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A CATEGORIAL APPROACH TO THE COMBINATION OF LOGICS[1]

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In this paper we propose a very general definition of combination of logics by means of the concept of sheaves of logics. We first discuss some properties of this general definition and list some problems, as well as connections to related work. As applications of our abstract setting, we show that the notion of possible-translations semantics, *introduced in previous papers by the first author, can be described in categorial terms. Possible-translations semantics constitute illustrative cases, since they provide a new semantical account for abstract logical systems, particularly for many-valued and paraconsistent logics.*

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1 SEMANTICS FOR ABSTRACT LOGICAL SYSTEMS: LOCAL AND GLOBAL LOGICS

The concept of *possible-translations semantics* (also called *non-deterministic semantics*) was introduced and discussed in (Carnielli (1990)) and (Carnielli (forthcoming)) as a new semantic approach to general logical systems, based on the idea of defining new forcing relations combining simple semantics by means of translations. Although several ideas on combinations of logics can be found in the literature, as described for example in (Blackburn & Rijke (1997)) and (Caleiro, Sernadas & Sernadas (manuscript)), this approach offers a different perspective to the question, which leads to new semantics for general logics, including several many-valued, paracomplete and paraconsistent logics. A special form of possible-translations semantics called *society semantics*, which is particularly apt for many-valued logics, has been presented in (Carnielli & Lima-Marques (1999)).

The idea behind possible-translations semantics is to encompass two or more basic semantic models (of the same similarity type) in such a way as to define a new logic which depends upon the basic ones by means of a collection of translations. The basic models can be distinct copies of classical models, or distinct many-valued models, or even Kripke models (for intuitionistic or modal logics). In this sense, the basic models can be regarded as *local logics*, while the more complex logic defined upon the basic ones via translations can be seen as a *global logic*.

In the present paper we intend to give a more abstract account of possible translations semantics, considering the basic models as organized through sheaf structures. Sheaves have been used in mathematics as a powerful tool for investigating relationships between local and global phenomena, and seem to be a very apt framework for generalizing the idea of possible translations.

In the case where sheaf theory is viewed in the abstract setting of category theory we can profit from the rich expressivity of categories, which includes diagrams, limits and co-limits, and which permits interesting interpretations of operations on sheaves. A similar account has been offered by J. Goguen in (Goguen (1992)) where he proposes a theoretical framework for concurrency in object-oriented programming systems. The idea of using categorial methods for modeling concurrent processes has also been proposed by L. Monteiro and F. Pereira in (Monteiro & Pereira (1986)), and the basic intuitions explaining the connections between object-oriented programming systems and sheaves go as follows:

1) objects can be viewed as sheaves,
2) inheritance relations between objects can be viewed as sheaf morphisms,
3) the systems correspond to diagrams of sheaves.

One of the most interesting consequences of this approach is that the limits of such diagrams correspond to the behavior of systems, modeling the dynamics of the underlying process.

We intend to show that an analogous treatment can be further generalized to the problem of providing semantics for abstract logical systems. Though we are more concerned with definitions and basic examples than with deep results, we hope to be able to motivate and illustrate the possibilities of the idea of combining logics by means of translations in the case where the set of translations are endowed with general sheaf structures.

In section 2 we review some generalities on abstract logical systems and the translations between them, and in section 3 we recall the main points on possible-translation semantics. Section 4 presents the main ideas of our approach, and is devoted to treating logics as presheaves and sheaves. Sections 5, 6 and 7 carefully explain several examples and their significance for our theory, and in section 8 we list some problems and related questions.

2 GENERALITIES ON LOGICAL SYSTEMS AND TRANSLATIONS

We recall here a general definition of logic, which has been used in previous articles (cf. (Carnielli & D'Ottaviano (1997))). A *consequence* (*closure*) operator over a set A is any function $C_A : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ satisfying the following conditions, for $X, Y \subseteq A$:

(i) $X \subseteq C_A(X)$

(ii) If $X \subseteq Y$, then $C_A(X) \subseteq C_A(Y)$

(iii) $C_A(C_A(X)) \subseteq C_A(X)$.

A set is *closed* (or is a *theory*) if $C_A(X) = X$. It follows trivially from these clauses that, for every $X \subseteq A$, $C_A(C_A(X)) = C_A(X)$.

A (monotonic) *logic* is a pair $\mathcal{L} = \langle A, C_A \rangle$ where A is a non-empty set, called the *domain* or the *universe* of $\mathcal{L}$, and C_A is a consequence operation. When clause (ii) does not apply, the resulting logic is *non-monotonic*. We restrict our treatment to monotonic logics.

The general notion of *translation between logical systems* regards translations as functions preserving consequence relations. Given two logics $\mathcal{L} = \langle A, C_A \rangle$ and $\mathcal{L}' = \langle B, C_B \rangle$ a *translation* from the logic $\mathcal{L}$ into the logic $\mathcal{L}'$ is a map $f : A \longrightarrow B$ such that, for any $X \subseteq A$,

$$f(C_A(X)) \subseteq C_B(f(X)).$$

We call the logic $\mathcal{L}$ the *source* of the translation, and $\mathcal{L}'$ the *target*. The map f is also called a *consequence-continuous* map because, if we regard closed sets as analogue to topologically closed sets, then f can be seen as the analogue of closed functions.

If f is a translation then, for any $a \in C_A(X)$, one has that $f(a) \in C_B(f(X))$, but the converse does not hold in general. In the particular case in which $\vdash_{\mathcal{L}}$ and $\vdash_{\mathcal{L}'}$ are syntactic consequence relations in the calculi $\mathcal{L}$ and $\mathcal{L}'$, respectively, one has that f is a translation if, and only if:

$$X \vdash_{\mathcal{L}} \varphi \quad \Rightarrow \quad f(X) \vdash_{\mathcal{L}'} f(\varphi).$$

Several other interesting properties of translations have been described in (Carnielli & D'Ottaviano (1997)).

In this paper we shall be concerned with logics regarded as objects in a special category, whose arrows are in many cases induced by translations between logical systems.

3 POSSIBLE-TRANSLATIONS SEMANTICS

Suppose we are given a logical system $\mathcal{L} = \langle A, C_A \rangle$ whose semantics we are interested in investigating, and suppose that we also have a class of systems $\mathcal{L}_\lambda = \langle B_\lambda, C_{B_\lambda} \rangle$ $(\lambda \in \Lambda)$, all having the same type of similarity. Suppose that each $\mathcal{L}_\lambda$ is characterized by a class of models $M_\lambda \in M$, and that we would like to analyze $\mathcal{L}$ in terms of $\{\mathcal{L}_\lambda : \lambda \in \Lambda\}$. We define a *possible-translation semantics* for $\mathcal{L}$ as a pair

$$PT = \langle M, T \rangle$$

where T is a class of translations $* : A \longrightarrow B_\lambda$.

Given a particular translation $* : A \longrightarrow B_\lambda$, the local forcing relation $\models^*_{PT}$ for $\mathcal{L}$ over $\mathcal{L}_\lambda$ is defined by:

$$\Gamma \models^*_{PT} \alpha \quad \Leftrightarrow \quad \Gamma^* \vdash_\lambda \alpha^*,$$

read as "Γ forces α under the translation $*$" or "α is possibly valid".

The concept of local forcing relation can be generalized considering not only one translation, but a collection of translations: given a collection of translations $T_\Delta \subseteq T$, $T_\Delta = \{* : A \longrightarrow B_\delta : \delta \in$

$\Delta\}$, the *semi-global forcing relation* $\models_{PT\Delta}$ for $\mathcal{L}$ over $\{\mathcal{L}_\delta : \delta \in \Delta\}$ is defined by:

$$\Gamma \models_{PT\Delta} \alpha \quad \Leftrightarrow \quad \Gamma \models^{*}_{PT} \alpha, \text{ for every } * \text{ in } T_\Delta$$

read as "Γ forces α under the translations in T_Δ" or "α is necessarily valid relative to T_Δ".

Finally, the *global forcing relation* $\models_{PT}$ *for* $\mathcal{L}$ *over the class* $\{\mathcal{L}_\lambda : \lambda \in \Lambda\}$ is defined by

$$\Gamma \models_{PT} \alpha \quad \Leftrightarrow \quad \Gamma \models^{*}_{PT} \alpha, \text{ for every } * \text{ in } T$$

read as "Γ forces α in PT" or "α is necessarily valid".

In intuitive terms, possible-translations can be seen as an abstraction of the possible-worlds in the usual Kripke semantics; in the most simple case we do not consider any relations between translations, but there are several possibilities of considering the collections of translations subject to topological structures or special relations (which we could see as an abstraction of accessibility relations between translations). In the most abstract form, as we discuss in this paper, translations can be regarded as sheaf morphisms.

We recall here some simple examples of possible-translations semantics. As a particularly motivating example, possible-translations semantics $PT = \langle M, T\rangle$, where M consists of two copies $\{C_1, C_2\}$ of classical logic, can be assigned to the many-valued logics I^1, P^1 and P^2 (cf. (Carnielli & Lima-Marques (1999)) and (Marcos (1999))).

Other examples are the possible-translations semantics for the paraconsistent calculi C_n (cf. (Carnielli (1990)), (Carnielli (forthcoming)) and (Marcos (1999))). In those cases, it can be proven that, using certain three-valued logics as basic systems, a new semantics can be given to these calculi. These semantics are particularly interesting because, even granted that the global (paraconsistent) logic is not truth-functional, the local (three-valued) logics are truth-functional. Several other properties and features of these paraconsistent logics are rendered clear by virtue of their possible-translations semantics, as discussed in the above mentioned papers.

We also recall the notion of *society semantics*, which are particular cases of possible-translations semantics. A *society* is a set of *agents* $Soc = \{Ag_1, Ag_2, ...\}$, where each Ag_i is a collection of propositional variables Var_i in an underlying logical system S_i.

An agent Ag_i *accepts* a formula α, denoted by $Ag_i \models \alpha$, if α is a semantical consequence of Var_i in S_i.

We consider here, as a particular example, the case where the logic of each agent is classical. A society is *open* (denoted by S^+) if it accepts α in case any of its agents accepts α.

A *satisfiability relation* for an open society S^+ is defined by:

(S1)	$S^+ \models p$	if there exists an agent Ag_i in Soc s.t. $Ag_i \models p$;
(S2)	$S^+ \models \neg p$	if there exists an agent Ag_i in Soc s.t. $Ag_i \not\models p$;
(S3)	$S^+ \models \neg\alpha$	if $S^+ \not\models \alpha$, for α non-atomic;
(S4)	$S^+ \models \alpha \wedge \beta$	if $S^+ \models \alpha$ and $S^+ \models \beta$;
(S5)	$S^+ \models \alpha \vee \beta$	if $S^+ \models \alpha$ or $S^+ \models \beta$;
(S6)	$S^+ \models \alpha \rightarrow \beta$	if $S^+ \not\models \alpha$ or $S^+ \models \beta$.

It has been proven that society semantics also provides a characterization for the three-valued systems P^1 (cf. (Carnielli & Lima-Marques (1999))) and P^2 (cf. (Marcos (1999))). These systems are interesting three-valued logics because they are at the same time paraconsistent and maximal, and have dual systems with intuitionistic flavor. They are however simple examples of the possibilities of possible-translations semantics. In (Carnielli (forthcoming)) and (Carnielli & Marcos (forthcoming)) it is also shown how possible-translations semantics can be assigned to other more complex systems, and the significance of such semantical accounts is discussed.

It should be remarked that possible-translations semantics can be viewed as a procedure which works on two different directions. First, analysing a logic system in terms of other systems, as in the example we hove mentioned: we call this procedure *splitting logics*. Second, possible translations semantics can also be used to define new systems, syntheting a new logic in terms of combinations of other logics: we call this procedure *splicing logics*.

4 LOGICS AS SHEAVES

In this section we will introduce the definitions necessary to achieve the main objective of this paper: to show that logics satisfying certain properties can be seen as objects (actually, as presheaves) in an appropriate category, and combinations of logics can be represented as diagrams in this category. As already mentioned, our approach is motivated by the paper of J. Goguen on sheaves and concurrent objects (cf. (Goguem (1992))).

Here we will adopt the following point of view: given a propositional logic L as defined in section 2, with a sound and complete semantics of valuations, the set of valuations over each $U \subseteq \mathbb{V}$ (where $\mathbb{V}$ is the set of propositional variables) give us a *local picture* of L. That is: if we regard U as a set of basic facts in which we are occasionally interested, then the set of L-valuations over U provides an atlas of the true propositions and inference relations in L concerning the facts in U (since we assume that the set of L-valuations is adequate).

This (semantic) approach defines a logic as *sets of observations* (each valuation v can be seen as an observation of the facts in U). A desirable property of these logics would be *coherence*: if two L-valuations v and v' over U and V, respectively, are *compatible* (that is, they coincide in $U \cap V$), then we expect that both v and v' could be extended to $U \cup V$, which we call a *join* of v and v'. Logics satisfying this property will be called *finite sheaves* in this category. If every set of L-valuations compatible pairwise have a join, then we have a *sheaf.* If, additionally, we require this join to be unique, then we have, respectively, *extensional finite sheaves* and *extensional sheaves.* In all cases, these logics can be seen metaphorically as *coherent sets of observations.* We will see throughout this paper examples of logics of each type described above.

Since the restriction of an L-valuation v over U to $V \subseteq U$ is again an L-valuation, then the logics considered are *presheaves* in the language of category theory. Morphisms, as expected, are defined as natural transformations between the presheaves or, more simply, as families of maps indexed by $U \subseteq \mathbb{V}$ which take L-valuations over U into L'-valuations over U preserving the restrictions to all the subsets $V \subseteq U$. Usually, translations between logics will induce morphisms of logics.

Finally, combinations of logics will be represented as diagrams in the category of logics, that is to say, typically a family of objects (logics) and arrows between them (relations). The *behavior* of that diagram will be a new object of the category, that is, a new logic, representing "the logic of the system of logics". This behavior can arise as the categorial limit or the categorial colimit of the diagram representing the system of combined logics. In general, the limit will represent the theorems of the resulting logic, whereas the colimit will represent deductions.

Definition 4.1 Let $\mathbb{V} = \{p_i \mid i \in \omega\}$ be a denumerable set of propositional variables. The category $\mathcal{L}_{\mathbb{V}}$ of logics over $\mathbb{V}$ is defined by the following:

• **Objects:** Let Σ be a propositional signature over $\mathbb{V}$ (that is, a set of logical constants and a set of n-ary connectives, for each $n \in \omega$), and let $F_\Sigma(U)$ be the set of Σ-formulas over $U \subseteq \mathbb{V}$. $F_\Sigma(U)$ is a free algebra of type Σ generated by U. Let T be a nonempty set (of truth-values) and $D \subseteq T$ a nonempty set (of designated values). An object $\mathcal{O}$ of $\mathcal{L}(\mathbb{V})$ is a family of sets

$$\mathcal{O}(U) \subseteq T^{F_\Sigma(U)} \quad (U \subseteq \mathbb{V}).$$

Elements in $\mathcal{O}(U)$ are called *valuations* (or *sections*) over U. Elements in $\mathcal{O}(\mathbb{V})$ are called *global sections* of $\mathcal{O}$. We will also assume that, if $U \subseteq V$ and $v : F_\Sigma(V) \longrightarrow T$ is a valuation, then

$$v_{|F_\Sigma(U)} : F_\Sigma(U) \longrightarrow T$$

is also a valuation. Objects in $\mathcal{L}_\mathbb{V}$ will be called $\mathbb{V}$-*logics* or simply *logics*.

• **Morphisms:** Let $\mathcal{O}$ and $\mathcal{O}'$ be logics. A morphism $\varphi : \mathcal{O} \longrightarrow \mathcal{O}'$ in $\mathcal{L}_\mathbb{V}$ from $\mathcal{O}$ to $\mathcal{O}'$ is a family of maps

$$\varphi_U : \mathcal{O}(U) \longrightarrow \mathcal{O}'(U) \quad (U \subseteq \mathbb{V})$$

satisfying:

$$\varphi_V(v)_{|F_\Sigma(U)} = \varphi_U(v_{|F_\Sigma(U)})$$

for every $v \in \mathcal{O}(V)$ and $U \subseteq V$. The class of morphisms in $\mathcal{L}_\mathbb{V}$ from $\mathcal{O}$ to $\mathcal{O}'$ will be denoted by $hom_\mathbb{V}(\mathcal{O}, \mathcal{O}')$, or simply $hom(\mathcal{O}, \mathcal{O}')$.

• **Composition:** The composition of morphisms is defined pointwise, that is: if $\varphi : \mathcal{O}_1 \longrightarrow \mathcal{O}_2$ and $\psi : \mathcal{O}_2 \longrightarrow \mathcal{O}_3$ are morphisms, then $\psi \circ \varphi : \mathcal{O}_1 \longrightarrow \mathcal{O}_3$ is defined by $(\psi \circ \varphi)_U = \psi_U \circ \varphi_U$ for each $U \subseteq \mathbb{V}$. $\square$

If $v \in \mathcal{O}(V)$ and $U \subseteq V$, then $v_{|U}$ will stand for $v_{|F_\Sigma(U)}$ (provided that the underlying signature Σ is clear). It is clear that we have the following:

$U \overset{i}{\hookrightarrow} V$ induces a mapping $\mathcal{O}(i) : \mathcal{O}(V) \longrightarrow \mathcal{O}(U)$ defined *via* $v \mapsto v_{|U}$

satisfying the following condition:

$$U \overset{j}{\hookrightarrow} V \overset{i}{\hookrightarrow} W \quad \text{induces} \quad \mathcal{O}(W) \xrightarrow{\mathcal{O}(i)} \mathcal{O}(V) \xrightarrow{\mathcal{O}(j)} \mathcal{O}(U)$$

in such a way that

$$\mathcal{O}(i \circ j) = \mathcal{O}(j) \circ \mathcal{O}(i), \qquad \mathcal{O}(id_U) = id_{\mathcal{O}(U)}.$$

This means that each logic is a (contravariant) functor $\mathcal{O} : (\mathbb{V}, \subseteq) \longrightarrow$ **Set**, that is, a presheaf. The morphisms in $\mathcal{L}_{\mathbb{V}}$ are simply presheaf morphisms.

Definition 4.2 A logic $\mathcal{O}$ is a *finite sheaf* if it satisfies: for a given $v \in \mathcal{O}(U)$ and $v' \in \mathcal{O}(V)$ such that $v_{|U \cap V} = v'_{|U \cap V}$, then there exists $w \in \mathcal{O}(U \cup V)$ such that $w_{|U} = v$ and $w_{|V} = v'$. If w is unique, then $\mathcal{O}$ is an *extensional finite sheaf.* $\mathcal{O}$ is a *sheaf* if it satisfies: for a given family $\{v_i\}_{i \in I}$ of valuations, $v_i \in \mathcal{O}(U_i)$ $(i \in I)$, such that $v_{i|U_i \cap U_j} = v_{j|U_i \cap U_j}$ $(i, j \in I)$ then there exists $w \in \mathcal{O}(\bigcup_{i \in I} U_i)$ such that $w_{|U_i} = v_i$ for all $i \in I$. If this w is unique then $\mathcal{O}$ is an *extensional sheaf.* □

Proposition 4.3 A logic $\mathcal{O}$ is a finite sheaf iff for every finite set $\{v_1, ..., v_n\}$ such that $v_i \in \mathcal{O}(U_i)$ and $v_{i|U_i \cap U_j} = v_{j|U_i \cap U_j}$ $(i, j = 1, ..., n)$ there exists $w \in \mathcal{O}(\bigcup_{i=1}^{n} U_i)$ such that $w_{|U_i} = v_i$ for every $i = 1, ..., n$.

Proof: By a straightforward induction. □

Definition 4.4 A logic $\mathcal{O}$ over Σ, T and $D \subseteq T$ is *algebraic* if T is an algebra similar to $F_\Sigma(\mathbb{V})$ and each $v \in \mathcal{O}(U)$ is a homomorphism; therefore v is determined by $\{v(p) \mid p \in U\}$. □

Definition 4.5 Let $\mathcal{O}$ be a logic, and $\Gamma \cup \{\alpha\} \subseteq F_\Sigma(U)$. We say that α is *valid in* $\mathcal{O}$ if $v(\alpha) \in D$ for all $v \in \mathcal{O}(U)$. We say that α *is a consequence of* Γ *in* $\mathcal{O}$ if, for all $v \in \mathcal{O}(U)$, $v(\Gamma) \subseteq D$ implies $v(\alpha) \in D$. $\models_{\mathcal{O}} \alpha$ and $\Gamma \models_{\mathcal{O}} \alpha$ will denote that α is valid in $\mathcal{O}$ and α is a consequence of Γ in $\mathcal{O}$, respectively. Of course $\models_{\mathcal{O}} \alpha$ iff $\emptyset \models_{\mathcal{O}} \alpha$. $\square$

We now define *combinations of logics* as (categorial) diagrams in the category $\mathcal{L}_{\mathbb{V}}$.

Definition 4.6 A *combination of logics* is a diagram in $\mathcal{L}_{\mathbb{V}}$, that is, a pair $\mathcal{D} = \langle O, M \rangle$ where $O = \{\mathcal{O}_i\}_{i \in I}$ is a (possibly empty) family of objects in $\mathcal{L}_{\mathbb{V}}$ and $M \subseteq \bigcup_{i,j \in I} hom(\mathcal{O}_i, \mathcal{O}_j)$ is a (possibly empty) family of morphisms. $\square$

Definition 4.7 Let $\mathcal{D} = \langle O, M \rangle$ be a diagram.
1) A *limit* for $\mathcal{D}$ is a logic $\mathcal{O}$ and a family of morphisms $\varphi_i : \mathcal{O} \longrightarrow \mathcal{O}_i$ ($i \in I$) such that $\varphi \circ \varphi_i = \varphi_j$ for each $\varphi : \mathcal{O}_i \longrightarrow \mathcal{O}_j$ in M. Additionally, the limit $\langle \mathcal{O}; \{\varphi_i\}_{i \in I} \rangle$ must satisfy the following universal property: given $\langle \mathcal{O}'; \{\psi_i\}_{i \in I} \rangle$ such that $\psi_i : \mathcal{O}' \longrightarrow \mathcal{O}_i$ ($i \in I$) and, for each $\varphi : \mathcal{O}_i \longrightarrow \mathcal{O}_j$ in M, $\varphi \circ \psi_i = \psi_j$, then there exists an unique $\psi : \mathcal{O}' \longrightarrow \mathcal{O}$ such that $\varphi_i \circ \psi = \psi_i$ for all $i \in I$. This implies that the limit of a diagram $\mathcal{D}$, if it exists, is unique (up to isomorphisms).
2) Dually, a *colimit* for $\mathcal{D}$ is a logic $\mathcal{O}$ and a family of morphisms $\lambda_i : \mathcal{O}_i \longrightarrow \mathcal{O}$ ($i \in I$) such that $\lambda_j \circ \varphi = \lambda_i$ for each $\varphi : \mathcal{O}_i \longrightarrow \mathcal{O}_j$ in M. Additionally, the colimit $\langle \mathcal{O}; \{\lambda_i\}_{i \in I} \rangle$ must satisfy the following universal property: given $\langle \mathcal{O}'; \{\psi_i\}_{i \in I} \rangle$ such that $\psi_i : \mathcal{O}_i \longrightarrow \mathcal{O}'$ ($i \in I$) and, for each $\varphi : \mathcal{O}_i \longrightarrow \mathcal{O}_j$ in M, $\psi_j \circ \varphi = \psi_i$, then there exists an unique $\psi : \mathcal{O} \longrightarrow \mathcal{O}'$ such that $\psi \circ \lambda_i = \psi_i$ for all $i \in I$. This implies that the colimit of a diagram $\mathcal{D}$, if it exists, is unique (up to isomorphisms). $\square$

Example 4.8 Let $\mathcal{D} = \langle \{\mathcal{O}_i\}_{i \in I}, \emptyset \rangle$ with an empty set of arrows. It follows that the limit of $\mathcal{D}$ (if it exists) is the product $\prod_{i \in I} \mathcal{O}_i$ in $\mathcal{L}_{\mathbb{V}}$, and the colimit of $\mathcal{D}$ is the coproduct $\coprod_{i \in I} \mathcal{O}_i$ in $\mathcal{L}_{\mathbb{V}}$. $\square$

Proposition 4.9 Let $\mathcal{D} = \langle\{\mathcal{O}_i\}_{i\in I}, M\rangle$ be a diagram such that $\Sigma_i = \Sigma$ for all $i \in I$. Under these conditions, the limit $\mathcal{O}$ of $\mathcal{D}$ exists, and moreover $\mathcal{O}$ is an extensional (respectively an extensional finite) sheaf if each logic $\mathcal{O}_i$ is also an extensional (respectively an extensional finite) sheaf.

Proof: Let $T = \prod_{i\in I} T_i$ and $D = \prod_{i\in I} D_i \subseteq T$ (products computed in **Set**). Let's consider, for each $U \subseteq \mathbb{V}$, families $v = \langle v_i\rangle_{i\in I}$ satisfying:

(i) $v_i \in \mathcal{O}_i(U)$ $(i \in I)$;
(ii) if there exists $\varphi : \mathcal{O}_i \longrightarrow \mathcal{O}_j$ in M, then $v_j = \varphi_U(v_i)$.

Item (ii) implies that, if $\varphi, \psi : \mathcal{O}_i \longrightarrow \mathcal{O}_j$ in M, then v_i satisfies: $\varphi_U(v_i) = \psi_U(v_i)$. Let's define

$$\mathcal{O}(U) = \{v : F_\Sigma(U) \longrightarrow T \mid v = \langle v_i\rangle_{i\in I} \text{ and } v \text{ satisfies (i)-(ii)}\}$$

(thus $\mathcal{O}(U)$ is possibly empty). Observe that for all $\alpha \in F_\Sigma(U)$ and $v \in \mathcal{O}(U)$, $v(\alpha) = \langle v_i(\alpha)\rangle_{i\in I}$. Since $v_{|V}(\alpha) = \langle v_{i|V}(\alpha)\rangle_{i\in I}$ then $v_{|V} \in \mathcal{O}(V)$ $(V \subseteq U)$, and thus $\mathcal{O}$ is a logic. We will show that $\mathcal{O}$ is the limit of $\mathcal{D}$. Define $\varphi_i : \mathcal{O} \longrightarrow \mathcal{O}_i$, $(\varphi_i)_U(v) = v_i$. If $\varphi : \mathcal{O}_i \longrightarrow \mathcal{O}_j$ in M, then

$$\varphi_U((\varphi_i)_U(v)) = \varphi_U(v_i) = v_j = (\varphi_j)_U(v),$$

so $\varphi \circ \varphi_i = \varphi_j$. Let $\mathcal{O}'$ be a logic and let $\psi_i : \mathcal{O}' \longrightarrow \mathcal{O}_i$ be a morphism for all $i \in I$ such that $\varphi \circ \psi_i = \psi_j$ for all $\varphi : \mathcal{O}_i \longrightarrow \mathcal{O}_j$ in M. Define $\psi : \mathcal{O}' \longrightarrow \mathcal{O}$ as $\psi_U(w) = \langle(\psi_i)_U(w)\rangle_{i\in I}$. If $\varphi : \mathcal{O}_i \longrightarrow \mathcal{O}_j$ in M, then $\varphi_U((\psi_i)_U(w)) = (\psi_j)_U(w)$, thus $\psi_U(w) \in \mathcal{O}(U)$. Since $(\varphi_i)_U(\psi_U(w)) = (\psi_i)_U(w)$, then $\varphi_i \circ \psi = \psi_i$ for all $i \in I$. Suppose now that $\psi' : \mathcal{O}' \longrightarrow \mathcal{O}$, $\psi'_U(w) = \langle v_i^w\rangle_{i\in I}$ such that $\varphi_i \circ \psi' = \psi_i$ for all $i \in I$. Then

$$(\psi_i)_U(w) = ((\varphi_i)_U \circ \psi'_U)(w) = (\varphi_i)_U(\psi'_U(w)) = v_i^w$$

for all $i \in I$, whence $\psi' = \psi$. This shows that $\langle \mathcal{O}; \{\varphi_i\}_{i\in I}\rangle$ is the limit of $\mathcal{D}$.

Finally, let's suppose that every $\mathcal{O}_i$ is an extensional finite sheaf; we will prove that $\mathcal{O}$ is, as well. In order to prove this, let $v \in \mathcal{O}(U)$ and $v' \in \mathcal{O}(V)$ such that $v_{|U\cap V} = v'_{|U\cap V}$; then $v_{i|U\cap V} = v'_{i|U\cap V}$ for all $i \in I$. Since each $\mathcal{O}_i$ is an extensional finite sheaf, then there exists a unique $w_i \in \mathcal{O}_i(U \cup V)$ such that $w_{i|U} = v_i$ and $w_{i|V} = v'_i$ for all $i \in I$. We will show that $w = \langle w_i\rangle_{i\in I}$ is in $\mathcal{O}(U \cup V)$. Let $\varphi : \mathcal{O}_i \longrightarrow \mathcal{O}_j$ in M; then $v_j = \varphi_U(v_i)$ and $v'_j = \varphi_V(v'_i)$. Thus

$$v_j = \varphi_U(v_i) = \varphi_U(w_{i|U}) = \varphi_{U\cup V}(w_i)_{|U}$$

and

$$v'_j = \varphi_V(v'_i) = \varphi_V(w_{i|V}) = \varphi_{U\cup V}(w_i)_{|V}.$$

Therefore $\varphi_{U\cup V}(w_i) = w_j$, since $\mathcal{O}_j$ is extensional, showing that $w \in \mathcal{O}(U\cup V)$. It is clear that $w_{|U} = v$, $w_{|V} = v'$, and that w is the unique valuation with this property, so $\mathcal{O}$ is an extensional finite sheaf. The proof for the case of extensional sheaves is similar. $\square$

Corollary 4.10 Let $\{\mathcal{O}_i\}_{i\in I}$ be a family of logics over a signature Σ. Under these conditions the product $\mathcal{O} = \prod_{i\in I} \mathcal{O}_i$ exists, and $\mathcal{O}$ is an extensional (respectively an extensional finite) sheaf provided that each $\mathcal{O}_i$ is an extensional (respectively an extensional finite) sheaf. Moreover, α is valid in $\mathcal{O}$ iff α is valid in $\mathcal{O}_i$ for all $i \in I$.

Proof: By proposition **4.9** there exists the limit $\mathcal{O}$ of $\mathcal{D} = \langle\{\mathcal{O}_i\}_{i\in I}, \emptyset\rangle$. Clearly $\mathcal{O} = \prod_{i\in I} \mathcal{O}_i$. By the construction described in the proof of proposition **4.9** we have that

$$\mathcal{O}(U) = \{v : F_\Sigma(U) \longrightarrow \prod_{i\in I} T_i \ \mid \ v = \langle v_i\rangle_{i\in I} \text{ and } v_i \in \mathcal{O}_i(U)$$

$$(i \in I)\}.$$

The rest of the proof is immediate. □

Proposition 4.11 Consider fixed Σ, T and $D \subseteq T$. If I is a linearly ordered set, consider a chain $\{\mathcal{O}_i\}_{i\in I}$ of logics over Σ, T, D such that, for all U, $\mathcal{O}_i(U) \subseteq \mathcal{O}_j(U)$ if $i \leq j$. Let φ_i^j : $\mathcal{O}_i \longrightarrow \mathcal{O}_j$ be the inclusion morphism ($i \leq j$), that is: $(\varphi_i^j)_U(v) = v$ if $i \leq j$. Let $\mathcal{D}$ be the diagram $\langle\{\mathcal{O}_i\}_{i\in I}, \{\varphi_i^j\}_{i\,\leq\,j}\rangle$. It follows that $\mathcal{O}(U) = \bigcup_{i\in I}\mathcal{O}_i(U)$ is the coproduct of $\mathcal{D}$, and $\mathcal{O}$ is a finite sheaf provided that every logic $\mathcal{O}_i$ is a finite sheaf. Moreover, if every nonempty subset of I has an upper bound in I then $\mathcal{O}$ is a sheaf (respectively, an extensional sheaf) provided that every logic $\mathcal{O}_i$ is a sheaf (respectively, an extensional sheaf).

Proof: $\mathcal{O}(U) = \bigcup_{i\in I}\mathcal{O}_i(U)$ is a logic over Σ, T and D. In fact, if $v \in \mathcal{O}(U)$ then there exists $i \in I$ such that $v \in \mathcal{O}_i(U)$. So, for all $V \subseteq U$, $v_{|V} \in \mathcal{O}_i(V)$ and then $v_{|V} \in \mathcal{O}(V)$. For all $i \in I$ let's consider $\lambda_i : \mathcal{O}_i \longrightarrow \mathcal{O}$, $(\lambda_i)_U(v) = v$. We will prove that $\langle\mathcal{O}; \{\lambda_i\}_{i\in I}\rangle$ is the coproduct of $\mathcal{D}$. It is not difficult to see that $\lambda_j \circ \varphi_i^j = \lambda_i$ for all $i \leq j$. Suppose that $\mathcal{O}'$ is a logic and $\psi_i : \mathcal{O}_i \longrightarrow \mathcal{O}'$ such that $\psi_j \circ \varphi_i^j = \psi_i$ for all $i \leq j$. If $v \in \mathcal{O}_i(U)$ and $i \leq j$ then $(\psi_i)_U(v) = (\psi_j)_U((\varphi_i^j)_U(v)) = (\psi_j)_U(v)$. Similarly, if $v \in \mathcal{O}_j(U)$ and $i \leq j$ then $(\psi_i)_U(v) = (\psi_j)_U(v)$. It follows that the maps $\psi_U : \mathcal{O}(U) \longrightarrow \mathcal{O}'(U)$, $\psi_U(v) = (\psi_i)_U(v)$ (if $v \in \mathcal{O}_i(U)$) are well-defined for all U and produce a morphism $\psi : \mathcal{O} \longrightarrow \mathcal{O}'$ such that $\psi \circ \lambda_i = \psi_i$ for all $i \in I$. Suppose that $\psi' : \mathcal{O} \longrightarrow \mathcal{O}'$ is such that $\psi' \circ \lambda_i = \psi_i$ for all $i \in I$, and let $v \in \mathcal{O}_i(U)$. It follows that

$$\psi'_U(v) = \psi'_U((\lambda_i)_U(v)) = (\psi'_U \circ (\lambda_i)_U)(v) = (\psi_i)_U(v),$$

whence $\psi' = \psi$. This shows that $\langle\mathcal{O}; \{\lambda_i\}_{i\in I}\rangle$ is the coproduct of $\mathcal{D}$.

Finally, we will prove that $\mathcal{O}$ is a finite sheaf if every $\mathcal{O}_i$ is a finite sheaf. Let $v \in \mathcal{O}(U)$ and $v' \in \mathcal{O}(V)$ such that $v_{|U\cap V} = v'_{|U\cap V}$. There exists $i, j \in I$ such that $v \in \mathcal{O}_i(U)$ and $v' \in \mathcal{O}_j(V)$.

Without loss of generality we can suppose that $i \leq j$, thus $v \in \mathcal{O}_j(U)$. Hence there exists $w \in \mathcal{O}_j(U \cup V)$ such that $w_{|U} = v$ and $w_{|V} = v'$. Since $w \in \mathcal{O}(U \cup V)$ we have that $\mathcal{O}$ is a finite sheaf. The proofs for the cases of sheaves and extensional sheaves are similar. This concludes the proof. □

5 FIRST EXAMPLE: BIASSERTIVE SOCIETIES

In this and the subsequent section we will give categorial examples of the process we have called splitting logics.

As a first application of our categorial approach to logics, we will rephrase in categorial terms the (open and closed) *biassertive societies* introduced in (Carnielli & Lima-Marques (1999)). In this example, an agent Ag can be identified with a set $C \subseteq \mathbb{V}$, viewed as the set of variables accepted by Ag; moreover, we suppose that Ag is endowed with classical logic. Throughout this section, Σ will denote the signature of the classical propositional calculus over $\mathbb{V}$. We can define an extensional sheaf $\mathcal{O}$ representing Ag as follows:

$$\mathcal{O}(U) = \{v^U\}, \quad v^U : F_\Sigma(U) \longrightarrow \mathbf{2} \text{ is a classical valuation s.t.}$$

$$v^U(p) = \mathbf{1} \quad \text{iff} \quad p \in U \cap C.$$

Given a closed biassertive society $S^- = \{C_i\}_{i \in I}$ (cf. (Carnielli & Lima-Marques (1999))), where C_i represents an agent Ag_i, we have that

$$\begin{array}{l} S^- \models p \quad \text{iff} \quad p \in \bigcap_{i \in I} C_i, \\ S^- \models \neg p \quad \text{iff} \quad p \notin \bigcup_{i \in I} C_i, \text{ and} \\ S^- \models \alpha \text{ is defined classically if } \alpha \neq p,\ \alpha \neq \neg p. \end{array}$$

In our setting, the systems S^- and S^+ can be presented as

$$\mathcal{O}_{S^-}(U) = \{v_-^U\}, \quad v_-^U : F_\Sigma(U) \longrightarrow \mathbf{2}, \quad v_-^U(\alpha) = \mathbf{1} \quad \text{iff} \quad S^- \models \alpha,$$

$$\mathcal{O}_{S^+}(U) = \{v_+^U\}, \quad v_+^U : F_\Sigma(U) \longrightarrow \mathbf{2}, \quad v_+^U(\alpha) = \mathbf{1} \ \text{ iff } \ S^+ \models \alpha.$$

Moreover, the fact that the internal logic of open (respectively closed) biassertive societies is P^1 (respectively I^1) is represented as follows: for all $U \subseteq \mathbb{V}$ set

$$\begin{aligned} \mathcal{O}^+(U) &= \{v : F_\Sigma(U) \longrightarrow \mathbf{2} \ \mid \ v = v_+^U \ \text{ for some } \ S^+\} \\ &= \bigcup\{\mathcal{O}_{S^+}(U) \ \mid \ \mathcal{O}_{S^+} \ \text{ is open biassertive}\}, \\ \mathcal{O}^-(U) &= \{v : F_\Sigma(U) \longrightarrow \mathbf{2} \ \mid \ v = v_-^U \ \text{ for some } \ S^-\} \\ &= \bigcup\{\mathcal{O}_{S^-}(U) \ \mid \ \mathcal{O}_{S^-} \ \text{ is closed biassertive}\}. \end{aligned}$$

Thus, if $\mathcal{OS}$ is the family of open biassertive societies and $\mathcal{CS}$ is the family of closed biassertive societies, then $\mathcal{O}^+$ is the coproduct of $\mathcal{D}^+ = \langle \mathcal{OS}, \emptyset \rangle$ and $\mathcal{O}^-$ is the coproduct of $\mathcal{D}^- = \langle \mathcal{CS}, \emptyset \rangle$.

Proposition 5.1 $\mathcal{O}^+$ and $\mathcal{O}^-$ are extensional sheaves.

Proof: Let $\{v_\lambda\}_{\lambda \in \Lambda}$ be a compatible family of valuations in $\mathcal{O}^+$, that is, $v_\lambda \in \mathcal{O}^+(U_\lambda)$ $(\lambda \in \Lambda)$ such that, for each $\lambda, \mu \in \Lambda$,

$$v_{\lambda|U_\lambda \cap U_\mu} = v_{\mu|U_\lambda \cap U_\mu}.$$

Let $U = \bigcup_{\lambda \in \Lambda} U_\lambda$. The map $w(p) = v_\lambda(p)$ if $p \in U_\lambda$, and $w(\neg p) = v_\lambda(\neg p)$ if $p \in U_\lambda$, is well-defined. Extend w to a map $w : F_\Sigma(U) \longrightarrow \mathbf{2}$ as follows:

$w(\neg\alpha) = \mathbf{1}$ iff $w(\alpha) = \mathbf{0}$, if $\alpha \notin U$;
$w(\beta \wedge \gamma) = \mathbf{1}$ iff $w(\beta) = w(\gamma) = \mathbf{1}$;
$w(\beta \vee \gamma) = \mathbf{1}$ iff $w(\beta) = \mathbf{1}$ or $w(\gamma) = \mathbf{1}$;
$w(\beta \rightarrow \gamma) = \mathbf{1}$ iff $w(\beta) = \mathbf{0}$ or $w(\gamma) = \mathbf{1}$.

Let $\{C_i^\lambda\}_{i \in I_\lambda}$ be the agents associated to v_λ; note that $C_i^\lambda \subseteq U_\lambda$ $(i \in I_\lambda)$ for each $\lambda \in \Lambda$. For $s \in \prod_{\lambda \in \Lambda} I_\lambda$ let $C_s = \bigcup_{\lambda \in \Lambda} C^\lambda_{\pi_\lambda(s)}$, where $\pi_\lambda : \prod_{\mu \in \Lambda} I_\mu \longrightarrow I_\lambda$ is the canonical projection. It is easy to see that w is the valuation associated to the family $\{C_s \mid s \in \prod_{\lambda\in\Lambda} I_\lambda\}$. Thus $w \in \mathcal{O}^+(U)$ such that $w_{|U_\lambda} = v_\lambda$ for all $\lambda \in \Lambda$. Obviously w is unique. The proof for $\mathcal{O}^-$ is analogous. $\square$

From the theorems of soundness and completeness given in (Lima-Marques (1992)) and (Carnielli & Lima-Marques (1999)) we obtain sheaf isomorphisms

$$\varphi : \mathcal{O}_{P^1} \longrightarrow \mathcal{O}^+, \qquad \psi : \mathcal{O}_{I^1} \longrightarrow \mathcal{O}^-$$

where $\mathcal{O}_{P^1}$ and $\mathcal{O}_{I^1}$ are the sheaf canonical representations of the logics P^1 (cf. (Sette (1973))) and I^1 (cf. (Carnielli & Lima-Marques (1999))), respectively. Since $\mathcal{O}^+$ and $\mathcal{O}^-$ are the colimits of the limits $\mathcal{O}_{S^+}$ and $\mathcal{O}_{S^-}$, we see the categorial realization of the fact that the internal logic of open societies is P^1 and I^1, respectively. Each $\mathcal{O}_{S^+}$ is a limit, that is, the behavior of a system (of agents). It follows that $\mathcal{D}^+ = \langle \mathcal{OS}, \emptyset\rangle$ is a diagram of system diagrams. According to (Goguen (1992)), the behavior of a diagram of system diagrams is the colimit of the diagram, therefore

$$\mathcal{O}_{P^1} \simeq \mathcal{O}^+ = colim\, \mathcal{D}^+,$$

says that P^1 is the (internal) logic of open biassertive societies. Dually,

$$\mathcal{O}_{I^1} \simeq \mathcal{O}^- = colim\, \mathcal{D}^-$$

expresses the fact that I^1 is the (internal) logic of closed biassertive societies.

The generalization of this technique to triassertive societies is straightforward, and analogous results can be obtained.

6 SECOND EXAMPLE: POSSIBLE-TRANSLATIONS SEMANTICS FOR C_1

The second application of our language is the possible-translations semantics for the paraconsistent calculus C_1 given in (Carnielli (forthcoming)) and (Marcos (1999)). In these papers it is shown that C_1 can be explained by a family TR of translations from C_1 to a three-valued logic W_3 which has a matrix semantics, in the framework of possible-translation semantics. Let's consider Σ', $T = \{\mathbf{T}, \mathbf{T}^-, \mathbf{F}\}$ and $D = \{\mathbf{T}, \mathbf{T}^-\}$ as the components of W_3; therefore

$$\mathcal{O}_3(U) = \{w : F_{\Sigma'}(U) \longrightarrow T \ \mid \ w \ \text{ is a homomorphism } \}$$

(for all $U \subseteq \mathbb{V}$) is an extensional sheaf in $\mathcal{L}_{\mathbb{V}}$ that represents W_3. If Σ is the (classical) signature for C_1, then

$$\mathcal{O}_1(U) = \{v : F_{\Sigma}(U) \longrightarrow \mathbf{2} \ \mid \ v \text{ is a } C_1\text{-valuation } \}$$

(for all $U \subseteq \mathbb{V}$) is a logic in $\mathcal{L}_{\mathbb{V}}$ that represents C_1, where $T = \mathbf{2}$ and $D = \{\mathbf{1}\}$.

Proposition 6.1 $\mathcal{O}_1$ is a sheaf.

Proof: Let $\{v_\lambda\}_{\lambda \in \Lambda}$ be a compatible family of valuations in $\mathcal{O}_1$, that is, $v_\lambda \in \mathcal{O}_1(U_\lambda)$ $(\lambda \in \Lambda)$ such that, for each $\lambda, \mu \in \Lambda$,

$$v_{\lambda | U_\lambda \cap U_\mu} = v_{\mu | U_\lambda \cap U_\mu}.$$

Let $U = \bigcup_{\lambda \in \Lambda} U_\lambda$. The map $w(\alpha) = v_\lambda(\alpha)$, if $\alpha \in F_\Sigma(U_\lambda)$, is well-defined. Extend w to a map $w : F_\Sigma(U) \longrightarrow \mathbf{2}$ as follows:

if $\alpha \notin \bigcup_{\lambda \in \Lambda} F_\Sigma(U_\lambda)$ then $w(\neg\alpha) = \mathbf{1}$ iff $w(\alpha) = \mathbf{0}$;
$w(\beta \wedge \gamma) = \mathbf{1}$ iff $w(\beta) = w(\gamma) = \mathbf{1}$;
$w(\beta \vee \gamma) = \mathbf{1}$ iff $w(\beta) = \mathbf{1}$ or $w(\gamma) = \mathbf{1}$;
$w(\beta \rightarrow \gamma) = \mathbf{1}$ iff $w(\beta) = \mathbf{0}$ or $w(\gamma) = \mathbf{1}$.

It is straightforward to prove that w satisfies the axioms of C_1-valuation. We obtain therefore $w \in \mathcal{O}_1(U)$ such that $w_{|U_\lambda} = v_\lambda$. This concludes the proof. $\square$

In (Carnielli (forthcoming)) and (Marcos (1999)) it is shown that, given a translation $* \in TR$ and $w \in \mathcal{O}_3(\mathbb{V})$, the map

$$v_w^* : F_\Sigma(\mathbb{V}) \longrightarrow \mathbf{2}, \quad v_w^*(\alpha) = \mathbf{1} \ \text{ iff } \ w(\alpha^*) \in D$$

is a C_1-valuation; moreover, every v is of the form $v = v_w^*$ for some $*$ and w. This is the adequacy theorem of the possible-translations frame for C_1. This theorem still holds locally (i.e., in all $U \subseteq \mathbb{V}$), because $p^* = p$ for all $p \in \mathbb{V}$. We will show how this representation theorem can be expressed in $\mathcal{L}_{\mathbb{V}}$.
For each $* \in TR$ we define a sublogic $\mathcal{O}_*$ of $\mathcal{O}_1$ as follows:

$$\mathcal{O}_*(U) = \{v \in \mathcal{O}_1(U) \ \mid \ v = v_w^* \ \text{ for some } w \in \mathcal{O}_3(U)\}.$$

For each $* \in TR$ and $U \subseteq \mathbb{V}$ let's define a map

$$(\varphi^*)_U : \mathcal{O}_3(U) \longrightarrow \mathcal{O}_*(U), \quad (\varphi^*)_U(w) = v_w^*.$$

It is clear that this produces a morphism $\varphi^* : \mathcal{O}_3 \longrightarrow \mathcal{O}_*$ for each $* \in TR$.

Proposition 6.2 The diagram $\mathcal{D} = \langle \{\mathcal{O}_3\} \cup \{\mathcal{O}_*\}_{*\in TR}, \{\varphi^*\}_{*\in TR}\rangle$ has a limit $\mathcal{O}_{lim}$. The limit $\mathcal{O}_{lim}$ is a sheaf, and $\mathcal{O}_{lim}$ validates the same formulas which are theorems of C_1, that is: $\models_{\mathcal{O}_{lim}} \alpha$ iff $\vdash_{C_1} \alpha$.

Proof: For $U \subseteq \mathbb{V}$, define the set

$$\mathcal{O}_{lim}(U) = \{\langle v_w^* \rangle_{*\in TR} \ \mid \ w \in \mathcal{O}_3(U)\}.$$

By considering the restriction pointwise, it is easy to see that $\mathcal{O}_{lim}$ is a logic over Σ, $T = \mathbf{2}^{TR}$ and $D = \{\mathbf{1}\}^{TR}$. Let's consider the maps

$$\psi_U : \mathcal{O}_{lim}(U) \longrightarrow \mathcal{O}_3(U), \quad \psi_U(\langle v_w^* \rangle_{*\in TR}) = w,$$

$$(\psi^*)_U : \mathcal{O}_{lim}(U) \longrightarrow \mathcal{O}_*(U), \quad (\psi^*)_U(\langle v_w^* \rangle_{*\in TR}) = v_w^*.$$

It is easy to see that these maps produce sheaves morphisms $\psi : \mathcal{O}_{lim} \longrightarrow \mathcal{O}_3$ and $\psi^* : \mathcal{O}_{lim} \longrightarrow \mathcal{O}_*$ such that $\varphi^* \circ \psi = \psi^*$ for all $* \in TR$. Let $\mathcal{O}$ be a logic and let $\lambda : \mathcal{O} \longrightarrow \mathcal{O}_3$, $\lambda^* : \mathcal{O} \longrightarrow \mathcal{O}_*$ be morphisms ($* \in TR$) such that $\varphi^* \circ \lambda = \lambda^*$ for all $* \in TR$. For $v \in \mathcal{O}(U)$ set $\varphi_U(v) = \langle \lambda^*(v) \rangle_{* \in TR}$. Observe that

$$\lambda^*(v) = \varphi^*(\lambda(v)) = v^*_{\lambda(v)} \quad \text{for all } * \in TR$$

therefore $\varphi_U(v) \in \mathcal{O}_{lim}(U)$, defining a morphism $\varphi : \mathcal{O} \longrightarrow \mathcal{O}_{lim}$ such that $\psi \circ \varphi = \lambda$ and $\psi^* \circ \varphi = \lambda^*$ for all $* \in TR$. It is clear that φ is the unique morphism from $\mathcal{O}$ to $\mathcal{O}_{lim}$ that commutes these triangles, therefore $\langle \mathcal{O}_{lim}; \{\psi\} \cup \{\psi^*\}_{* \in TR} \rangle$ is the limit of $\mathcal{D}$. We now prove that $\mathcal{O}_{lim}$ is a sheaf. Let $v_\lambda = \langle v^*_{w_\lambda} \rangle_{* \in TR}$ in $\mathcal{O}_{lim}(U_\lambda)$ ($\lambda \in \Lambda$) such that $v_{\lambda | U_\lambda \cap U_\mu} = v_{\mu | U_\lambda \cap U_\mu}$. This implies that $(v^*_{w_\lambda})_{| U_\lambda \cap U_\mu} = (v^*_{w_\mu})_{| U_\lambda \cap U_\mu}$ for all $* \in TR$. Thus $w_\lambda(\alpha^*) \in D$ iff $w_\mu(\alpha^*) \in D$ for all $*$ and all $\alpha \in F_\Sigma(U_\lambda \cap U_\mu)$. Suppose (without loss of generality) that there exists $p \in U_\lambda \cap U_\mu$ such that $w_\lambda(p) = \mathbf{T}$ and $w_\mu(p) = \mathbf{T}^-$. Let $* \in TR$ such that $(\neg p)^* = \neg_c p$, where $\neg_c$ is one of the negations of W_3 (cf. (Carnielli (forthcoming)) or (Marcos (1999))). Consequently

$$w_\lambda((\neg p)^*) = \neg_c w_\lambda(p) = \neg_c \mathbf{T} = \mathbf{F},$$

$$w_\mu((\neg p)^*) = \neg_c w_\mu(p) = \neg_c \mathbf{T}^- = \mathbf{T}^-,$$

a contradiction. Thus $w_{\lambda | U_\lambda \cap U_\mu} = w_{\mu | U_\lambda \cap U_\mu}$ and thus the map $w(p) = w_\lambda(p)$ if $p \in U_\lambda$, is well-defined over $U = \bigcup_{\lambda \in \Lambda} U_\lambda$. Now we can extend w to a valuation $w \in \mathcal{O}_3(U)$ such that $w_{|U_\lambda} = w_\lambda$ for each $\lambda \in \Lambda$. Then, $v = \langle v^*_w \rangle_{* \in TR}$ is a valuation in $\mathcal{O}_{lim}(U)$ such that $v_{|U_\lambda} = v_\lambda$ for each $\lambda \in \Lambda$. Thus, $\mathcal{O}_{lim}$ is a sheaf. Finally, we observe that, for all $U \subseteq \mathbb{V}$ and $\alpha \in F_\Sigma(U)$,

$$\models_{\mathcal{O}_{lim}} \alpha \quad \text{iff} \quad (\forall\, w)(\forall\, *)(v^*_w(\alpha) = 1) \quad \text{iff} \quad (\forall v)(v(\alpha) = 1)$$

$$\text{iff} \quad \models_{\mathcal{O}_1} \alpha.$$

However $\models_{\mathcal{O}_1} \alpha$ iff α is a theorem of C_1, by the adequacy of the semantics of C_1-valuations. This completes the proof. □

This shows that the possible-translations framework for C_1 is represented as a limit in $\mathcal{L}_{\mathbb{V}}$. On the other hand, we can obtain the whole logic $\mathcal{O}_1$ (including deductions) as a colimit.

Proposition 6.3 Consider, for each $*$ and w, the sheaf $\mathcal{O}^*_w$ given by $\mathcal{O}^*_w(U) = \{(v^*_w)_{|U}\}$. Let $\mathcal{D}$ be the diagram

$$\mathcal{D} = \langle \{\mathcal{O}^*_w \mid * \in TR,\ w \in \mathcal{O}_3(\mathbb{V})\}, \emptyset \rangle.$$

Then $\mathcal{O}_1 = colim\ \mathcal{D}$, that is: $\mathcal{O}_1 = \coprod_{*,w} \mathcal{O}^*_w$.

Proof: Straightforward. $\square$

7 THIRD EXAMPLE: THE LIMIT OF C_n $(n \in \omega)$

The results in the previous sections about the possibility of representing possible-translations semantics by means of categorial constructions can be generalized to the whole hierarchy of paraconsistent logics C_n. On the other hand we can use possible-translations semantics to define a limit C_{min} of the C_n $(n \in \omega)$ in the directium of what we have called splicing logics. This example has been proposed in (Marcos & Carnielli (manuscript)). The calculus C_{min} is a kind of minimal paraconsistent logic which preserves positive classical logic and can be characterized by a simple semantics of valuations and by means of a possible-translation semantics $\langle \mathcal{W}, TR_0 \rangle$ (for further details see (Marcos & Carnielli (manuscript))). Using the same methods as in the last section we can prove that the logic $\mathcal{O}_{\mathcal{W}}$ associated to $\mathcal{W}$ is an extensional sheaf, and

$$\mathcal{O}_{min}(U) = \{v : F_{\Sigma}(U) \longrightarrow \mathbf{2} \mid v \text{ is a } min\text{-valuation}\}$$

is a sheaf. Let's consider the sublogics of $\mathcal{O}_{min}$

$$\mathcal{O}_*(U) = \{v \in \mathcal{O}_{min}(U) \mid v = v^*_w \text{ for some } w \in \mathcal{O}_{\mathcal{W}}(U)\}.$$

For each $* \in TR_0$ and $U \subseteq \mathbb{V}$ consider the map

$$(\varphi^*)_U : \mathcal{O}_{\mathcal{W}}(U) \longrightarrow \mathcal{O}_*(U), \quad (\varphi^*)_U(w) = v^*_w.$$

It is clear that this produces a morphism $\varphi^* : \mathcal{O}_{\mathcal{W}} \longrightarrow \mathcal{O}_*$ for each $* \in TR_0$.

Proposition 7.1 The diagram $\mathcal{D} = \langle \{\mathcal{O}_{\mathcal{W}}\} \cup \{\mathcal{O}_*\}_{*\in TR_0}, \{\varphi^*\}_{*\in TR_0}\rangle$ has a limit $\mathcal{O}_{lim}$. The limit $\mathcal{O}_{lim}$ is a sheaf, and $\mathcal{O}_{lim}$ validates the same formulas which are valid in C_{min}, that is: $\models_{\mathcal{O}_{lim}} \alpha$ iff $\models_{C_{min}} \alpha$. $\square$

Proposition 7.2 Consider, for each $*$ and w, the sheaf $\mathcal{O}^*_w$ given by $\mathcal{O}^*_w(U) = \{(v^*_w)_{|U}\}$. Let $\mathcal{D}$ be the diagram

$$\mathcal{D} = \langle \{\mathcal{O}^*_w \mid * \in TR_0, \; w \in \mathcal{O}_{\mathcal{W}}(\mathbb{V})\}, \emptyset\rangle.$$

Then $\mathcal{O}_{min} = colim\,\mathcal{D}$, that is: $\mathcal{O}_{min} = \coprod_{*,w} \mathcal{O}^*_w$. $\square$

Finally, we recall that the limit C_{lim} of the hierarchy C_n ($n \in \omega$) is characterized in terms of theories by

$$Th(C_{lim}) = \bigcap_{n\in\omega} Th(C_n)$$

(cf. (Marcos & Carnielli (manuscript))). It is interesting to note that this system C_{lim} has not yet been defined axiomatically, but even so can be characterized naturally in our categorial approach. Observe that, if $\mathcal{O}_n$ is the sheaf corresponding to C_n ($n \in \omega$) then $\mathcal{O}_n \overset{\varphi^m_n}{\hookrightarrow} \mathcal{O}_m$ if $n \leq m$, where each φ^m_n is the inclusion morphism. By proposition **4.11** we obtain:

Corollary 7.3 The logic $\mathcal{O}_\infty$ associated to C_{lim} is the colimit of the diagram $\langle \{\mathcal{O}_n\}_{n\in\omega}, \{\varphi^m_n\}_{n \leq m}\rangle$. The logic $\mathcal{O}_\infty$ is a finite sheaf. $\square$

This shows that the conception of C_{lim} as a limit of theories (in a certain sense) is adequate, because the categorial counterpart of this fact is precisely a colimit.

8 CONCLUDING REMARKS

The results and examples we have obtained strongly suggest the interest of using categorial constructions and methods of the sheaf theory as conceptual tools for combining logical systems in general. As explained in the introduction, our principal motivation in this paper is to define the most abstract setting where the fruitful idea of possible-translations semantics can be generalized.

Many problems and questions are still to be explored, and further development to be done, but our examples highlight the interest of this approach. We have not yet treated, for instance, the question of computing products and coproducts of logics with heterogeneous signatures, neither quantified languages. Those seem to be natural questions to be investigated.

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