

INFESTATION OR PEST CONTROL: THE INTRODUCTION OF GROUP THEORY INTO QUANTUM MECHANICS¹

OTÁVIO BUENO

*Department of Philosophy,
California State University, Fresno,
2380 East Keats Avenue,
FRESNO, CA 93740-0105
USA*

phloab@arts-01.novell.leeds.ac.uk

STEVEN FRENCH

*School of Philosophy,
University of Leeds,
LEEDS LS2 9JT
UK*

s.r.d.french@leeds.ac.uk

*The applicability of mathematics to science is a fundamental issue
in both the philosophy of mathematics and the philosophy of science.
In this paper we offer a structuralist approach to this issue which ac-*

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knowledges the often complex moves which have to be made to bring mathematical and physical structures into an appropriate relationship. Our overall representational framework is that of the partial structures approach of Newton da Costa and we present as a case study the application of group theory to atomic and nuclear physics. We conclude with some remarks on the implications of our work for realism in both the philosophy of mathematics and the philosophy of science.

INTRODUCTION

The issue of the applicability of mathematics to science is very well known but, surprisingly perhaps, it has only recently come under serious attention within the philosophy of mathematics. On the one hand, many philosophers pay lip service to this issue in the context of a formal approach to the philosophy of mathematics in general but fail to provide supporting details from actual practice (e.g. Shapiro (1997), Hellman (1989)). On the other hand, others have presented detailed case studies but in the absence of any overarching framework (e.g. Steiner (1998)). Many commentators have taken as their starting point Wigner's famous remark about the 'unreasonable effectiveness' of mathematics and have then sought to explain and hence dispel the apparent mystery. Thus Grattan-Guinness (1992) has insisted that if one pays close attention to practice the effectiveness of mathematics can, in fact, be seen to be rather more reasonable than Wigner's remark might lead us to expect. More recently, and also focusing on the details of practice, Wilson (preprint) has emphasised the unreasonable *uncooperativeness* of mathematics, noting how attempts to bring mathematics and science together have involved considerable efforts on both sides.

In what follows we shall attempt to advance the debate by presenting details from the history of twentieth century physics in the context of a particular formal framework. The case study we have chosen concerns the introduction of group theory into quan-

tum mechanics in the late 1920s and early 1930s (further details can be found in French (1999); French (forthcoming); and Bueno & French (forthcoming)). This episode is not much discussed in standard histories of quantum theory although aspects of these developments feature quite prominently in the recent work of Steiner (1998) and Peressini (1997) for example.

1. APPLYING GROUP THEORY TO ATOMS

Two intertwined programmes of application can be discerned in this history (see Mackey (1993)): Wigner's, which was concerned with the solution, or side-stepping, of dynamical problems by focusing on the underlying invariances of the situation; and Weyl's, which attempted to bring some order to the disparate models and principles of the new quantum physics by introducing group theory at the very foundations. The programme of application as a whole can be traced back to Wigner's own early work on symmetry in crystals for which he learned some group theory. But the turning point came when von Neumann gave him a reprint of the famous 1906 paper of Frobenius and Schur (see Wigner in Doncel *et. al.* (eds.) (1987), p. 633). This gave Wigner a set of powerful mathematical tools which he could then bring to bear on quantum physics, following Heisenberg and Dirac's work of 1926 and 1927 on the quantum statistics of indistinguishable particles. In particular, this work emphasised the connection between such statistics and the symmetry characteristics of the relevant states of the particle assemblies, where such symmetry characteristics arise because of the non-classical indistinguishability of the particles.

This 'permutation invariance' is the first fundamental symmetry of atomic systems and the central problem in this context concerned the effect of some (small) perturbation of the Hamiltonian of a system of indistinguishable particles, such as electrons in an atom, on the known eigenvalues of that Hamiltonian. For 3 or fewer particles this problem could be solved by elementary means, but for greater than 3 Wigner noted that the theory of

group representations as applied to the permutation group could be used to determine the splitting of the eigenvalues of the original Hamiltonian under the effect of the perturbation (Mackey (1993), pp. 242-246). Multi-dimensional representations give rise to multiple eigenvalues of the appropriate Hamiltonian which split under the effect of the perturbation. This whole application of group theory hinges on the fundamental relationship between the irreducible representations of the group and the subspaces of the Hilbert space representing the states of the system. Under the action of the permutation group the Hilbert space of the system decomposes into mutually orthogonal subspaces corresponding to the irreducible representations of this group. The most well known are the symmetric and antisymmetric representations, corresponding to Bose-Einstein and Fermi-Dirac statistics respectively, but others, corresponding to so-called 'parastatistics' are also possible, although not, it seems, exemplified in nature².

The second fundamental atomic symmetry is rotational, if inter-electronic interactions are ignored, giving our first crucial idealisation. Again group representations can be appropriately utilised to label the relevant eigenstates and here Wigner appealed to results established by Schur and Weyl who had extended the theory of group representations from finite groups to compact Lie groups. In his three classic papers of 1925 and 1926, Weyl established the complete reducibility of linear representations of semi-simple Lie algebras. In particular, this allowed the irreducible representations of the three-dimensional pure rotation (or orthogonal) group to be deduced. The history, as represented in perhaps its most apprehensible manifestation, namely the *dates* of publication of these papers, is of fundamental significance here: not only was the relevant physics under construction at this time,

² For a period in the late 1960s and early 1970s quarks were described in terms of a form of parastatistics, but this approach was superseded by an alternative, based on a new property termed 'colour', which led to the quantum chromodynamics programme (see French (1995)).

but so was the appropriate mathematics³. Both can be seen as aspects of programmes open to further development.

In another 1927 paper, Wigner presented a systematic account of the application of the mathematics of group theory to the physics of the energy levels of an atom which covered both the permutation and rotation groups. He acknowledged, however, that his model of the atom was simplified insofar as it did not take account of the newly proposed notion of 'spin'. In his three part paper co-authored with von Neumann and published the following year, spin was incorporated into the analysis using Weyl's 'double valued representations' of the rotation group (see Wigner (1931), pp. 157-170, and Judd (1993), pp. 19-21). These results were then presented in systematic fashion in Wigner's 1931 book *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra*. It is to this work that elementary particle physicists returned in the 1950s when they 'rediscovered' Lie algebras' and group-theoretical techniques in general (Mehra (1971), pp. 329-330).

Wigner's works are cited by Weyl in the latter's 1927 paper on group theory and quantum mechanics (Weyl (1927)), and several years later, in 1939, Weyl refers to Wigner's 'leadership' in this regard⁴ ((1968), p. 679). However, Weyl was careful to point out that his work takes a completely different direction from Wigner's. In Weyl's classic 1928 work, *The Theory of Groups and*

³Of course the relevant physics was already articulated mathematically, although in non-group-theoretical terms. What this gave was an assemblage of models, principles and heuristic rules (including, for example, the 'aufbauprinzip', Heisenberg's Uncertainty Principle, Pauli's Exclusion Principle and so forth) which, as Weyl subsequently noted, could be brought under a unifying mathematical framework via group theory. An alternative framework was, of course, provided by von Neumann's introduction of Hilbert spaces. These contrasting developments are examined further in Bueno and French (forthcoming).

⁴ Specifically with reference to the decomposition into irreducible invariant sub-spaces using Young's symmetry operators.

Quantum Mechanics, one can find both the ‘Wigner’ and ‘Weyl’ programmes represented⁵. Thus, with regard to the latter, Weyl’s aim was to base Heisenberg’s commutation relations on certain fundamental symmetry principles, expressed group-theoretically (see Mackey (1993), pp. 249-251). The central idea was to represent the ‘kinematical structure’ of a physical system via an irreducible Abelian group of unitary ray rotations in Hilbert space, with the real elements of the algebra of this group representing the physical quantities of the system ((1931), p. 275). Heisenberg’s formulation then follows ‘automatically’ from the requirement that the group be continuous and, in particular, the requirement of irreducibility gives the relevant pairs of canonical variables. Weyl concludes that only one irreducible representation of a two-parameter continuous Abelian group exists, namely the one which leads to Schrödinger’s equation. Thus, the fundamentals appear to simply drop out of the group-theoretic approach⁶.

However, what is particularly significant for our discussion here is Weyl’s emphasis on the intimate relation which holds between the representations of the group of all unitary transformations or the group of all homogenous linear transformations and those of the symmetric group of permutations of f things:

⁵ It is probably fruitless to speculate which of these books, Weyl’s or Wigner’s, was the more influential. On the one hand, Eckart’s important paper ‘The Application of Group Theory to the Quantum Dynamics of Monoatomic Systems’ (Eckart (1930)) relies heavily on Weyl, and the latter’s book is the only work cited by Dirac in the Introduction to his *The Principles of Quantum Mechanics*. (We’d like to thank James Ladyman for pointing this out.) On the other hand, many people found Weyl’s work difficult to penetrate and, as we have already indicated, the resurgence of group-theoretic considerations in the 1960s can be traced back to Wigner.

⁶ For a discussion of the significance of Weyl’s results and its connection with subsequent important work in group theory, see Mackey (1993), pp. 249-251 and pp. 274-275.

the substratum of a representation of the former consists of the linear manifold of all tensors of order f which satisfy certain symmetry conditions, and the symmetry properties of a tensor are expressed by linear relations between it and the tensors obtained from it by the $f!$ permutations. (Weyl (1931), p. 281.)

This correspondence between the representations is referred to as the 'bridge' leading from the character of the unitary group to that of the permutation group (Weyl (1968), pp. 286-287).

In particular the reduction of the state space of equivalent particles into irreducible subspaces noted above, 'parallels' (Weyl (1931), p. 321) the complete reduction of the total group space of the symmetric permutation group into invariant subspaces. This 'reciprocity' between the symmetric permutation group and the algebra of symmetric transformations is referred to as 'the guiding principle' (*ibid.*, p. 377) and elsewhere Weyl writes,

The theory of groups is the appropriate language for the expression of the general qualitative laws which obtain in the atomic world. In particular the reciprocity laws between the representations of the symmetry group σ_v and the unitary group Γ are the most characteristic feature of the development which I have here indicated; they have not previously come into their own in the physical literature, in spite of the fact that quantum physics leads very naturally to this relation. ((1968), p. 291.)

It is important to note that the application of group theory to quantum physics was crucially dependent on relationships internal to the former. We shall return to this 'bridge' shortly.

The upshot, then, is that the group-theoretic approach appeared to deliver an embarrassment of riches: on the Weyl side, it gave both the Heisenberg commutation relations and Schrödinger's equation; on Wigner's, it not only provided the classification of atomic line spectra, taking into account the exclu-

sion principle and spin (Ch. V)⁷, but also a formal understanding of the nature of the homopolar molecular bond ((1931), pp. 341-342) and chemical valency in general (*Ibid.*, pp. 372-377), leading Heitler to declare, now '[w]e can ... eat Chemistry with a spoon' (Gavroglu (1995), p. 54).

Of course, not everyone was so taken with this approach. Hartree, for example, was famously less than enthusiastic, although he eventually admitted, 'Is it really going to be necessary for the physicist and chemist of the future to know group theory? I am beginning to think it may be' (quoted by Gavroglu (1995), p. 56). Slater was even hailed as having 'slain the Gruppenpest' by Condon and Shortley in their 1935 work, *Theory of Atomic Spectra*. However, the linear operators of angular momenta appealed to by Slater, Condon and Shortley are simply the generators of the Lie algebra of the rotation group $SO(3)$ (Mackey (1993)). Likewise, Dirac's 1929 presentation of a non-group-theoretic rewriting of Weyl's results in terms of permutation operators also foundered:

In 1928 Dirac gave a seminar, at the end of which Weyl protested that Dirac had said he would make no use of group theory but that in fact most of his arguments were applications of group theory. Dirac replied, 'I said that I would obtain the results without *previous* knowledge of group theory!' (Coleman (1997), p. 13.)

2. APPLICATION OF GROUP THEORY TO NUCLEI (ISOSPIN)

The application of the theory of group representations in physics was not restricted to atomic spectra and chemical valency, of course (see Mackey (1993), pp. 254-278). In particular, in his well known 1937 paper, Wigner applied group theory to nuclear

⁷ Referring to developments in spectroscopy, Weyl writes 'The theory of groups offers the appropriate mathematical tool for the description of the order thus won.' ((1931), p. 245.)

spectroscopy, following, again, the work of Heisenberg who had earlier introduced a new internal symmetry in terms of which neutrons and protons could be regarded as two different states of a single particle, the nucleon (see Miller (1987), pp. 316-320). This then provided Wigner with the opportunity to construct an analogy between atomic and nuclear structure by making certain idealisations which allowed him to treat these 'nucleons' in a manner similar to that of electrons.

The first pair of idealisations were empirically based, namely the approximately equality of the forces between protons and neutrons and the approximate equality of their masses. Ignoring the electrostatic repulsion between the protons, it is these twin idealisations which ground the treatment of these particles as different states of the nucleon. On this basis, the nucleus can be regarded as an assembly of indistinguishable particles and by analogy with the atom, a description in terms of symmetry groups can be given. In particular, Heisenberg's internal symmetry was recast by Wigner in terms of 'isotopic spin', drawing on the analogy with electron spin, which, as we have noted, was dealt with group-theoretically by Wigner himself with von Neumann. The decomposition of the Hilbert space for a nucleon into proton and neutron subspaces is then analogous to the decomposition of the two dimensional Hilbert space for the spin of an electron into a direct sum of two one-dimensional subspaces corresponding to spin 'up' and 'down' respectively⁸.

However, it is important to note that this analogy is only partial (see Mackey (1993), p. 259). First of all, in the case of electron spin, the above decomposition depends on choosing an axis, but not in the case of the proton/neutron decomposition. Secondly, and more fundamentally, in addition to isospin, nucleons themselves possess spin (Wigner *op. cit.*, p. 107). Hence, whereas the representations of the rotation group is irreducible in the elec-

⁸ The relevant groups have isomorphic Lie algebras.

tron case, it is the direct sum of two equivalent irreducible representations in that of the nucleons. The appropriate representations were therefore those of the four-dimensional unitary group, giving instead of multiplets, as in the atomic case, nuclei 'super-multiplets'. We see, therefore, that the introduction of isospin, on the physics side, requires, on the mathematical side, the use of an appropriate symmetry group which is more complicated than in the atomic case (Mackey (1993), p. 259).

On the basis of this pioneering work, in 1938, Kemmer predicted the existence of the pion triplet which was subsequently discovered (nine years later). More significantly perhaps, as is well known, it was the attempt to combine the $SU(2)$ group of isospin and the $U(1)$ group of strangeness which led Gell-Mann and Ne'eman, independently, to propose $SU(3)$ as the fundamental group of the quark model. Within this model, in which $SU(3)$ was viewed as the symmetry of three quarks, two of them (the 'up' and 'down' quarks) generate the isospin symmetry, and the latter was reduced to colour symmetry.

Of course, there is a great deal of further history to discuss and the above is only a summary of the more important details, but the following points are worth noting:

- 1) both the physics and the mathematics were developing at the time, in an open-ended manner. And both Wigner and, in particular, Weyl were well placed to span the two fields, drawing on recently obtained results from one to illuminate the equally recently obtained results of the other;
- 2) the relationship between these two fields depends on relationships 'internal' to the mathematical: in particular the reciprocity 'laws' between the representations of the symmetry group and the unitary group, referred to by Weyl, as we have seen, as the fundamental 'bridge' and 'guiding principle';
- 3) not all of the mathematics of group theory is brought to bear on the physics; there is a significant 'surplus' on the mathematical side (cf. Redhead (1975));

- 4) Wigner's development of isospin can be seen as an *extension* by analogy with the atomic case;
- 5) that analogy rode on the back of certain crucial idealisations; and
- 6) the analogy was significantly incomplete, or partial.

An account of the applicability of group theory to physics that has any pretensions as to adequacy with regard to the actual practice of science is going to have to accommodate these points.

3. THREE ISSUES

We shall now consider three important issues concerning the relationship between mathematics and science, in the light of the above history.

A. Representation

We insist that any attempt to represent the relationship between mathematics and science must accommodate the points we have noted above. Elsewhere we have argued that such an appropriate representation can be found in the semantic or model-theoretic approach in philosophy of science, which represents theories in terms of families of mathematical models. This approach can be extended to mathematical theories also (Bueno (1999a) and (1999c)), and thus provides a means for capturing the relationship between the mathematical and scientific. Clearly, this relationship will be represented as one which holds between *structures*.⁹

⁹ It is an important issue, of course, to characterise the nature of these structures. For our purposes here, we shall assume that they are *mathematical* structures, formulated in a convenient formal setting (such as that provided by set theory). This is clearly a controversial assumption, especially for those who are sympathetic to nominalism in mathematics. (For some steps towards meeting nominalist demands in the context of the partial structures approach, see Bueno (1999a).)

Furthermore, we have argued that the open-ended nature of developments in both science and mathematics can be accommodated by introducing da Costa's partial structures (for further details, see Mikenberg, da Costa & Chuaqui (1986); da Costa (1986); da Costa & French (1989), (1990) and (1993); Bueno (1997) and (1999b)). A (first-order) partial structure can be represented as follows:

$$\mathbf{A} = \langle A, R_i, P \rangle_{i \in I}$$

where A is the set of individuals of the domain under consideration; R_i , $i \in I$, is a family of partial relations defined on A , and P is a set of accepted sentences about the structure \mathbf{A} (P may be empty)¹⁰. If A is a non-empty set, then an n -place *partial relation* R over A is a triple $\langle R_1, R_2, R_3 \rangle$, where R_1 , R_2 , and R_3 are mutually disjoint sets, with $R_1 \cup R_2 \cup R_3 = A^n$, and such that: R_1 is the set of n -tuples that (we know that) belong to R , R_2 is the set of n -tuples that (we know that) do not belong to R , and R_3 is the set of n -tuples for which it is not known whether they belong or not to R . A partial structure \mathbf{A} can then be extended into a total structure via so-called \mathbf{A} -normal structures, where the structure $\mathbf{B} = \langle A', R'_i, P \rangle_{i \in I}$ is said to be an \mathbf{A} -normal structure if (i) $A = A'$, (ii) every constant of the language in question is interpreted by the same object both in \mathbf{A} and in \mathbf{B} , and (iii) R'_i extends the corresponding relation R_i , in the sense that each R'_i (supposed of arity n) is defined for every n -tuples of objects of A' (Mikenberg, da Costa and Chuaqui *op. cit.*). A sentence S is then said to be quasi-true (in \mathbf{A} according to \mathbf{B}) if: (1) \mathbf{A} is a partial structure; (2) \mathbf{B} is an \mathbf{A} -normal structure; and (3) S is true in \mathbf{B} in conformity with Tarski's definition of truth (for

¹⁰ Depending on the interpretation of science which is adopted, different sentences are introduced in the set P . Realists will typically include laws and observation statements, whereas empiricists will focus mostly on the latter.

further details, see da Costa & French (1989), (1990), (1993) and Bueno (1997) and (1999b)).

The openness of both scientific and mathematical theories to further developments is expressed in terms of those members of the R_i for which it is not known whether they hold in the given domain or not (da Costa & French (1990); French (1997); Bueno (1997) and (1999b)). This provides a unitary account of the various sorts of models used in the sciences 'iconic', 'analogue', 'theoretical' or whatever (French & Ladyman (1999); da Costa & French (forthcoming)). Furthermore, analogy can be straightforwardly accommodated: Hesse's famous reductive analysis into 'positive', 'negative' and 'neutral' analogies maps nicely onto the R_1 , R_2 and R_3 of the partial structure representation, for example (Hesse (1963)). This can then be extended to cover the above analogy between the atomic and nucleic cases which, as we have indicated, was critically dependent on certain empirically driven idealisations.

Let us consider, first of all, how these idealising moves can be represented. We have seen that the applicability of group theory to nuclear physics depended on the nucleus being regarded as an assembly of indistinguishable particles. How might we represent the relationship between this idealised model and the 'non-idealised' form? In terms of the partial structures approach, what we have is a relationship between parts of the structures in each case: putting it rather crudely, certain of the R_i which, strictly speaking, are different for protons and neutrons, are set the same. Elsewhere, this relationship has been represented in terms of 'partial isomorphisms' between the relevant structures (Bueno (1997); French & Ladyman (1999)). Formally, if we have two partial structures,

$$A = \langle A, R_i \rangle_{i \in I}$$

and

$$A' = \langle A', R'_i \rangle_{i \in I}$$

(where R_i and R'_i are (binary) partial relations as above, so that $R_i = \langle R_{i1}, R_{i2}, R_{i3} \rangle$ and $R'_i = \langle R'_{i1}, R'_{i2}, R'_{i3} \rangle$), then a (partial) function f from A to A' is a partial isomorphism between \mathbf{A} and \mathbf{A}' if (1) f is bijective, and (2) for all x and y in A , $R_{i1}xy$ iff $R'_{i1}f(x)f(y)$, and $R_{i2}xy$ iff $R'_{i2}f(x)f(y)$. If $R_{i3} = R'_{i3} = \emptyset$, so that we no longer have partial structures but 'total' ones, then we recover the standard notion of isomorphism.

This then gives us the basis for representing the relationship between mathematical and scientific theories, viewed structurally. In applying a mathematical theory to physics, we are often 'bringing in' structure from the mathematical level to the physical. What the existence of a partial isomorphism establishes is that *some* structure is carried over in this process; namely, that involving the R_{i1} and R_{i2} components, for which enough information is available. However, there might not be a complete structural preservation when we move from one level to the other. As noted above, typically mathematics bring a 'surplus structure' with respect to physics. But a partial isomorphism, as defined above, requires a *bijection* between the domains of the structures under consideration. So only structures of the *same cardinality* can be partially isomorphic. As a result, it is difficult to accommodate the existence of *structural surplus* in terms of partial isomorphisms: a broader notion is required.

To motivate this, let us recall the three points noted above: that both quantum physics and group theory were developing in an *open-ended* manner during the late 1920's; that the applicability of group theory to quantum theory depended on certain relationships *within the mathematics itself*; and, finally, that *not all* of the mathematics is brought down to the physical level, as it were. Our claim is that the appropriate way of accommodating these three points is to modify the above account by introducing 'partial homomorphisms'. Let

$$\mathbf{A} = \langle A, g, R_i \rangle_{i \in I}$$

and

$$\mathbf{A}' = \langle A', g', R_i \rangle_{i \in I}$$

be partial structures, where each R_i and R'_i is a (binary) partial relation, of the form $\langle R_1, R_2, R_3 \rangle$ and $\langle R'_1, R'_2, R'_3 \rangle$ respectively, and g and g' are partial functions. We say that $f: A \rightarrow A'$ is a *partial homomorphism* from \mathbf{A} to \mathbf{A}' if for every x and every y in A where g is defined,

- (i) $R_1 xy \rightarrow R'_1 f(x)f(y)$;
- (ii) $R_2 xy \rightarrow R'_2 f(x)f(y)$;
- (iii) $f(g(x,y)) = g'(f(x), f(y))$

(adapted from Bueno, French & Ladyman (forthcoming)).

Having indicated already how idealisations can be represented, we can now accommodate the first three of our points above. Firstly, the open-ended nature of the development of both group theory and quantum mechanics can be represented by the use of *partial* structures. Both in physics and in mathematics, in a given instant of time, a number of issues are known to be the case, others are known not to be, and a huge amount is simply not known at all. A partial structure represents this situation, indicating, in particular, the accepted information in that context via its R_1 and R_2 components whereas the R_3 component points out where further research must be done. In terms of this latter component, the open-ended nature of quantum mechanics and group theory can then be accommodated.

Secondly, a partial homomorphism is a particular *mathematical* relationship between structures: it is a map that 'transfers' some relations from a structure into another. As we saw, the application of group theory to quantum mechanics crucially depended on the reciprocity relation between the representations of the

symmetry group and the unitary group. This reciprocity provides a mathematical relation between such structures, one which allows us to 'transfer structure' from one domain into another. Partial homomorphisms capture this kind of transfer.

Thirdly, and crucially, it's not *all* of group theory that is used in quantum mechanics. Some parts of the theory are crucial, but some are not.¹¹ And it is because of this surplus provided by the mathematics with respect to the physics that 'partial homomorphisms' become decisive. Intuitively, there is 'more structure' at the mathematical level than at the physical. It's no wonder then that structures are 'mapped' from the former to the latter.

The representation of aspects of the physical world by mathematical structures is, of course, a particular case of the general issue of representation in science. Two questions immediately come to mind: How should the notion of representation be understood? Is there anything special about the use of *mathematical structures* in the representation of the phenomena? Without any claim to completeness, we shall briefly address these issues, since they will help us to systematise a few points.

In a recent work, van Fraassen has examined the issue of representation in an interesting way (van Fraassen (forthcoming b)). In his view, whatever falls under the rubric of representation should have four components: an *object* (which is represented), a *language* or a *medium* (through which the representation is achieved), a *manner* (in which the language is used in the repre-

¹¹ It is tempting to speculate *why* some parts of group theory happen to be more crucial than others in quantum mechanics. Part of the answer arises from the structure of quantum mechanics itself, and the relations it imposes on the objects studied. As we saw, symmetry considerations are crucial for the representation of quantum particles. This is, of course, one of the most crucial features of group theory. The results of the latter that *don't* depend on symmetry considerations *might* not be so crucial. (We insist on the 'might', since such results may turn out to be important for other reasons.)

sentation), and an *aim* or *function* (of the representation).¹² Of course, this is not a definition of representation, but it indicates some items that should be considered in any discussion of this concept.

In particular, regarding the use of mathematical structures in the representation of phenomena, the following account emerges: What is represented is the physical world (*object*), through the employment of a mathematical language; in particular, of mathematical structures (*medium*). These structures are formulated in a convenient formal setting, that is, using an appropriate logic and a given set theory (*manner*). And the *aim* of representing the phenomena in this way is to better formulate and solve empirical problems. (This highlights the heuristic fruitfulness of the use of mathematical structures; see French (1997) and Bueno (1999c).)

To be more specific, with regard to the use of group theory in quantum mechanics, the *object* of the representation are aspects of the subatomic world (e.g. the study of the quantum statistics of indistinguishable particles). The *medium* is provided, of course, by a mathematical language in particular, by the use of group-theoretic structures. The relevant structures are those which concern symmetry features, and they are formulated in a way that highlights this aspect (*manner*).¹³ And the group-theoretic representation is formulated in order to allow the solution of new empirical problems (*aim*). In Wigner's case, one of these problems was accounting for the physics of the energy levels of an atom (covering both the permutation and rotation groups).

Thus, in terms of van Fraassen's account, we can say that (parts of) group theory provide a *representation* of the subatomic

¹² Van Fraassen's account was motivated by an analysis of the Aristotelian notion of tragedy (for details, see van Fraassen (forthcoming b)).

¹³ As we saw, the decisive feature of the *use* of group theory in quantum mechanics arises from the representation of symmetry that it allows.

world. And the use of partial structures allows us to accommodate the open-ended nature of this representation, its (intrinsic) mathematical character, and the surplus structure provided by the introduction of group-theoretic structures.

Moreover, with regard to the isospin case, the partial structures approach also provides an interesting account. As noted at the end of section 2, (i) the introduction of isospin was made via an *extension* by analogy with the atomic case; (ii) that analogy depended on decisive idealisations; and (iii) the analogy was significantly incomplete, or partial. With regard to (i), we can represent the extension provided by Wigner with the notion of isospin in terms of the formulation of a convenient *A-normal structure*, which *extends* the (partial) information about the atom into the nucleon itself. (ii) The idealisations involved in this process, as noted above, can be straightforwardly accommodated in terms of a partial homomorphism between the relevant structures. The R_1 and R_2 relations that are known to hold (such as the decomposition of the Hilbert space for a nucleon into proton and neutron subspaces and the decomposition of the two dimensional Hilbert space for the spin of an electron into a direct sum of two one-dimensional subspaces) are carried over by the partial homomorphism. Finally, (iii) that the analogy between atomic and nuclear structure is *partial* is not surprising, for the relationship between the relevant structures is similarly partial; only a *partial* homomorphism holds between them. After all, as we noted, whereas the above decomposition for protons and neutrons doesn't depend on choosing an axis, it does in the case of electron spin. In this way, the nature of the argument used by Wigner in the introduction of the isospin (a *partial analogy* between atomic and nuclear structure) can also be accommodated.

We insist that a proper account of the relationship between mathematics and science can only proceed once this issue of representation has been broached. In what follows we shall consider a well-known discussion of this relationship which also draws on the

group-theoretic examples sketched above but which, through historical misrepresentation, crucially fails to account for precisely what is important about these examples.

B. Applicability

In a series of works, leading up to his recent book, *The Applicability of Mathematics as a Philosophical Problem*, Steiner understands Wigner's point about the 'Unreasonable Effectiveness of Mathematics' in terms of the unreasonable effectiveness of mathematics in *scientific discovery* (Steiner (1998)). Here he sets up his central mystery, giving the example of the move from spin to isospin and isospin to $SU(3)$ as one of his examples of what he calls a 'formal argument' from analogy:

Suppose we have effected a successful classification of a family of 'objects' on the basis of a mathematical structure S . Then we project that this structure, or some related mathematical structure T , should be useful in classifying other families of objects, even if (a) structure S is not known to be equivalent to any physical property, and (b) the relationship between structures S and T is not known to be reducible to a physical relation. ((1989), p. 460; (1998), p. 84)

But, 'such formal analogies appear to be irrelevant analogies and irrelevant analogies should not work at all' (1989, p. 454). So, the success of such analogies is puzzling and 'unreasonable'.

What is meant here becomes apparent through the example of the introduction of isospin ((1998), pp. 86-88). Thus Steiner claims that Heisenberg 'conjectured boldly' that the proton and neutron could be regarded as two states of the same particle and reasoned that the nucleus was invariant under the $SU(2)$ group. Therefore, there had to exist a new conserved quantity, *mathematically* analogous to spin, which came to be known as isospin. The emphasis here is crucial: it is the mathematics, Steiner claims, which does all the work in the analogy:

physicists see no *physical* analogy between [...] 'spin' and 'isospin' and therefore have no explanation for the success of Heisenberg's reasoning. ((1998), p. 90.)

Steiner calls this form of analogy 'Pythagorean', in the sense that it is mathematical and not paraphrasable into non-mathematical language. And, in precisely this sense, he goes on to insist, it is 'anthropomorphic' rather than 'naturalist', in that it depends on certain aesthetic qualities (which we select when we construct mathematical structures).

Setting aside Steiner's reconstruction of the relevant history for the moment, one's first response might be that the success of such analogies is not puzzling or unreasonable, since group theory is grounded on symmetry, regarded as an empirical notion, and thus its use in discovery is empirically justifiable (Steiner (1989), pp. 464-466). However, Steiner counters, not all of the groups used in modern elementary particle physics express empirical or perceptual or geometrical symmetries (Steiner (1989), pp. 464-466; and (1998), pp. 150-151). Thus the grounding is not straightforwardly empirical.

The second response might be to accept this point and, indeed, to take it further by noting that there was a fundamental non-perceptual and non-geometric symmetry present at the very beginning of the introduction of group theory into quantum physics, namely the permutation symmetry, which is a result of the non-classical indistinguishability of the particles. Hence, any mysterious unreasonableness cannot be a function of the move from isospin to unitary spin. What our story above illustrates is that the introduction of isospin was dependent on the empirical 'facts' of the approximate equality of forces and masses for protons and neutrons, and the resulting idealisations and (partial) analogies. That group theory worked is no surprise or mystery at all given the

fundamental idealisation of regarding the nucleus in terms of an assembly of indistinguishable particles¹⁴.

Here again Steiner has a possible counter:

This explanation removes one mystery [that of a particular mathematical concept describing a particular phenomenon] but leaves another. Mathematicians, not physicists, developed the $SU(3)$ concept, for reasons unconnected to particle physics. They were attempting to classify continuous groups, for their own sake. ((1998), p. 152.)

Thus Wigner's issue is taken to be an epistemological one: 'how does the mathematician, closer to the artist than to the explorer, by turning away from nature, arrive at its most appropriate descriptions?' (*Ibid.*, p. 154).

But now we can hold back our historical considerations no longer. Steiner's last remark surely misposes the issue: the mathematician does not 'arrive' at a description of nature. Recall the story above: A variety of theoretical moves were made and at the highest theoretical (deepest metaphysical) level, electrons came to be regarded as (non-classically) indistinguishable - very much like mathematical points in fact! This invites a connection (represented by a homomorphism) with the permutation group and the whole family of structures making up group theory. Further developments - idealisations in particular - at the theoretical level, elaborated on the basis of the analogies then permitted, allow us to draw on further members of this family. Steiner is simply wrong: it is not the mathematics which does all the work in these analogies; rather it is the combination of empirical 'facts' and idealising moves leading to indistinguishability. If we pay attention to the historical details, the explanation of Heisenberg's

¹⁴ And at this point it is worth recalling that in the context of $SU(3)$ spontaneous symmetry breaking must be introduced to get the difference in masses.

and Wigner's reasoning is readily apparent, as we have indicated above with the use of partial structures¹⁵.

C. Realisms

Before closing the paper, let us point out some implications of the arguments just discussed, about the use of group theory in quantum mechanics, to the debates about realism both in the philosophy of science and of mathematics.

(i) *Implications for realism in philosophy of science*

The current focus of attention in the realism/anti-realism debate, on the realist side at least, is on what has come to be known as 'structural realism'. This has a noble pedigree in being traced back to the work of Russell, Cassirer, Duhem and Poincaré. In its modern incarnation, it appears to offer a response to the anti-realist's pessimistic meta-induction by shifting perspective from the fluctuating ontologies of scientific theories, to their underlying mathematical structures (Worrall (1989)). Thus, although the ontological aspect of light has changed dramatically, from that of a wave in some sort of aether to that of an oscillating electromagnetic field, its structural relationships are preserved as we move from Fresnel's equations to Maxwell's.

When we come to quantum physics, group theory presents structures which describe some of the most fundamental features of particles, such as their division into the most basic natural kinds of bosons and fermions as represented by symmetrical and anti-symmetrical wave-functions. From the perspective of structural realism, it is these structures we should be realists about. Weyl himself wrote that,

¹⁵ For further discussion of Steiner's approach, see Kattau (forthcoming).

F. KLEIN considered the group concept as most characteristic of nineteenth century mathematics. Until the present, its most important application to natural science lay in the description of the symmetry of crystals, but it has recently been recognised that group theory is of fundamental importance for quantum physics; it here reveals the essential features which are not contingent on a special form of the dynamical laws nor on special assumptions concerning the forces involved. We may well expect that it is just this part of quantum physics which is most certain of a lasting place. (Weyl (1931), p. xxi.)

Note the claim that group theory reveals the 'essential features' which are not contingent on the particular forms of dynamical laws or forces. It reveals the most fundamental structures.

This form of structural realism has been called 'epistemic', since the structures are what we can *know*, and there remains a residual ontology of objects whose ultimate nature is beyond our reach, as it were¹⁶. More recently, Ladyman has advocated an 'ontic' version which drops the ontology of objects altogether and suggests a realist metaphysics of structures (Ladyman (1998); for anti-realist responses, see Bueno (forthcoming) and van Fraassen (forthcoming a)). This form of structural realism draws, in part, on recent discussions of the metaphysics of quantum objects, which have concluded that the physics can support two alternative packages: one in which the objects are individuals and one in which they are not (French & Redhead (1988); French (1989); van Fraassen (1991)). In the face of such metaphysical underdetermination regarding the ontology of objects, the ontic structural realist urges us to focus entirely on the structures themselves, as represented by group theory.

It is through group theory that we may retain objectivity even in the absence of objects, as the theory describes the most

¹⁶ We are grateful to Mary Domski for uncovering the neo-Kantian origins of this view.

fundamental characteristics of particles as represented by sets of invariants. This gives rise to a structural conception of objects which effectively renders such objects more mathematical than physical (French & Ladyman (forthcoming); see also Castellani (1998)). As Heitler wrote,

The individual atom cannot even be pictured in space and time. Even a bare description of it demands profound mathematical concepts. So it can hardly be thought of as something of a purely material nature: its 'mathematical aspect', that is, its non-material aspect, is even more strikingly prominent than is the case with an object of classical physics. (Heitler (1963), p. 53.)

(ii) *Implications for realism in philosophy of mathematics*

Realism about mathematical objects has typically appealed to (a form of) the 'Indispensability Argument' which, put crudely, runs as follows: The use of mathematical concepts and theories is indispensable for the formulation of scientific theories. So, if the latter theories are true, the indispensable mathematical theories used in their formulation should be true as well. After all, scientific theories quantify over *both* physical and mathematical objects, and there is no way of even formulating such theories without referring to the latter objects. In other words, just as we have good reason (according to the realist) for believing in unobservable physical objects, such as electrons, so we have good reason for believing in abstract mathematical ones. However, Peressini has recently pointed out that issues concerning applicability generate complications for such indispensability accounts, drawing, again, on the example of group theory (Peressini (1997)).

He begins by making a clear distinction between what he calls 'pure' mathematics and its 'formal analog' in a physical theory:

What is the relationship between pure group theory and its 'formal analog' at work in the physical theory? It is clear

enough that pure mathematical group theory is not the *same* theory as the analog present in the theory of the spin of quantum particles, for they are about different things. The groups/members that appear in quantum theory are taken to be specific groups/members that are interpreted as the physical properties (spin) of physical objects (particles). The propositions of pure group theory, on the other hand, lack any such physical interpretation. This physical interpretation, which is at the heart of the difference between pure theory and physical application, is far from trivial as a look at the details of mathematical application will reveal. (Peressini (1997), p. 213.)

Furthermore, he insists, the physical application of pure group theory requires 'empirical bridge principles' to underwrite the physical interpretation:

These principles are what distinguish pure mathematics from mathematized physical theory and enable claims about the physical world to be deduced from the latter. (Peressini (1997), p. 214.)¹⁷

Clearly, Peressini's picture is very different from the model-theoretic one presented above which effectively blurs the pure/applied distinction. However 'bridge principles' ignore the location of the 'object vocabulary', in this case at a very high level in the hierarchy of models, idealisations etc., and presuppose that, on one side of the bridge, we have a clear grasp of the physical object, when in fact, we may not. In particular, they presuppose a mathematically independent grasp but, as Heitler put it, the objects of quantum physics may be more mathematical than material. If the fundamental basis of what grasp we have of the fermionic nature of electrons lies with group theory, how can we

¹⁷ In linking pure mathematical vocabulary to the physical object/property vocabulary these bridge principles supply a semantics (one applicability problem according to Steiner (1998), who adopts a Fregean approach).

separate the 'pure' mathematical vocabulary from the physical object vocabulary? It is not clear in this case that the physics and the mathematics are indeed 'about different things'.

Nevertheless, Peressini makes a useful point, namely that indispensability arguments typically assume a form of confirmational holism which encourages 'global' realism about the objects referred to by the theory. However, on the science side, Glymour (1980) has long urged resistance to such holism, arguing that his 'bootstrap' theory of confirmation allows us to adopt a piecemeal approach. For the realist, this has the obvious advantage of selectivity in what has to be accepted into her ontology. For the anti-realist, this selectivity is similarly welcome, since it allows her to avoid ontological commitment to (at least some) unwanted entities.

On the mathematical side, in the kinds of applications that we have sketched above, typically only some of the mathematical structure is 'carried over'; thus, not all will be confirmed. Furthermore, as we have emphasised, the applicability of mathematics in certain cases may depend crucially on appropriate idealisations being made on the physical side. Any attempt to run a form of indispensability argument in these cases is first going to have to come up with an account of confirmation which can accommodate such idealising moves. In effect, the question has to be answered: why should we be realists about those mathematical objects or structures which feature in idealised models of the physical phenomena? Given that scientific realists are typically cautious about the existence of idealised entities, such as mass points, rigid rods and, indeed, nucleons, so the mathematical realist should adopt a similar attitude. And for obvious reasons, these considerations are also welcomed by the mathematical anti-realist.

4. CONCLUSION

What we have given here is an example of what Wilson (preprint) calls 'mathematical opportunism' at work: the mathe-

matics and physics had to be wrestled into agreement. And, as we have emphasised, the process is both dynamic, on both sides of the applicability divide, and involves crucial idealisations - aspects of the applicability problem which have not been given due attention in the literature. By paying attention to the actual details of this process within an appropriate representational framework, we can dispel the air of mystery about applicability and understand how the effectiveness of mathematics is not so unreasonable after all.

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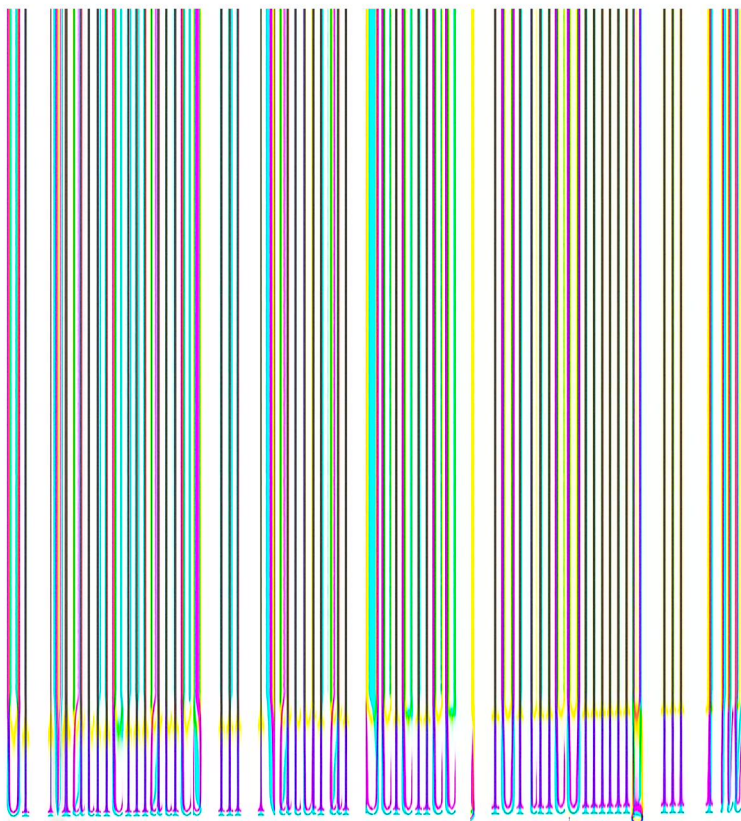
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