle altogether: to maintain it, but declare that it does not vindicate the procedure Wright has in mind; and to formulate a restriction upon it that distinguishes the cardinality operator from the abstraction operator. Wright does none of these things: he maintains the context principle in full generality, understood as he interprets it, and defends the appeal to it to justify ascribing a reference to numerical terms, considered as introduced in the foregoing manner, without stopping to explain why apparently similar manner of introducing value-lange terms should have led to contradiction. He owes us such an explanation ((1991), p. 189).

One way of answering Dummett's challenge would be to establish the recursive decidability of the consistency of contextual definitions. This, however, is impossible, as Heck has shown in another paper,[17] where he proves that there is a re-

[17] Heck (1992).

cursive characterization of the consistency of contextual definitions if and only if the consistency of arbitrary second-order sentences is decidable. But the latter cannot be the case (within a consistent theory that contains Peano's arithmetic), as we know from Gödel's second incompleteness theorem.

One could argue that there is no *prima facie* reason for considering the contextual strategy as wrong in general solely because one (or possibly more) of its applications is inconsistent. In the case of Axiom V, the contextual strategy was not effectively applied, for the resulting sentence is inconsistent. But Hume's principle, which is what really interests us here, is consistent (and possibly true?). Why is this not enough?[18] I shall return to this point in section VI.

5. WRIGHT ON THE JULIUS CAESAR PROBLEM

The second fundamental question that arises for Wright's neo-Fregeanism is an obvious one for anyone who has followed Frege's argument in *GLA* §§ 66-7. Let us recall that Frege rejects Hume's principle as a definition for numbers, alleging that we would have no means of deciding, using this definition, whether or not the number 3 is identical with an object like Julius Caesar. Now apparently, if Wright wants to take Hume's principle as the definition of numbers, he has to

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[18] In "The Limits of Abstraction" (Fine (1998)), Kit Fine lays down some requirements for the acceptability of abstraction principles, such as the requirement of *non-inflationism, i. e.*, the abstraction principle should not introduce a greater number of objects than the number of objects in the domain and the requirement of *logicality*, in other words, the equivalence relation occurring in the principle should be a logical one. But Fine's criteria can hardly help here, since it is precisely the notion of logicality of an abstraction principle that is still unclear.

provide some different solution for the Julius Caesar problem. The viability of his project depends on the effectiveness of his alternative solution.

This problem is attacked by Wright in section xiv of *Frege's Conception of Numbers as Objects*. The question he tries to answer there is whether or not, by laying down the identity condition for numbers (Hume's principle), Frege has done all that is required to differentiate numbers from objects of all other kinds. Given two sortal concepts F and G, there are two distinct questions to be asked regarding the identity of the objects that instantiate them: first, given two instances, a and b, of F, we can ask in which circumstances a is identical to b. Second, there is the question of whether a or b is also an instance of G. In the present case, the second question is whether the number 3 is also an instance of the concept corresponding to 'Roman emperor'.

Now, Wright's suggestion is that the answer to the first question (which seems to be less problematic) can be exploited in order to get an answer to the second question as well. His hypothesis is that a or b can only be regarded as instances of G if the criterion of identity between instances of F can also be intelligibly employed as a criterion of identity between at least some instances of G:

> As a solution to the Caesar problem for the special case of number, I thus propose:
>
> N^d: Gx is a sortal concept under which numbers fall (if? and) only if there are, or could be, singular terms 'a' and 'b' purporting to denote instances of Gx such that the truth-conditions of '$a = b$' could adequately be explained as those of some statement to the effect that a 1-1 correlation obtains between a pair of concepts ((1983), pp. 116-7).

In our case, the number 3 can legitimately be regarded as an instance of the concept of Roman emperor if the criterion of identity for numbers (equinumerosity) can plausibly be used as the criterion of identity for some Roman emperors. Since this seems to be absurd, it follows by Wright's N^d that the number 3 does not instantiate the concept of Roman emperor.

It should be noticed that the formulation of Wright's solution is sufficiently cautious to leave room for the possibility that numbers do fall under some other sortal concepts such as, *e. g.*, extension. For – and this seems to be Wright's idea – there may be some extensions for which equinumerosity could be a sufficient condition of identity.

It is not entirely clear what the status of Wright's N^d thesis above is. But there is one reason for suspicion about it: it is not clear whether the test recommended by N^d can be applied independently of our having already at our disposal a solution for the Julius Caesar problem. In other words, how can we be certain that, in the case of the Roman emperors, there are no two instances of this sortal which could involve equinumerosity as a criterion of identity? It seems to be already presupposed that we know that no numbers can be identical with Julius Caesar or any other Roman emperor. Hence, it seems, it is not clear that Wright's solution does not involve some circularity.

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A crucial question for a reconstruction of logicism along the lines that Wright suggests is to what extent it can still be seen as logicism. More precisely, how confident can we be that Hume's principle is a logical law? It implies, like Axiom V, the existence of objects. Indeed, it implies the existence of infinitely many objects, since the infinity of the number series is, as we saw before, a consequence of Hume's principle plus second-order logic. There are two explanations to be found in Frege for why the principle is logical. First, when Frege presents the principle as a possible definition of numbers in *GLA* § 62, he says that what occurs on the right-hand side of it as explanation of the identity between the number belonging to two concepts – equinumerosity – is a purely logical notion. Second, after the principle was dropped as a definition, it was proved as a theorem. Since it follows only from the definition of numbers together with Axiom V and second-order logic, it is analytic, according to Frege's characterization of analyticity in *GLA* § 3, and it is about logical entities – hence it is logical. Obviously, after the discovery of Russell's paradox, the second explanation turned out to be inadequate, since, given the fact that Axiom V is inconsistent, everything follows from it – both logical and non-logical principles.

Wright tries to give an account of the logicality of Hume's principle in section xvii of *Frege's Conception of Numbers as Objects*, by inquiring whether the explanation of "$Nx{:}Fx = Nx{:}Gx$" contains anything but logical notions or concepts. One of the worries that Wright tries to eliminate is the one generated by the presence of second-order quantification in Hume's Law.[19] This is meant as a reply to those philosophers who, following Quine, claim that the use of higher-order quantification indicates that we are no longer in the realm of pure logic, but rather in the field of mathematics. Wright's argument against this claim relies on a broad criterion for the classification of a concept as logical:

[19] The second-order quantification is involved in the claim that two concepts *F* and *G* are equinumerous since, as we saw, this claim is, for Frege, equivalent to the claim that there is a one-one correlation between the instances of *F* and the instances of *G*.

> [...] *logical* concepts are precisely those concepts which we need to use in order to describe what we intuitively apprehend as the common form of valid inferences, which may otherwise have quite diverse subject-matters [...] ((1983), p. 134).

But this criterion is indeed so broad that it can hardly work without some substantial qualification. For the set-theoretical paradoxes have shown that sometimes "what we intuitively apprehend as the common form of valid inferences" is not valid at all (like the universal transition from a concept to its extension).

But there are further problems for the view that Hume's principle is a logical law. First, as we saw, the principle is not compatible with the existence of only finitely many objects. But it seems natural to think that a logical law should be true independently of the number of objects existing. This has been regarded by some philosophers as a decisive refutation of the claim that Hume's principle is logical.[20] Now this conclusion is admittedly not entirely justified, for Frege's project, had it been successful, would have been strong evidence that our intuitive view of logical laws as ontologically neutral is wrong, and that indeed logic is only compatible with the existence of infinitely many logical objects (although the existence of these objects is independent of the existence of any other kind of objects). But Frege can provide a natural explanation of the existence of extensions as logical objects, since the existence of these objects seems to follow[21] from the existence of concepts, while the neo-Fregean project – which take numbers to be logical objects without reducing them to extensions – owes us

[20] Cf. Boolos (1990), p. 249.

[21] Russell's paradox showed that this transition from concepts to their extensions cannot always be made.

an analogous explanation. A second problem was pointed out by Boolos ((1990), pp. 250-251). Boolos constructs an abstraction principle that is consistent, but incompatible with Hume's principle. Let us say that two concepts *F* and *G* differ evenly if the number of objects that fall under *F* but not under *G* or *vice-versa* is even. Now "the parity of" designates a function from concepts into objects governed by what Boolos calls the "Parity principle": the parity of *F* is identical with the parity of *G* if and only if *F* and *G* differ evenly. This is, like Hume's principle, a consistent abstraction principle since a model can easily be constructed. Moreover, it seems to be built using only logical notions. But, as Boolos demonstrates (*ibid*, p. 250), this principle is true in no infinite domain. Hence, both the Parity principle and Hume's principle are consistent and employ only logical notions, but they are inconsistent when taken jointly. It follows that one of them must be true and the other false, although both are "logical" by the same criterion that Wright uses to advocate the logicality of Hume's principle.

7. CLOSING REMARKS

Even leaving aside the problems faced by the neo-Fregean project that I outlined above, the main question that emerges from my discussion is why Frege did not pursue the lines of this reconstruction. As we saw, in order to avoid Russell's paradox, Frege had at his disposal the technical option of rejecting Axiom V and adopting instead Hume's principle as an axiom. It is hard to imagine that he was unaware of this possibility. Frege gives all signs that he was aware of the possibility of building his system taking other basic laws as axioms (*e. g.*, in *BS* § 13, *GGA* xxvi, *WB* 248). But, as it seems, no alternative would be tolerable for him that did not include Axiom

V as a basic law. What is missing from the current discussion concerning the neo-Fregean project is, in my view, an explanation of why Frege regards Axiom V (instead of Hume's principle) as playing such a fundamental role in arithmetic (and hence in logic). As I said before, many of the most influential Frege scholars tend to see this choice as just a convenient way of avoiding the Julius Caesar problem. I think this is mistaken for two basic reasons. First, as it seems to me, the primary role of the Julius Caesar problem in Frege's course of thought in *GLA* is not to point out the residual indeterminacy unsolved by the contextual definition of §§ 63-5, but rather to call attention for the fact that this definition fails to make evident the logical nature of numbers (*i. e.*, it fails to distinguish them from ordinary, non-logical entities). If the primary role of the Julius Caesar problem were to point out the indeterminacy, then it would be rather mysterious (as Dummett and others have pointed out) why on earth Frege thought that his definition of numbers as extensions solved the problem, for obviously it didn't, *i.e.*, there is still a problem of indeterminacy of the identity of *extensions* regarding other – ordinary – objects. Second, given the privileged status of extensions as logical objects in Frege's thought, he would have to come to the definition of numbers as extensions anyway – quite independently of the considerations about the Julius Caesar problem. For, otherwise, he would owe us an explanation of the logical status of numbers and why they are on a par with extensions without being themselves extensions. The assumption that Frege opted for a definition of numbers *just* to solve the Julius Caesar problem (and this assumption, as I mentioned before, seems to be shared by many of the most influential Frege scholars) is simply incorrect.

A corollary of my considerations in the last paragraph is that the main thing missing from the project outlined by Wright, Heck and Boolos is a correct account of Frege's very idea of a logical object. I have no space here for a detailed discussion of the relevant passages of Frege's texts,[22] but there is plenty of textual evidence that, for Frege, not only extensions of concepts were the paradigmatic case of logical objects, but also the only case of logical objects. For example, in a letter to Russell from 1902, written shortly after the discovery of Russell's paradox, Frege says that he saw no other way of apprehending numbers as logical objects except by apprehending them as extensions of concepts (*WB* 223). It follows that, in Frege's perspective, adopting Hume's principle instead of the definition of numbers as extensions (and hence Axiom V) would by no means show that numbers are logical objects since they were not reduced to objects that were recognized by logicians and mathematicians of Frege's time as being logical in nature. In other words, if Hume's principle were to replace Axiom V as the basic law of arithmetic, then the truth of Frege's logicism would depend on a blind acceptance of numbers as logical objects. But given the standards of rigor, clarity and ontological economy of Frege's philosophical enterprise, this strategy would be unacceptable.

Resumo: *No presente artigo, apresento uma reconstrução das principais etapas do desenvolvimento do Logicismo de Frege; sugiro também possíveis soluções para algumas das dificuldades interpretativas que surgiram na literatura sobre o procedimento Fregeano. Em seguida, discuto alguns dos problemas que se colocam para a moderna recon-*

[22] Although I have discussed them elsewhere (cf. Ruffino (1996)). See also Burge (1984) and (1986).

strução da aritmética Fregeana defendida por Crispin Wright (entre outros). Por fim, devo indicar aquilo que me parece estar faltando na moderna discussão sobre a rejeição por parte de Frege do Princípio de Hume como lei fundamental da aritmética.

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