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INTUITIONISTIC EQUIVALENCE

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Dedicated to O. Chateaubriand

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It appears to be a characteristic of Western Philosophy that it always searches for a small number of primitives with which to explain everything. Sometimes this was carried to extremes, as was the case in Parmenides' "One"[1]. Occasionally success is

[1] Which seems to be returning to Cosmology; see [Fer96].

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achieved for a while; for example consider when it was theorized that the Cosmos could be reduced to *protons, neutrons* and *electrons*; or in the case of Mathematics, when it was thought that it could be reduced to Set Theory with just a binary relation "ε". In the field of Logic similar events took place.

Once the primitives have been reduced to a small finite number, then the question arose whether it could be done with just "one"; in the case of Logic that meant one axiom, one rule, one connective. Sometimes the problem, although interesting in its own right, was, together with its solution, quickly forgotten[2]. On the other hand, the reduction to a *small* finite number has often led to either new fields or to simplifications in other areas. An example of the latter is Prawitz-Russell-Scott discovery that the traditional intuitionistic connectives and quantifiers could be generated from the second order universal quantifier and the conditional[3]. The reduction in the number of logical particles made it then possible to give much simpler formulations of semantical notions as well as obtaining new proof-theoretical results and interpretations. For example, the Prawitz-Russell-Scott result justified the common observation that the conditional in intuitionistic logic has a much more important role than, say, conjunction or disjunction [4].

One could then further reduce the number of logical particles by combining the universal quantifier and the conditional into a version of Russell's *material implication* and thus it could be claimed that it had been reduced to just *one* logical particle. However that result could (and would) easily be forgotten since it only made the semantical and proof-theoretical concepts

[2] A formulation of the classical propositional calculus with one axiom and one rule of inference was given in by Nicod, see Wajsberg's [Waj00], in 1917.

[3] [Sco79], [Pra65].

[4] In more than one instance, when measuring the complexity of a formula it has been advantageous to consider the conditional on the par with the quantifiers.

more complicated, adding very little to the understanding of the subject.

Now there is a propositional connective which has had a long and meritorious history, namely *equivalence*. Amongst its many important roles, is its role in *definitions*. The founders of modern logic (*i.e. mathematical logic*) placed a great deal of importance in definitions and thus it is not surprising that the question arose whether the "conditional" could not be replaced by "equivalence" in classical logic. The question was posed first by Stanisław Leśniewski and answered by Alfred Tarski. In his paper "On the primitive term of logistic"[5], which constitutes an essential part of his doctoral dissertation submitted to the University of Warsaw in 1923, Tarski shows that it is possible:

> to construct a system of logistic in which the sign of equivalence is the only primitive sign (in addition of course to the quantifiers).

The quantifiers used in "On the primitive term of logistic" were quantifiers over propositions and propositional functions[6]. At the beginning of the century, the Polish School were obtaining fundamental results in many areas of Logic, in particular in the traditional propositional calculus as well as in the extended propositional calculus (that is; allowing quantifiers over propositions)

Jan Łukasiewicz, Stanisław Leśniewski and Alfred Tarski were amongst the first contributors to that field. The works of Łukasiewicz and Tarski are well known, however the work of Leśniewski is relatively unknown outside of Poland. The reasons why Leśniewski's work has not had a wider acceptance are varied. Much of it was lost during the 1944 fire of Warsaw and the

[5] Translated in [Tar56], also appearing as "Sur le terme primitif de la logique", [Tar23b], and as "O wyrazie pierwotnym logistyki", [Tar23a].

[6] Concepts introduced by Bertrand Russell in [Rus03].

few articles that have survived are in Polish Journals not readily available outside of Poland. Fortunately the Ballieu Library and the Philosophy Department of the University of Melbourne have formed the *Leśniewski Collection*. This collection comprises all materials published by Stanisław Leśniewski during his lifetime, and some unpublished materials in their original languages[7].

All these developments came about before Gödel had shown that *truth* and *derivation* need not be co-extensive. Consequently the *axioms* for the sentential connectives were determined purely on their truth-value interpretation; furthermore only two truth values were considered[8]. On the whole *derivations* were considered as secondary objects; they were just a means to discover the *truths*. Almost universally, the only rule of inference was *modus ponens*. The particular choice of axioms was motivated mainly by aesthetical considerations (the least, shortest etc.).

After Gödel's incompleteness theorems it became evident that *derivations* could, and should, be studied on their own right. Gentzen's systems of Natural Deduction showed that *pure logic* could be adequately analyzed through the use of various *rules of inference* rather than relying on cleverly chosen *axioms*. In fact Gentzen went much further and showed that the Natural Deduction Systems could be set up so that each *logical atom* (i.e. each connective, quantifier etc.) had its own set of rules (in which no other logical atom was explicitly mentioned); in addition he observed that the rules of inference for the logical atoms could be separated into two types. One type, now traditionally called an Introduction-rule, acted as a *definition*[9] of the logical atom; the

[7] The works of Leśniewski are being translated into English and published in the Nijhoff International Philosophy Series by Kluwer Academic Publishers, (see the Introduction to [SS88]).

[8] Łukasiewicz introduced the many valued logics at more or less the same time.

[9] That is, gave sufficient conditions to derive a formula with the logical atom as principal connective (quantifier).

other type, called an Elimination-rule, gives sufficient conditions in order to infer from a formula with the logical atom.

Another event that took place at the turn of the century that reduced the over-emphasis on *truth-values* was L. E. J. Brouwer's *Intuitionistic Mathematics.* Now although Brouwer insisted that his interest lay in the study of Mathematics and not in the study of this or that particular logic, Arend Heyting observed that in intuitionistic mathematics the linguistic expressions *and, or, for all,* etc., were being used with certain regularities[10]. Thus in spite of Brouwer's dislike of formal logic, there arose a well defined "*Intuitionistic Logic*" and in particular an "*Intuitionistic Propositional Calculus*". Gentzen's investigations on logical inference include intuitionistic logic – the *NJ* and *LJ* systems – and in fact he found that as far as the Natural Deduction Systems were concerned the intuitionistic inference was far more amenable to his analysis in that he was able to derive his famous *Haupsatz* for *NJ*[11]. With 20/20 hindsight this is not surprising since both intuitionism and Gentzen's systems place much more importance on *proofs* than on *truth-values*; although in intuitionism the proofs are represented in the (ideal) mathematician's mind, while in Gentzen's systems the proofs are represented by finite trees of formulas.

Eventually it came to be recognized[12] that the abstract notion of proof could be made the subject of mathematical analysis, just as it had occurred with the concept of set. Consequently in the middle of the century different theories of *constructions* were put forward[13]; the aim usually being to show that a sentence is

[10]For the history of the formalization the reader is recommended to read Troelstra's [Tro89] and Ruitenburg's [Rui91].

[11]And 30 years passed before D. Prawitz, in [Pra65], obtained it for the classical system *NK*.

[12]Principally because of the many articles by G. Kreisel supporting that view.

[13]See, for example, Goodman's [Goo70], Laüschi's [Läu70] and Scott's

an intuitionistic theorem iff there is a(n abstract) construction justifying it.

Now although the idea of an abstract proof arose in the intuitionistic mathematics, it need not be so restricted; it can be used whenever one is more interested in a *dynamic* rather than *static* viewpoint[14]. And even in classical mathematics, the problem of the *identity of proofs* is wide open; in particular there are conflicting views between *proofs* and their linguistic representations as *derivations*.

1 Constructive Equivalence

Tarski had shown that a considerable portion of classical logic could be based on the classical propositional connective of equivalence "≡" (and suitable quantifiers). The original equivalence connective (used by Leśniewski and Tarski) had its roots in the two valued truth table; however in this manuscript we base it upon an abstract concept of construction.

Since we wish to be sure that the discussion can be carried out in a constructive metatheory (and will so do on many occasions) we start by considering a subtheory of the propositional minimal calculus whose only connective is "≡" and name it the **M**inimal **E**quivalence **C**alculus; **MEC**.

For the *propositional parameters*[15] of **MEC** we will use: $p, q, r, \ldots$. The *formulas* of **MEC** are defined so that they are either propositional parameters or expressions of the form $(\mathcal{A} \equiv \mathcal{B})$, where $\mathcal{A}$, $\mathcal{B}$ are themselves formulas.

The crucial step, which determines the nature of the logic, is

[Sco69].

[14] A viewpoint shared by the Categorists, see Lawvere's [Law75].

[15] Sometimes referred to as atomic formulas

the understanding of when one may *assert* a formula of the form $(\mathcal{A} \equiv \mathcal{B})$. Following the Brouwer-Heyting-Kolmogorov interpretation (also known as the *BHK interpretation*. In the authors' view it would be more accurate to call it the "BHK^2 interpretation" because of the many contributions of Kreisel, for example, [Kre62]) we will assert such a formula only when there is an abstract proof (a.k.a. construction) that justifies it. Keeping in mind the view of equivalence as a biconditional, one argues that a *construction c proves* or *justifies* a formula $(\mathcal{A} \equiv \mathcal{B})$ when c consists of a pair of constructions (c_1, c_2) where c_1 takes any proof of $\mathcal{A}$ into a proof of $\mathcal{B}$ and c_2 takes a proof of $\mathcal{B}$ into a proof of $\mathcal{A}$ Note also that one could put other additional conditions, *e.g.* that c_1 and c_2 be in some sense *equivalent* (equal complexity, similar structure, built up from the same components and so on).

The simplest formalization[16] to represent such an interpretation is one in the style of Gentzen's *NJ*. That is, the derivations are to be trees of formulas. The formulas at the top of the tree are *assumptions* and (some) of the rules of inference may *close* or *discharge* assumption formulas. As Gentzen himself observed, the beauty of his Natural Deduction Systems is that the rules of inference deal explicitly with one logical atom at a time. Furthermore the rules on inference for any given logical atom can be partitioned into two classes, one that acts as a *definition* (called **I**-rules) and the other which gives sufficient conditions for drawing inferences from formulas containing the logical atom (called **E**-rules).

There is only one **I**-rule of inference. It *defines*[17] the propositional connective $\equiv$ and it represents a possible reading of the above interpretation using a pair of constructions:

[16] But by no means the only one. Nevertheless we find it to be the simplest one that does not explicitly involve terms from a construction calculus.

[17] In the sense of Gentzen's [Gen36].

$$\frac{\begin{matrix}[\mathcal{F}] \\ \mathcal{C}\end{matrix} \qquad \begin{matrix}[\mathcal{C}] \\ \mathcal{F}\end{matrix}}{(\mathcal{F} \equiv \mathcal{C})}$$

(We are following the convention of enclosing within "[ ]" the formula occurrences discharged by the rule.)

In Gentzen's [Gen36] it is stated it should be possible to obtain the **E**-rules from the **I**-rule:

> Durch Präzisierung dieser Gedanken dürfte es möglich sein, die B-Schlüsse auf Grund gewisser Anforderung als eindeutige Funktionen der zugehöringen E-Schlüsse nachzuweisen.

In López-Escobar's [Lóp95] a method is given on how the **E**-rules can be effectively obtained from the **syntactical form** of the **I**-rule. Following the algorithm one would obtain the following two rules:

$$\frac{(\mathcal{F} \equiv \mathcal{C}) \quad \mathcal{F} \quad \begin{matrix}[\mathcal{C}] \\ \mathcal{B}\end{matrix}}{\mathcal{B}} \qquad \text{and} \qquad \frac{(\mathcal{F} \equiv \mathcal{C}) \quad \mathcal{C} \quad \begin{matrix}[\mathcal{F}] \\ \mathcal{B}\end{matrix}}{\mathcal{B}}$$

However, since we are interested in a *specific* Natural Deduction System—instead of obtaining results about many diverse systems—the above rules suggest the following simpler **E**-rules for the connective $\equiv$:

$$\frac{(\mathcal{F} \equiv \mathcal{C}) \qquad \mathcal{F}}{\mathcal{C}} \qquad \text{and} \qquad \frac{(\mathcal{F} \equiv \mathcal{C}) \qquad \mathcal{C}}{\mathcal{F}}$$

In other words, the biconditional form of *modus ponens*. The formula explicitly mentioning the logical atom $\equiv$ is known as the *major premise*. An important characteristic of the biconditional forms of modus ponens is that the conclusion (of an application) of the rule is a proper subformula of the major premise.

It should perhaps be emphasized that **intuitionistic equivalence is not associative**, as is the case for classical propositional calculus. In fact, an axiomatization of classical equivalence can be obtained by adjoining a rule that guarantees associativity.

If Γ is a set of formulas and $\mathcal{B}$ is a formula then by

$$\Gamma \vdash \mathcal{B}$$

is to be understood that there is a formal derivation Π of the formula $\mathcal{B}$ in which the formulas which have an open assumption occurrence in Π belong to Γ.

The following observations express the principal properties of the **MEC** derivability relation. We call them propositions not because they are in any way difficult to prove but rather because they could be used as a starting point for other logics. We write them in a "sequent like" manner

Proposition 1.1 (Finiteness) *If* $\Gamma \vdash \mathcal{B}$*, then there is* $\Delta \subseteq_{Finite} \Gamma$ *such that* $\Delta \vdash \mathcal{B}$*.*

Proposition 1.2 (Monotonicity) $\dfrac{\Gamma \vdash \mathcal{B}}{\Gamma, \Delta \vdash \mathcal{B}}$

Proposition 1.3 (Transitivity)

$$\frac{\Gamma \vdash \mathcal{B} \qquad\qquad \Delta, \mathcal{B} \vdash \mathcal{F}}{\Gamma, \Delta \vdash \mathcal{F}}$$

Proposition 1.4 (Deduction Theorem)

$$\frac{\Gamma, \mathcal{F} \vdash \mathcal{C} \qquad\qquad \Delta, \mathcal{C} \vdash \mathcal{F}}{\Gamma, \Delta \vdash (\mathcal{F} \equiv \mathcal{C})}$$

Proposition 1.5 (Modus Ponens)

$$\frac{\Gamma \vdash (\mathcal{F} \equiv \mathcal{C}) \qquad\qquad \Delta \vdash \mathcal{F}}{\Gamma, \Delta \vdash \mathcal{C}}$$

Using the above tools one can obtain the following results about the system (we shall omit parentheses when there is no risk of confusion and "$\top$" is an abbreviation for "$(p\equiv p)$"):

Theorem 1.6

1. $\vdash (\mathcal{F}\equiv\mathcal{C})\equiv(\mathcal{C}\equiv\mathcal{F})$.
2. $\mathcal{F}\equiv\mathcal{C}, \mathcal{C}\equiv\mathcal{B} \vdash \mathcal{F}\equiv\mathcal{B}$.
3. $\vdash \mathcal{F}\equiv(\top\equiv\mathcal{F})$.
4. $\mathcal{F}_1\equiv\mathcal{A}_1, \mathcal{F}_2\equiv\mathcal{A}_2 \vdash (\mathcal{F}_1\equiv\mathcal{F}_2)\equiv(\mathcal{A}_1\equiv\mathcal{A}_2)$.
5. $\mathcal{F}, \mathcal{A} \vdash (\mathcal{F}\equiv\mathcal{A})$.
6. $\mathcal{F}, \mathcal{A} \vdash ((\mathcal{F}\equiv\mathcal{D})\equiv(\mathcal{A}\equiv\mathcal{D}))$.
7. $(\mathcal{F}\equiv\mathcal{A})\equiv\mathcal{D}, \mathcal{F} \vdash (\mathcal{A}\equiv\mathcal{D})$.

If q is a propositional parameter and $\mathcal{F}$ and $\mathcal{B}$ are formulas then $[q=\mathcal{B}]\mathcal{F}$ is the *expression* obtained by replacing *all* occurrences of q in $\mathcal{F}$ by $\mathcal{B}$ and it is called the **replacement of q by $\mathcal{B}$**. In the case of **MEC**, since there are no (bound) variables, $[q=\mathcal{B}]\mathcal{F}$ is also a *formula* and thus we say that it is the formula obtained by **substitution of q by $\mathcal{B}$ in $\mathcal{F}$**. We write the *substitution* as '$\mathcal{F}\ulcorner q/\mathcal{B}\urcorner$'. If the q is obvious from context then we may simply write "$\mathcal{F}\ulcorner\mathcal{B}\urcorner$". On the other hand, if we wish to call attention to the propositional parameter q, then we may write "$\mathcal{F}\ulcorner q\urcorner$" instead of "$\mathcal{F}$". Correspondingly for *simultaneous substitutions*: $\mathcal{F}\ulcorner\mathcal{D}_1,\ldots,\mathcal{D}_k\urcorner$.

Proposition 1.7 (Substitutivity)

$$\mathcal{D}_1\equiv\mathcal{B}_1,\ldots,\mathcal{D}_k\equiv\mathcal{B}_k,\ \mathcal{F}\ulcorner\mathcal{D}_1,\ldots,\mathcal{D}_k\urcorner \vdash \mathcal{F}\ulcorner\mathcal{B}_1,\ldots,\mathcal{B}_k\urcorner.$$

Proof : By induction on the complexity of the formula $\mathcal{F}$. It is immediate for atomic formulas and for compound formulas it follows from 1.6. □

Corollary 1.8 $\mathcal{B}{\equiv}\mathcal{D} \vdash \mathcal{F}\ulcorner\mathcal{B}\urcorner{\equiv}\mathcal{F}\ulcorner\mathcal{D}\urcorner$.

We extend the notation for substitution to sets of formulas; that is "$\Gamma\ulcorner q\urcorner$" and "$\Gamma$" represent the same set, while "$\Gamma\ulcorner q/\mathcal{B}\urcorner$" and "$\Gamma\ulcorner\mathcal{B}\urcorner$" represent the set of formulas obtained by substituting in each formula in Γ the formula $\mathcal{B}$ for all the occurrences of the propositional parameter q.

Proposition 1.9 (Invariance) *If* $\Pi\ulcorner q\urcorner$ *is a derivation of* $\mathcal{F}\ulcorner q\urcorner$ *from* $\Gamma\ulcorner q\urcorner$*, then for all formulas* $\mathcal{B}$*:* $\Pi\ulcorner\mathcal{B}\urcorner$ *is a derivation of* $\mathcal{F}\ulcorner\mathcal{B}\urcorner$ *from* $\Gamma\ulcorner\mathcal{B}\urcorner$.

Proof : This time the proof is on the length of the *derivation*. □

Corollary 1.10 *If* $\Gamma\ulcorner q\urcorner \vdash \mathcal{F}\ulcorner q\urcorner$ *then for all formulas* $\mathcal{B}$*:* $\Gamma\ulcorner\mathcal{B}\urcorner \vdash \mathcal{F}\ulcorner\mathcal{B}\urcorner$.

In view of the invariance theorem, we shall give, whenever convenient, the results in terms of the propositional variables.

Let us now **add** to the propositional language of **MEC** a propositional constant for *intuitionistic absurdity*, "$\perp$". The essence of intuitionistic absurdity is that all formulas are derivable from $\perp$. It turns out that it suffices to have all the atomic formulas derivable from $\perp$; hence the following rule of inference:

Rule for Intuitionistic Absurdity $\dfrac{\perp}{\mathcal{A}}$

where $\mathcal{A}$ is an atomic formula other than $\perp$ (thus in the present context, a propositional parameter).

The extension of **MEC** obtained by adding $\bot$ to the language (as an atomic formula) and the rule of intuitionistic absurdity to the rules of inference will be called *Intuitionistic Equivalence*, in symbols: **MECn**. A simple induction on the complexity of the formula gives us:

Lemma 1.11 *For every formula $\mathcal{F}$ of* **MECn**, $\bot \vdash_{MECn} \mathcal{F}$.

Abbreviation: We will abbreviate "$(\mathcal{F} \equiv \bot)$" by "$\neg\mathcal{F}$".

The following results contain some of the basic inferences in **MECn**, where "$\vdash$" is to represent "$\vdash_{\text{MECn}}$".

Lemma 1.12 *a)* $\vdash \neg\bot$ *and* $\vdash (\neg\bot \equiv \top)$.

b) $\vdash (\neg p \equiv \neg q) \equiv (\neg\neg p \equiv \neg\neg q)$.

c) $\vdash (\neg p \equiv \neg\neg q) \equiv (\neg q \equiv \neg\neg p)$.

Making use of the fact that from $\bot$ any formula may be derived we obtain the following:

Theorem 1.13 *If* $\Gamma, \mathcal{F} \vdash \bot$, *then* $\Gamma \vdash \neg\mathcal{F}$.

Corollary 1.14 *a)* $p, \neg p \vdash q$.

b) If $\Gamma, \mathcal{F} \vdash \mathcal{G}$ *then* $\Gamma, \neg\mathcal{G} \vdash \neg\mathcal{F}$.

c) $\vdash \neg(p \equiv \neg p) \vdash \neg p$.

d) $\vdash \neg(p \equiv \neg p)$.

e) $\vdash \neg(p \equiv q) \equiv \neg(\neg p \equiv \neg q)$.

The following require a little more work:

Lemma 1.15 *a)* $\vdash \neg(p \equiv q) \equiv (\neg p \equiv \neg\neg q)$.

b) $\vdash \neg(p\equiv q)\equiv\neg(\neg p\equiv\neg q)$.

c) $\vdash \neg\neg p\equiv((\neg\neg p\equiv q)\equiv q)$.

d) $\vdash \neg p\equiv\neg(q\equiv(p\equiv q))$.

e) $\vdash \neg\neg p\equiv(\neg q\equiv\neg(p\equiv q))$.

f) $\vdash \neg\neg(p\equiv q)\equiv(\neg\neg p\equiv\neg\neg q)$.

The following theorem could be called **Associativity under negation**:

Theorem 1.16 $\vdash \neg(p\equiv(q\equiv r))\equiv\neg((p\equiv q)\equiv r)$.

We shall present algebraic proofs of 1.15 and 1.16 when we discuss **equivalence algebras with negation** in section 7 (see Proposition 7.4).

2 Heyting and Complete Heyting Algebras

Since the systems we are considering are subsystems of the Extended Intuitionistic Propositional Calculus, we shall characterize *soundness* and *completeness* in latter sections in terms of Heyting and complete Heyting Algebras. The aim of this section is to set down the basic terminology for the topics in the title for the convenience of the reader and latter reference. Proofs of our statements can be found in [BD74] and [FS79].

If L is a partially ordered set and x , $y \in L$, write

* $x^{\rightarrow} = \{y \in L : x \leq y\}$;
* $x^{\leftarrow} = \{y \in L : y \leq x\}$;
* $x \wedge y$ (the meet of x and y) for inf $\{x, y\}$, whenever it exists;
* $x \vee y$ (the join of x and y) for sup $\{x, y\}$, whenever it exists;

* $\top$ and $\bot$ for the largest and smallest element of L, whenever these exist.

A partially ordered set L is a **lattice** if every pair of elements in L has a meet and a join. A lattice is **distributive** iff

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

It is easily verified that this last condition is equivalent to its dual,

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Let L be lattice with $\top$. A subset F of L is a **filter** if $\top \in F$, it is closed under finite meets and $x \in F$ implies $x^{\rightarrow} \subseteq F$. A filter is **proper** iff it is distinct from L. Hence, if L has $\bot$, F is proper iff $\bot \notin F$. It is easily established that the property of being a filter is preserved by arbitrary intersections and directed unions. If $S \subseteq L$, the **filter generated** by S is the intersection of all filters in L that contain S.

Lemma 2.1 *Let L be a lattice with $\top$. Let S be a subset of L. Then, the filter generated by S in L is given by*

$$[S] = \{x \in L : \exists\, a_1, \ldots, a_n \subseteq S, \text{ such that } x \geq a_1 \wedge \ldots \wedge a_n\}.$$

If L has $\bot$, $[S]$ is proper iff S has the finite intersection property (fip), that is, the meet of any finite subset of S is distinct from $\bot$. ◇

A **Heyting algebra (Ha)** is a distributive lattice with $\top$ and $\bot$, H, such that for all $x, y \in H$

[Ha] The set $\{z \in H : x \wedge z \leq y\}$ has a *maximum* in H.

We write

$$x \rightarrow y =_{def} \max \{z \in H : x \wedge z \leq y\},$$

called the **implication operation** in H. Hence, for all x, y,

$z \in H$,

$$[\to] \qquad x \wedge y \le z \;\text{ iff }\; x \le (y \to z).$$

Lemma 2.2 *Let H be a Ha and let $x, y, z \in H$. Then,*

a) $x \le y$ *iff* $x \to y = \top$.

b) $x \wedge (x \to y) = x \wedge y$.

c) $x \wedge (y \to z) = x \wedge ((x \wedge y) \to (x \wedge z))$.

d) If F is a filter in H, then

(i) $x \in F$ and $(x \to y) \in F$ imply $y \in F$.

(ii) $x \in F$ implies $y \to x \in F$.

e) If F is a proper filter in H and $(x \to y) \notin F$, then there is a proper filter G in H, such that $x \in G$ and $y \notin G$.

Proof : We prove only (e). If $y \in [F \cup \{x\}]$, 2.1 yields $t \in F$ such that $x \wedge t \le y$, and the adjointness relation $[\to]$ implies $t \le (x \to y)$. Since $t \in F$, we get $(x \to y) \in F$, a contradiction. Hence, the filter generated by F and x is the proper extension of F separating x and y. □

If x, y are elements of a Ha H, define

$$[\leftrightarrow] \qquad x \leftrightarrow y = (x \to y) \wedge (y \to x),$$

called the **equivalence operation** in H; its basic properties are stated in

Lemma 2.3 *Let H be a Ha and let x, y, z be elements of H. then,*

a) $x = y$ *iff* $x \leftrightarrow y = \top$.

b) $x \le (y \leftrightarrow z)$ *iff* $x \wedge y = x \wedge z$.

c) $x \wedge (y \leftrightarrow z) = x \wedge ((x \leftrightarrow y) \leftrightarrow (y \leftrightarrow z))$.

d) If F is a filter in H,

(i) $x \in F$ and $(x \leftrightarrow y) \in F$ imply $y \in F$.

(ii) $x \in F$ and $y \in F$ imply $(x \leftrightarrow y) \in F$.

e) If F is a filter in H, then

$$(x \leftrightarrow y) \in F \quad \textit{iff} \quad [F \cup \{x\}] = [F \cup \{y\}]$$

f) If F is a proper filter in H and $(x \leftrightarrow y) \notin F$, then there is a proper filter G in H, such that $F \subseteq G$ and $x \notin G$ or $y \notin G$.

Proof : We prove only (e). Suppose that $[F \cup \{x\}] = [F \cup \{y\}]$; by 2.1, there are a, $b \in F$, such that

$$a \wedge x \leq y \quad \text{and} \quad b \wedge y \leq x.$$

But then, $a \wedge b \wedge x = a \wedge b \wedge y$ and so, by item (b), $(a \wedge b) \leq x \leftrightarrow y$, as desired. The converse is immediate from item (i) in (d). □

Recall that a lattice is **complete** if all subsets of L have meets and joins. For $S \subseteq L$, write $\bigvee S$ and $\bigwedge S$ for the join and meet, respectively, of S in L. A **complete Heyting algebra (cHa)** is a complete lattice that satisfies the following distributive law :

$[\wedge, \bigvee]$ For all $a \in L$ and $S \subseteq L$, $a \wedge \bigvee S = \bigvee_{s \in S} a \wedge s$.

Some authors use *frame* or *locale* for what here is called a cHa. Important examples of cHa's are furnished by topologies on any set. If L is a cHa, the implication operation in L is defined by

$$a \to b = \bigvee \{x \in L : x \wedge a \leq b\}.$$

Because L satisfies $[\wedge, \bigvee]$, it is easily established that the defining property of implication, $[\to]$, is verified in L.

3 The Lindenbaum algebras of MEC and MECn

Another tradition started by Lindenbaum, Łukasiewicz, Tarski *et alia*, is the association of algebraic structures to logical theories[18]. This technique has been extensively developed so that it is now possible to construct complete algebraic semantics for many important logics, see for example H. Rasiowa [Ras74] or D. Scott [Sco74]. We shall discuss in this section how these ideas apply to the systems **MEC** and **MECn**.

Let Γ be a set of **MEC** (respectively, **MECn**) formulas.

Definition 3.1 *If $\mathcal{A}$ is a formula of* **MEC** *or* **MECn**, *set*

* $\mathcal{A}_\Gamma = \{\mathcal{B} : \Gamma \vdash (\mathcal{A} \equiv \mathcal{B})\}$.
* $\mathcal{A}_\Gamma \leq \mathcal{B}_\Gamma$ *iff* $\Gamma, \mathcal{A} \vdash \mathcal{B}$.
* $\underline{\Gamma} = \{\mathcal{B}_\Gamma : \mathcal{B}$ *is a propositional formula*$\}$.
* $\top = (p \equiv p)_\Gamma$.
* *In the case of* **MECn**, *we identify* $\bot$ *with* $\overline{\bot}$.

* When $\Gamma = \emptyset$, write $\overline{\mathcal{A}}$ for $\mathcal{A}_\Gamma$.

Using the derivability properties of **MEC** and **MECn**, the following are straightforward :

Lemma 3.2 *With notation as above*

a) $\mathcal{A}_\Gamma = \mathcal{B}_\Gamma$ *iff* $\Gamma \vdash (\mathcal{A} \equiv \mathcal{B})$.

b) $\leq$ *is a partial order on the set* $\underline{\Gamma}$ *and* $\top$ *is its largest element.*

[18] At first this was quite distinct from the *algebrization of Logic*, a method started by G. Boole and continued by Tarski, Halmos *et alia*. However, with the advent of *categories*, the demarcation line has just about disappeared.

c) For **MECn**, $\perp$ *is the smallest element in the partial order* $\leq$.

d) If $\mathcal{A}_{1\Gamma} = \mathcal{A}_{2\Gamma}$ *and* $\mathcal{B}_{1\Gamma} = \mathcal{B}_{2\Gamma}$, *then* $(\mathcal{A}_1 \equiv \mathcal{B}_1)_\Gamma = (\mathcal{A}_2 \equiv \mathcal{B}_2)_\Gamma$.

Because of 3.2.(d), we may define a **non-associative binary operation $*$ on $\underline{\Gamma}$** by :

$$\mathcal{A}_\Gamma * \mathcal{B}_\Gamma = (\mathcal{A} \equiv \mathcal{B})_\Gamma.$$

Note that because neither in **MEC** nor in **MECn** do we have a propositional connective for the *conditional*, there is a fundamental distinction in $\underline{\Gamma}$ between $\leq$ and $*$. Thus, we consider it appropriate that Lindenbaum algebras include both of them.

Definition 3.3 *With notation as above,*

a) The Lindenbaum algebra *of a set* $\Gamma \subseteq$ **MEC** (**MECn**) *is the structure* $\mathcal{L}_\Gamma = \langle \underline{\Gamma}, \leq, \top, * \rangle$, *(resp.,* $\mathcal{L}_{n\Gamma} = \langle \underline{\Gamma}, \leq, \top, \perp, * \rangle$*).*

b) The **Lindenbaum algebra of MEC** (**MECn**), $\mathcal{L}$ *(resp.,* $\mathcal{L}_n$*), is the Lindenbaum algebra of the empty set of formulas.*

From the derivability properties of **MEC** and **MECn**, including the results on negation (1.15), one can show

Lemma 3.4 *The Lindenbaum algebra* $\mathcal{L}_\Gamma$ *of a* **MEC** (**MECn**) *set of formulas* Γ *satisfies the following rules :*

(1) $\leq$ *is a partial order in* $\underline{\Gamma}$ *with* $\top$ *as its largest element and, in the case of* **MECn**, *with* $\perp$ *as its least element.*

(2) $*$ *is a binary operation on* $\underline{\Gamma}$ *such that, universally in* $\underline{\Gamma}$

$[* \; 1]$: $x * y = y * x$.

$[* \; 2]$: $x * \top = x$.

$[* \; 3]$: $x * y = \top$ *iff* $x = y$.

$[* \; 4]$: *If* $a \leq x * y$ *and* $a \leq b * c$, *then,* $a \leq (x * a) * (y * c)$.

$[* \, 5] : x * y = ((x * y) * y) * y.$

$[* \, 6]$: *If* $x \leq y \leq z$ *then,* $x * z \leq x * y$.

$[* \, 7] : (x * y) * z \leq [x * z] * [(y * z) * z].$

(3) *In the case of* **MECn**, *we also have*

$[neg] : \bot * (x * \bot) = \bot.$

The remainder of this section is dedicated to the proof of Theorem 3.9, that will be important for the completeness result of the next chapter (Theorem 4.4). If Λ is a set of formulas in **MEC** or **MECn**, write

$$\overline{\Lambda} = \{\overline{\mathcal{A}} : \mathcal{A} \in \Lambda\}.$$

The relation of provability in **MEC** and **MECn** induces a relation from $2^{\mathcal{L}}$ to $\mathcal{L}$, indicated by the same symbol, and defined as follows:

For $S \cup x \subseteq \mathcal{L}$ (or $\mathcal{L}_n$),

$$S \vdash x \quad \text{iff} \quad \begin{cases} \text{There is } \Lambda \cup \{\mathcal{A}\} \subseteq \mathbf{MEC} \text{ (resp., } \mathbf{MECn}) \\ \text{such that } \overline{\Lambda} \subseteq S, \overline{\mathcal{A}} = x \text{ and } \Lambda \vdash \mathcal{A}. \end{cases}$$

Definition 3.5 *A subset* T *of* $\mathcal{L}$ *(resp.,* $\mathcal{L}_n$*) is a* **theory** *iff it closed under* $\vdash$ *, that is, if* $T \vdash x$*, then* $x \in T$*. A theory is proper if* $T \neq \mathcal{L}$ *(resp.,* $\mathcal{L}_n$*).*

Note that a theory in $\mathcal{L}_n$ is proper iff $\bot \notin T$.

Proposition 3.6 *With notation as above, let* T *be a theory in* $\mathcal{L}$ *(resp.,* $\mathcal{L}_n$*) and let* $x, y, z \in \mathcal{L}$ *(resp.,* $\mathcal{L}_n$*).*

a) (i) $\top \in T$;

(ii) $x * y, y * z \in T$ *implies* $x * z \in T$.

(iii) $x, x * y \in T$ *implies* $y \in T$.

b) For all $x \in \mathcal{L}$ *(resp.,* $\mathcal{L}_n$*),* $x^{\rightarrow}$ *is a theory.*

c) The property of being a theory is preserved under arbitrary intersections and directed unions.

d) For $U \subseteq \mathcal{L}$ *(resp.,* $\mathcal{L}_n$*), let* U^t *be the intersection of all theories containing* U *(the theory generated by* U*). Then*

$$U^t = \{x \in \mathcal{L} : U \vdash x\},$$

and analogously for $\mathcal{L}_n$*. In particular,* $\{x\}^t = x^{\rightarrow}$*, for all* $x \in \mathcal{L}$ *(resp.,* $\mathcal{L}_n$*).*

e) If $U \subseteq \mathcal{L}$ *(resp.,* $\mathcal{L}_n$*), then*

$$(U \cup \{x\})^t = (U \cup \{y\})^t \quad \textit{iff} \quad x * y \in U^t.$$

f) The operation of taking theories satisfies the following properties, where $U, V \subseteq \mathcal{L}$ *(resp.,* $\mathcal{L}_n$*) :*

- $*$ $U \subseteq U^t$ *(inflationary);*
- $*$ $U \subseteq V$ *implies* $U^t \subseteq V^t$ *(increasing);*
- $*$ $(U^t)^t = U^t$ *(idempotent).*

Proof : Items (a) and (b) are straightforward. For (c), let T_i, $i \in I$, be theories and write $T = \bigcap_{i \in I} T_i$. If $T \vdash x$, there is $\Lambda \cup \{\mathcal{A}\} \subseteq$ **MEC** such that $\overline{\Lambda} \subseteq T$, $\overline{\mathcal{A}} = x$ and $\Lambda \vdash \mathcal{A}$. From this, it follows immediately that $x \in T_i$, for all $i \in I$, and so, $x \in T$.

Now suppose that $\{T_i : i \in I\}$ is up-directed and $T \vdash x$, with $T = \bigcup_{i \in I} T_i$. Thus, there is $\Lambda \cup \{\mathcal{A}\} \subseteq$ **MEC**, such that $\overline{\Lambda} \subseteq T$, $\overline{\mathcal{A}} = x$ and $\Lambda \vdash \mathcal{A}$. By the compactness of **MEC**, there is a finite $\Gamma \subseteq \Lambda$, such that $\Gamma \vdash \mathcal{A}$ in **MEC**. Since Γ is finite, $\overline{\Gamma} \subseteq T_i$, for some $i \in I$. But then, $x \in T_i \subseteq T$, as desired.

d) It is sufficient to show that the right-hand side of the equality is a theory. Write $T = \{x \in \mathcal{L} : U \vdash x\}$ and assume that $T \vdash y$. As above, there is $\Lambda \cup \{\mathcal{A}\} \subseteq$ **MEC**, such that $\overline{\Lambda} \subseteq T$,

$\overline{\mathcal{A}} = x$ and $\Lambda \vdash \mathcal{A}$ in **MEC**. Let Σ be a set of formulas such that $\overline{\Sigma} \subseteq U$ and $\Sigma \vdash \Lambda$. But then, in **MEC**,

$$\Sigma \vdash \Lambda \quad \text{and} \quad \Lambda \vdash \mathcal{A},$$

and so transitivity of proof yields $\Sigma \vdash \mathcal{A}$, i.e., $x \in T$, as desired.

e) By (b) above, the left-hand side of the equivalence implies its right-hand side. For the converse, the hypothesis means that U, $x \vdash y$ and U, $x \vdash y$. It follows easily from the definition of $\vdash$ in $\mathcal{L}$ and the $\equiv$ introduction rule that $U \vdash x * y$, as needed. Item (f) is left to the reader.

Clearly, all of the above arguments will also work for $\mathcal{L}_n$. □

Definition 3.7 *Let S be a set, P a subset of S and x, y be distinct elements of S. We say that P* **separates x and y** *if both P and its complement have non-empty intersection with $\{x, y\}$, that is,*

$$\textit{Either } (x \in P \textit{ and } y \notin P) \quad \textit{or} \quad (y \in P \textit{ and } x \notin P).$$

A collection $\mathcal{U}$ of subsets of S **separates points** *in S iff all distinct points in S can be separated by elements of $\mathcal{U}$.*

One of the most important properties of theories is described by

Corollary 3.8 (*Separation*) *Let T be a proper theory in $\mathcal{L}$ or $\mathcal{L}_n$. If $x, y \in \mathcal{L}$ (or $\mathcal{L}_n$) are such that $x * y \notin T$, then there is a proper theory that extends T and separates x and y.*

Proof : By 3.6.(e) we have $(T \cup \{x\})^t \neq (T \cup \{y\})^t$, that is, either

$$x \notin (T \cup \{y\})^t \quad \text{or} \quad y \notin (T \cup \{x\})^t.$$

If the first alternative holds, $(T \cup \{y\})^t$ is a proper extension of T, separating x and y; if the second alternative holds, then

$(T \cup \{x\})^t$ is the extension of T separating x and y. Clearly, this reasoning also applies to $\mathcal{L}_n$. □

We are now in a position to prove that $\mathcal{L}$ and $\mathcal{L}_n$ embed in a cHa (in fact a topology), in a special way.

Let $\mathcal{T}(\mathcal{L})$ ($\mathcal{T}(\mathcal{L}_n)$) be the set of all proper theories in $\mathcal{L}$ (resp., $\mathcal{L}_n$). For each $x \in \mathcal{L}$ (resp., $\mathcal{L}_n$), set

$$\tau_x = \{T \in \mathcal{T}(\mathcal{L}) : x \in T\},$$

with a similar definition for $\mathcal{L}_n$. Note that $\tau_\top = \mathcal{T}(\mathcal{L})$ and that, in the case of $\mathcal{L}_n$, $\tau_\bot = \emptyset$. We take

$$\mathcal{B} = \{\tau_x : x \in \mathcal{L}\} \cup \{\emptyset\} \quad \text{and} \quad \mathcal{B}_n = \{\tau_x : x \in \mathcal{L}_n\},$$

as a sub-basis for a topology on $\mathcal{T}(\mathcal{L})$ and $\mathcal{T}(\mathcal{L}_n)$, respectively. Thus, a set is open in these spaces iff it can be written as a union of finite intersections of sets of the form τ_x. Let Ω and Ω_n be the topologies on $\mathcal{T}(\mathcal{L})$ and $\mathcal{T}(\mathcal{L}_n)$. Write $\leftrightarrow$ for the notion of equivalence in the cHa's Ω and Ω_n. The statement that follows is for $\mathcal{L}$, but it also holds for $\mathcal{L}_n$, with the same proof.

Theorem 3.9 *With notation as above, the map*

$$\varepsilon : \mathcal{L} \longrightarrow \Omega, \text{ given by, } \varepsilon(x) = \tau_x$$

has the following properties :

(a) For all $x, y \in \mathcal{L}$,
$$\begin{cases} x \le y \text{ iff } \varepsilon(x) \le \varepsilon(y); \\ \varepsilon(x * y) = (\varepsilon(x) \leftrightarrow \varepsilon(y)). \end{cases}$$

In particular, ε is injective. In the case of $\mathcal{L}_n$, the corresponding map takes $\bot$ to $\bot$.

(b) If T is a proper theory in $\mathcal{L}$ and F is the filter generated by T in Ω, then F is a proper filter in Ω and $T = \varepsilon^{-1}(F)$.

Proof : Since for all $x \in \mathcal{L}$, $x^{\rightarrow}$ is a theory (3.6.(b)) and $x^{\rightarrow} \in \tau_x$, we have

$$x \leq y \text{ iff } y^{\rightarrow} \subseteq x^{\rightarrow} \text{ iff } \tau_x \subseteq \tau_y,$$

establishing the first of the properties stated in (a). For the second, note that for all x, $y \in \mathcal{L}$, item (iii) in 3.6.(a) implies that

$$\tau_x \cap \tau_{x*y} = \tau_y \cap \tau_{x*y},$$

and so, by 2.3.(b) we have $\tau_{x*y} \subseteq (\tau_x \leftrightarrow \tau_y)$. To verify equality, it is enough, by 2.3.(b) and the fact that finite intersections of elements of $\mathcal{B}$ are a basis for Ω, to check that if $\{t_1, \ldots, t_n\} \subseteq \mathcal{L}$, then

$$\textstyle\bigcap_{i=1}^{n} \tau_{t_i} \cap \tau_x = \bigcap_{i=1}^{n} \tau_{t_i} \cap \tau_y \quad \text{implies} \quad \bigcap_{i=1}^{n} \tau_{t_i} \subseteq \tau_{x*y}.$$

Let T be a theory in $\mathcal{L}$, such that $\{t_1, \ldots, t_n\} \subseteq T$. If $x * y \notin T$, 3.8 yields a theory S, containing T, and separating x and y. Assume, for example that $x \in S$ and $y \notin S$; then, $S \in \bigcap_{i=1}^{n} \tau_{t_i} \cap \tau_x$, although $S \notin \bigcap_{i=1}^{n} \tau_{t_i} \cap \tau_y$, which is impossible. Hence, $T \in \tau_{x*y}$, completing the proof of (a).

Let T be a theory in $\mathcal{L}$. The filter generated by $\varepsilon(T)$ in Ω is given by (2.1)

$$F = \{u \in \Omega : \exists \{t_1, \ldots, t_n\} \subseteq T, \text{ such that } u \geq \textstyle\bigcap_{i=1}^{n} \tau_{t_i}\}.$$

Notice that the set $\varepsilon(T)$ has the fip, because $T \in \varepsilon(t)$, for all $t \in T$. Hence, F is a proper filter in Ω. It remains to verify that $\varepsilon^{-1}(F) = T$; if $x \in \mathcal{L}$ is such that $\varepsilon(x) \in F$, then there is $\{t_1, \ldots, t_n\} \subseteq T$ such that

$$\textstyle\bigcap_{i=1}^{n} \tau_{t_i} \subseteq \tau_x,$$

and so $T \in \tau_x$, that is, $x \in T$, as desired. □

4 Algebraic Completeness of MEC and MECn

From now on we shall follow the algebraic tradition of using the same symbol (and font) for a structure and the set of elements of the structure.

Definition 4.1 *Let L be a Heyting algebra. Then,*

∗ *An* **assignment in** L *is a function from the set of propositional parameters into L. In the case of* **MECn**, *the propositional parameter $\perp$ is assigned to $\perp \in L$.*

∗ *A* **MEC-valuation in** L *is a function v from the set of all formulas of* **MEC** *into L, such that for all formulas $\mathcal{A}$, $\mathcal{B}$ of* **MEC** *:*

$$v(\mathcal{A} \equiv \mathcal{B}) = v(\mathcal{A}) \leftrightarrow v(\mathcal{B}).$$

∗ *A* **MECn-valuation** *in L is a* **MEC**-*valuation v such that $v(\perp) = \perp$.*

∗ $\mathrm{VAL}(L)$ *is the set of valuations in L.*

The usual induction on complexity yields

Lemma 4.2 *If L is a Heyting algebra, then $\mathrm{VAL}(L) \neq \emptyset$. In fact, any assignment in L can be extended to a unique valuation.*

We extend the concept of a valuation to sets of formulas; if Γ is a set of equivalence formulas[19] and $v \in \mathrm{VAL}(L)$, then we set

$$v(\Gamma) = \{v(\mathcal{F}) : \mathcal{F} \in \Gamma\}.$$

Definition 4.3 *Let L be a Heyting algebra and let v be a L-valuation of* **MEC** *or* **MECn**. *Let $\Gamma \cup \{\mathcal{A}\}$ be a set of formulas of* **MEC** *or* **MECn**.

[19] By that we mean formulas of either **MEC** or **MECn**.

(1) $\mathcal{A}$ *is an* **L, v-algebraic consequence of Γ**, *in symbols*

$$\Gamma \models_{\Gamma} \mathcal{A}[\![v]\!],$$

iff $v(\mathcal{A})$ *belongs to the filter generated by* $v(\Gamma)$ *in* L.

(2) $\mathcal{A}$ *is an* **L-algebraic consequence of Γ**, *in symbols,* $\Gamma \models_L \mathcal{A}$, *if for all L-valuations v,* $\Gamma \models_L \mathcal{A}[\![v]\!]$.

(3) $\mathcal{A}$ *is* **an algebraic consequence of Γ**, *in symbols,* $\Gamma \models \mathcal{A}$, *iff for all Heyting algebras L,* $\Gamma \models_L \mathcal{A}$.

Theorem 4.4 (*Completeness*) *Let* $\Gamma \cup \{\mathcal{A}\}$ *be a set of formulas in* **MEC** *or* **MECn**. *Then,*

$$(1)\ \Gamma \vdash \mathcal{A} \quad iff \quad (2)\ \Gamma \models \mathcal{A}.$$

Proof : We shall treat the case of **MEC**, leaving the straightforward modifications needed for **MECn** to the reader.

The proof of (1) $\Rightarrow$ (2) (soundness) is by induction on the *length of the derivation*, Π, of $\mathcal{A}$ from Γ. To handle the induction step corresponding to an application of the elimination rule of $\equiv$, one makes use of item (i) in 2.2.(d). To treat the induction step corresponding to the introduction rule for $\equiv$, the crucial result is the separation property in 2.2.(e), reasoning exactly as in the proof of the preservation of $*$ by the embedding ε in Theorem 3.9.(a).

To show that (2) $\Rightarrow$ (1), let Ω be the cHa of opens in the space of (proper) theories of **MEC**, as in Theorem 3.9; with notation as therein, define, for a formula $\mathcal{A}$ in **MEC**,

$$v(\mathcal{A}) = \varepsilon(\overline{\mathcal{A}}).$$

By Lemma 3.2 and Theorem 3.9.(a), v is a Ω-valuation. Let $T = \overline{\Gamma}^t$ be the theory generated by $\overline{\Gamma} = \{\overline{\mathcal{F}} \in \mathcal{L} : \mathcal{F} \in \Gamma\}$ in $\mathcal{L}$; we may as well assume that T is proper. Let F be the filter generated by $\varepsilon(T) = v(\Gamma)$ in Ω. By (2), we have $v(\mathcal{A}) \in F$, and so, 3.9.(b) yields $\overline{\mathcal{A}} \in T$, that is, $\overline{\Gamma} \vdash \overline{\mathcal{A}}$ in $\mathcal{L}$. But this is implies that $\Gamma \vdash \mathcal{A}$ in **MEC**. $\square$

From soundness we get

Corollary 4.5 *Let L be a Heyting algebra and let $\Gamma \cup \{\mathcal{A}, \mathcal{B}\}$ be a set of formulas in* **MEC** *or* **MECn**. *If v is a L-valuation, then*

a) $\mathcal{A} \vdash \mathcal{B}$ *implies* $v(\mathcal{A}) \leq v(\mathcal{B})$.

b) If $\Gamma \vdash \mathcal{A}$ *and* $v(\mathcal{F}) = \top$ *for all* $\mathcal{F} \in \Gamma$, *then* $v(\mathcal{A}) = \top$.

c) If $\mathcal{B}$ is a **MEC** *or* **MECn** *thesis, then* $v(\mathcal{B}) = \top$.

For a finite set of formulas Γ in **MEC** or **MECn**, Lemma 2.1 and Theorem 4.4 yield

Corollary 4.6 *If $\Gamma \cup \{\mathcal{A}\}$ is a finite set of formulas in* **MEC** *or* **MECn**, *then the following are equivalent :*

(1) $\Gamma \vdash \mathcal{A}$;

(2) *For all Ha's L and all L-valuations v,* $\bigwedge_{\mathcal{F}\in\Gamma} v(\mathcal{F}) \leq v(\mathcal{A})$.

5 Weak Equivalence Algebras

From this section on we endeavor to construct the algebraic counterparts of **MEC** and **MECn**. As it will become clear, there is an important distinction between the systems developed here and the usual ones associated to logic : the partial order that represents provability **is not** definable in terms of the operations, as is the case with lattices, Boolean or Heyting algebras. Hence, we shall have work harder to lay hands on the analogues of theories and filters.

The following Definition should be compared with 3.4 :

Definition 5.1 *A* **weak equivalence algebra (wEa)**, $\langle L, \leq, \top, * \rangle$, *is a set* L, *together with a partial order* $\leq$ *and a binary operation* $*$ *on* L, *such that, for all* $a, x, y, t, z \in L$:

$[*\ 0]$: $\top$ *is the largest element of* L *in the partial order* $\leq$;

$[*\ 1] : x * y = y * x;$

$[*\ 2] : x * \top = x;$

$[*\ 3] : x * y = \top \;\; iff \;\; x = y;$

$[*\ 4] : a \leq x * y$ *and* $a \leq t * z$ *implies* $a \leq (x * t) * (y * z)$.

A **weak equivalence algebra with negation (wEan)** *is a wEa* L, *that has a least element,* $\bot$, *satisfying*

$[neg\ 1]$: *For all* $x \in L$, $x * (x * \bot) = \bot$.

If L *and* R *are wEa's a map* $f : L \longrightarrow R$ *is a* **wEa-morphism** *if it is increasing and preserves* $\top$ *and* $*$. *We say that* f *is a* **wEa-embedding** *if it is a wEa-morphism such that for all* $x, y \in L$,

$$x \leq y \;\; iff \;\; f(x) \leq f(y).$$

The definitions of morphism and embedding of wEan's is analogous, adding the requirement that f *take* $\bot$ *to* $\bot$. *Write* $Hom(L, R)$ *for the set of wEa-morphisms (or wEan-morphisms) from* L *to* R.

Whenever convenient, we write xy *for* $x * y$ *and* $\neg x$ *for* $x * \bot$.

Remark 5.2 Note that, in general, the operation $*$ **is not** associative. One of the main distinctions between the equivalence algebras and the usual algebraic structures associated to Logic is that the partial order $\leq$ **is not** definable by a connective. This introduces the need to be careful in defining concepts such as <u>embedding</u>. In the category of meet-semilattices (or joinsemilattices), an injective increasing map is an isomorphism onto its image; for, in this case we have

$$x \leq y \quad \text{iff} \quad x \wedge y = x \text{ (resp., } x \vee y = y).$$

In the category of posets and increasing maps, it is an entirely different matter. Just consider $P = \{\{1\}, \{2\}, \{1, 2\}\}$ with its natural partial order (containment) and $Q = \{1, 2, 3\}$ with the order induced by the natural numbers. Then, $f(\{1\}) = 1$, $f(\{2\}) = 2$ and $f(\{1, 2\}) = 3$ is a bijective increasing map, but it is clear that P and Q are not isomorphic. This is at the root of the definition of *embedding of wEa* : it must be required that order be strictly preserved, for an injective increasing map to be an isomorphism onto its image.

Lemma 5.3 *For x, y, z in a wEa L,*

a) $x \leq y$ *and* $x \leq z$ *implies* $x \leq y * z$.

b) $x \leq (x * y) * y$.

c) $x * y \leq (x * z) * (y * z)$.

d) $x * y \leq y$ *iff* $x * y \leq x$.

e) $x * z \leq x * y$ *iff* $x * z \leq y * z$.

Proof : a) Using [* 1] and [* 2], we may write our hypothesis as

$$x \leq y * \top \quad \text{and} \quad x \leq \top * z.$$

An application of [* 4] yields the desired conclusion.

b) From [* 4], $x \leq x * \top$ and $x \leq y * y$, we conclude

$$x \leq (x * y) * (\top * y) = (x * y) * y.$$

c) Since $x * y \leq x * y$ and $x * y \leq z * z$, [* 4] yields the desired result.

d) From [* 4], $xy \leq \top * y$ and $xy \leq x * y$, we get $xy \leq x$. The converse is clear.

e) [$*$ 4], $x * z \leq x * y$ and $x * z \leq x * z$ yield

$$x * z \leq (x * x) * (y * z) = (y * z).$$

The converse is similar, ending the proof. □

Definition 5.4 *A subset F of a wEa L is a* **filter** *if $\top \in F$ and for all $x, y, z \in L$*

[*fil* 1] $x \in F$ *implies* $x^{\rightarrow} \subseteq F$;

[*fil* 2] $x * y \in F$ *and* $y * z \in F$ *implies* $x * z \in F$.

A filter is **proper** *iff $F \neq L$.*

Note that if L has a least element $\perp$, then F is proper iff $\perp \notin F$.

Lemma 5.5 *If F is a filter in a wEa L, then for all x, y, t, $z \in L$*

a) $x \in F$ *and* $x * y \in F$ *implies* $y \in F$.

b) $x, y \in F$ *implies* $x * y \in F$.

c) $x * y \in F$ *and* $t * z \in F$ *implies* $(xt) * (yz) \in F$.

Proof : (a) and (b) are a consequence of [*fil* 2] and [$*$ 2]. For (c), note that [*fil* 1] and 5.3.(b) yield

$$\begin{cases} xy \in F \quad \text{implies} \quad (xt) * (yt) \in F; \\ tz \in F \quad \text{implies} \quad (yt) * (yz) \in F, \end{cases}$$

and so [*fil* 2] yields $(xt) * (yz) \in F$, as desired. □

Example 5.6 1. The set $\{\top\}$ is a filter in any wEa. The only condition that needs verification is [*fil* 3]; recalling that $x * y = \top$ iff $x = y$ ([$*$ 3]), it is easily seen that it satisfies [*fil* 3]. $\{\top\}$ is the smallest filter (with respect to containment) in any wEa.

2. More generally, let x be an element of a wEa L. It is clear that $x^{\rightarrow}$ satisfies [*fil* 1] and [*fil* 2]. That it is, in fact, a filter,

is a consequence of [$*$ 4]. The filter $x^{\rightarrow}$ is the **principal filter** generated by x.

3. If F_i, $i \in I$, is a family of filters in L, then $\bigcap_{i \in I} F_i$ is a filter in L.

4. If F_i, $i \in I$, is a right-directed family of filters, that is,

For all $i, j \in I$, there is $k \in I$ such that $F_i, F_j \subseteq F_k$,

then $\bigcup_{i \in I} F_i$ is a filter in L.

Because the property of being a filter is preserved by intersections, we may define **the filter generated** by a subset S of an wEa L as

$$[S] = \bigcap \{F : F \text{ is a filter in } L \text{ and } S \subseteq F\}.$$

Clearly, $S \subseteq T$ implies $[S] \subseteq [T]$. The following is straightforward :

Lemma 5.7 *Let S_i, $i \in I$, be a right-directed collection of subsets of L. If $S = \bigcup_{i \in I} S_i$, then*

$$[S] = \bigcup_{i \in I} [S_i].$$

In particular, if $A \subseteq L$ and 2^A_ω is the set of finite subsets of A, then

$$[A] = \bigcup_{\alpha \in 2^A_\omega} [\alpha].$$

6 T-operators. Equivalence Algebras

Let D be a set. Recall that a map $\mu : 2^D \longrightarrow 2^D$ is

- **Inflationary** if $A \subseteq \mu(A)$, for all $A \subseteq D$;
- **Increasing** if for all $A, B \in 2^D$, $A \subseteq B$ implies $\mu(A) \subseteq \mu(B)$;
- **Idempotent** if $\mu \circ \mu = \mu$.

Let L be a wEa. Define $\mu_0 : 2^L \longrightarrow 2^L$ by

$\mu_0(A) = \bigcup \{(x * y)^{\rightarrow} : \exists\ t \in L$ such that $(x * t), (t * y) \in A \cup \{\top\}\}$.

Lemma 6.1 *With notation as above, μ_0 is increasing and inflationary. Moreover, for all $x \in L$ and $A \subseteq L$,*

$$x \in \mu_0(A) \quad \textit{implies} \quad x^{\rightarrow} \subseteq \mu_0(A).$$

Proof : Clearly μ_0 is increasing and $x \in \mu_0(A)$ implies $x^{\rightarrow} \subseteq \mu_0(A)$. For $a \in A$, note that $a = a * \top$, with $(a * \top)$ and $(\top * \top)$ both in $A \cup \{\top\}$. Hence, $A \subseteq \mu_0(A)$. □

For $A \subseteq L$, define a sequence of subsets of L, $\sigma_n(A)$, $n \geq 0$, by induction on n, as follows:

$$\sigma_0(A) = A \quad \text{and} \quad \sigma_{n+1}(A) = \mu_0(\sigma_n(A)).$$

Now, set $\tau_0(A) = \bigcup_{n \geq 0} \sigma_n(A)$. Then,

Proposition 6.2 *For all $A \subseteq L$, $\tau_0(A)$ is the filter generated by A in L.*

Proof : It is clear that any filter containing A must contain $\tau_0(A)$. By 6.1, $A \cup \{\top\} \subseteq \mu_0(A) \subseteq \tau_0(A)$ and $x \in \mu_0(A)$ implies $x^{\rightarrow} \subseteq \tau_0(A)$. To verify [*fil* 2], let $(x * t)$ and $(t * y)$ be in $\tau_0(A)$. Since the sequence $\sigma_n(A)$ is increasing, there is $n \geq 0$ such that $(x * t), (t * y) \in \sigma_n(A)$. But then, $x * y \in \sigma_{n+1}(A) \subseteq \tau_0(A)$. □

From here on we shall use, interchangeably, the notation $[A]$ and $\tau_0(A)$ for the filter generated by A.

Definition 6.3 *Let L be a wEa. A* **T-operator** *on L is a map $\beta : 2^L \longrightarrow 2^L$, satisfying :*

[T 1] : β is inflationary, increasing and idempotent;

[T 2] : For all $A \subseteq L$, $\beta(A)$ is a filter in L;

[T 3] : *For all* $x, y \in L$ *and* $A \subseteq L$,

$$\beta(A \cup \{x\}) = \beta(A \cup \{y\}) \quad \textit{iff} \quad (x * y) \in \beta(A).$$

Write $T_{op}(L)$ *for the set of* T*-operators on* L. *Define a partial order in* $T_{op}(L)$ *by*

$$\alpha \leq \beta \quad \textit{iff} \quad \textit{For all } A \subseteq L,\ \alpha(A) \leq \beta(A).$$

For $\beta_i \in T_{op}(L)$, $i \in I$, *and* $A \subseteq L$, *set*

[T∧] $$[\textstyle\bigwedge_{i\in I} \beta_i](A) = \bigcap_{i\in I} \beta_i(A)$$

Remark 6.4 Let L be a wEa. With notation as above,

a) The map $A \in 2^L \mapsto L \in 2^L$ is largest T-operator on L.

b) $T_{op}(L)$ is a complete lattice, with meets as in [T∧] of 6.3.

It will be important in what follows to obtain an explicit description of the bottom of the complete lattice $T_{op}(L)$. To this end, we construct an increasing sequence of inflationary and increasing maps,

$$\tau_0 \leq \tau_1 \leq \dots \leq \tau_n \leq \tau_{n+1} \leq \dots$$

such that for all $A \subseteq L$, all $n \geq 0$ and all $x, y \in L$

[c 1] : $\tau_n(A)$ is a filter in L;

[c 2] : $\tau_n(A \cup \{x\}) = \tau_n(A)$ iff $x \in \tau_n(A)$;

[c 3] : $\tau_n(A \cup \{x\}) = \tau_n(A \cup \{y\})$ implies $(x * y) \in \tau_{n+1}(A)$.

For $n = 0$, $\tau_0(A)$ is the filter generated by A in L. Assume that τ_n has been constructed, and define $\mu_{n+1} : 2^L \longrightarrow 2^L$ as follows:

$$\mu_{n+1}(A) = \bigcup \{(x * y)^{\rightarrow} : \tau_n(A \cup \{x\}) = \tau_n(A \cup \{y\})\}.$$

Lemma 6.5 *With notation as above,* μ_{n+1} *is increasing and for all* $x \in L$ *and* $A \subseteq L$, $x \in \mu_{n+1}(A)$ *implies* $x^{\rightarrow} \subseteq \mu_{n+1}(A)$. *Moreover, for all* $A \subseteq L$, $\tau_n(A) \subseteq \mu_{n+1}(A)$.

Proof : It is clear that $x \in \mu_{n+1}(A)$ implies $x^{\rightarrow} \subseteq \mu_{n+1}(A)$. To show that μ_{n+1} is increasing, let $A \subseteq B \subseteq L$. It is enough to verify that for $x, y \in L$

$\tau_n(A \cup \{x\}) = \tau_n(A \cup \{y\})$ implies $\tau_n(B \cup \{x\}) = \tau_n(B \cup \{y\})$. (I)

Since τ_n is inflationary and increasing, we have

$$x \in \tau_n(A \cup \{y\}) \subseteq \tau_n(B \cup \{y\}).$$

It follows from [c 2] that

$$\tau_n(B \cup \{y\}) = \tau_n(B \cup \{y\} \cup \{x\}).$$

Similarly, one shows that $\tau_n(B \cup \{x\}) = \tau_n(B \cup \{x\} \cup \{y\})$, proving (I). It remains to check that $\tau_n(A) \subseteq \mu_{n+1}(A)$. Since $\tau_n(A)$ is a filter, $\top \in \tau_n(A)$, and so [c 2] yields $\tau_n(A \cup \{\top\}) = \tau_n(A)$. Similarly, [c 2] yields

$$x \in \tau_n(A) \quad \text{implies} \quad \tau_n(A \cup \{x\}) = \tau_n(A) = \tau_n(A \cup \{\top\}).$$

Thus, $x = x * \top \in \mu_{n+1}(A)$, ending the proof. □

For $A \subseteq L$, define, by induction on $k \geq 0$, a sequence of subsets of L, $\sigma_k^{n+1}(A)$, as follows:

$$\sigma_0^{n+1}(A) = A \quad \text{and} \quad \sigma_{k+1}^{n+1}(A) = \mu_{n+1}(\sigma_k^{n+1}(A)).$$

Clearly, $\sigma_k^{n+1}(A)$ is increasing. Now set

$$\tau_{n+1}(A) = \bigcup_{k \geq 0} \sigma_k^{n+1}(A).$$

It is clear that $\tau_n(A) \leq \tau_{n+1}(A)$ (i.e., $\tau_n \leq \tau_{n+1}$) and that τ_{n+1} is increasing.

Proposition 6.6 *The map τ_{n+1} satisfies* [c 1], [c 2] *and* [c 3].

Proof : [c 1] : Fix $A \subseteq L$. Clearly, $\tau_n(A)$ satisfies [*fil* 1] and $\top \in \tau_{n+1}(A)$. Now assume that $(x * t)$, $(t * y) \in \tau_{n+1}(A)$. Then, there is $k \geq 0$ such that $(x * t)$, $(t * y) \in \sigma_k^{n+1}(A)$. It is enough to show that

$$\tau_n(\sigma_k^{n+1}(A) \cup \{x\}) = \tau_n(\sigma_k^{n+1}(A) \cup \{y\}), \tag{I}$$

because then $(x * y) \in \mu_{n+1}(\sigma_k^{n+1}(A)) = \sigma_{k+1}^{n+1}(A) \subseteq \tau_{n+1}(A)$.

Since τ_n satisfies [c 2], to get (I) it is sufficient to check that

$$\begin{cases} \text{(i) } y \in \tau_n(\sigma_k^{n+1}(A) \cup \{x\}) \\ \text{(ii) } x \in \tau_n(\sigma_k^{n+1}(A) \cup \{y\}) \end{cases}$$

Since $(x * t), (t * y) \in \sigma_k^{n+1}(A)$ and $\tau_n(\sigma_k^{n+1}(A) \cup \{x\})$ is a filter containing x, 5.5.(a) implies that (i) is verified. A similar argument proves (ii), completing the proof of [c 1].

[c 2] : First notice that for all $k \geq 0$

$$\tau_{n+1}(\sigma_k^{n+1}(A)) = \tau_{n+1}(A), \tag{II}$$

because for all $l \geq 0$

$$\sigma_l^{n+1}(\sigma_k^{n+1}(A)) = \mu_{n+1}^l(\mu_{n+1}^k(A)) = \mu_{n+1}^{k+l}(A) = \sigma_{k+l}^{n+1}(A) \subseteq \tau_{n+1}(A).$$

Hence, if $x \in \tau_{n+1}(A)$, then there is $k \geq 0$ such that $x \in \sigma_k^{n+1}(A)$ and so (II) yields

$$\tau_{n+1}(A \cup \{x\}) \subseteq \tau_{n+1}(\sigma_k^{n+1}(A)) \subseteq \tau_{n+1}(A),$$

and equality follows from the fact that τ_{n+1} is increasing. That τ_{n+1} satisfies [c 3] follows from the fact that

$$\tau_n(A \cup \{x\}) = \tau_n(A \cup \{y\}) \text{ implies } (x * y) \in \mu_{n+1}(A) \subseteq \tau_{n+1}(A),$$

ending the proof. □

Proposition 6.6 completes the construction of the τ_n's.

Proposition 6.7 *With notation as above*

a) For all integers $l > n \geq 0$, $\tau_l \circ \tau_n = \tau_l$.

b) For all $n \geq 0$ and all right-directed families of subsets of L, B_i, $i \in I$,

$$\tau_n(\bigcup B_i) = \bigcup_{i \in I} \tau_n(B_i).$$

In particular, for all $A \subseteq L$ and $n \geq 0$, $\tau_n(A) = \bigcup_{\alpha \in 2^A_\omega} \tau_n(\alpha)$.

Proof : a) We first verify that for all $n \geq 0$ and all $A \subseteq L$

$$\tau_{n+1}(\tau_n(A)) = \tau_{n+1}(A). \qquad (*)$$

Since $A \subseteq \tau_n(A)$, it follows that $\tau_{n+1}(A) \subseteq \tau_{n+1}(\tau_n(A))$. On the other hand, by Lemma 6.5, $\tau_n(A) \subseteq \mu_{n+1}(A) = \sigma_1^{n+1}(A)$. But in the proof of 6.6 (see (II)), we have shown that $\tau_{n+1}(\sigma_1^{n+1}(A)) = \tau_{n+1}(A)$, and the equality in (*) follows. Then, induction on $k \geq 1$ yields

$$\begin{aligned}\tau_{n+k+1} &= \tau_{n+k+1} \circ \tau_{n+k} = \tau_{n+k+1} \circ (\tau_{n+k} \circ \tau_n) \\ &= (\tau_{n+k+1} \circ \tau_{n+k}) \circ \tau_n = \tau_{n+k+1} \circ \tau_n,\end{aligned}$$

ending the proof of (a).

b) Write $A = \bigcup_{i \in I} B_i$. It is enough to verify that if $x \in \tau_n(A)$, there is $i \in I$, such that $x \in \tau_n(B_i)$. Proceed by induction on $n \geq 0$. For $n = 0$, the result follows from 5.7. Assume the result true for n. We first prove

Fact *For all $k \geq 0$,* $\sigma_k^{n+1}(A) = \bigcup_{i \in I} \sigma_k^{n+1}(B_i)$.

Proof For $k = 0$ there is nothing to prove. Assume the result true for k and that $y \in \sigma_{k+1}^{n+1}(A)$. Then, there are $u, v \in L$ such that

$$\tau_n(\sigma_k^{n+1}(A) \cup \{u\}) = \tau_n(\sigma_k^{n+1}(A) \cup \{v\}),$$

and $y \geq (u * v)$. Note that the family $(\sigma_k^{n+1}(B_i) \cup \{u\})$, $i \in I$, is directed. Moreover, by induction,

$$\sigma_k^{n+1}(A) \cup \{u\} = \bigcup_{i \in I} (\sigma_k^{n+1}(B_i) \cup \{u\})$$

Since $v \in \tau_n(\sigma_k^{n+1}(A) \cup \{u\})$, the (first) induction hypothesis yields $k \in I$ such that $v \in \tau_n(\sigma_k^{n+1}(B_k) \cup \{u\})$. By a similar argument, there is $j \in I$ such that $u \in \tau_n(\ \sigma_j^{n+1}(B_i) \cup \{u\})$. If we select $i \in I$ such that $B_j,\ B_k \subseteq B_i$, we conclude that

$$u \in \tau_n(\sigma_k^{n+1}(B_i) \cup \{v\}) \quad \text{and} \quad v \in \tau_n(\sigma_k^{n+1}(B_i) \cup \{u\}).$$

It follows that

$$\tau_n(\sigma_k^{n+1}(B_i) \cup \{u\}) = \tau_n(\sigma_k^{n+1}(B_i) \cup \{v\}),$$

and so $y \in (u * v)^{\rightarrow} \subseteq \sigma_{k+1}^{n+1}(B_i)$, ending the proof of the Fact.

If $x \in \tau_{n+1}(A)$, then $x \in \sigma_k^{n+1}(A)$, for some $k \geq 0$. By the Fact, there is $i \in I$ such that $x \in \sigma_k^{n+1}(B_i) \subseteq \tau_{n+1}(B_i)$, as desired. □

For $A \subseteq L$, set

$$\tau_L(A) = \bigcup_{n \geq 0} \tau_n(A).$$

Whenever clear from context, we omit the name of the wEa L from the notation. Since this is a directed union of filters containing A, $\tau(A)$ is a filter containing A. Moreover, $A \mapsto \tau(A)$ is increasing, because the same is true of each τ_n.

Proposition 6.8 *For $A \subseteq L$ and $x,\ y \in L$*

a) $x \in \tau(A)$ iff $\tau(A) = \tau(A \cup \{x\})$.

b) $\tau(A \cup \{x\}) = \tau(A \cup \{y\})$ iff $x \in \tau(A \cup \{y\})$ and $y \in \tau(A \cup \{x\})$.

c) The operation $A \mapsto \tau(A)$ satisfies [T 1] and [T 2].

d) For all $n \geq 0$ and all $B \subseteq L$

$$B \subseteq \tau(A) \quad \textit{implies} \quad \tau(B) \subseteq \tau(A).$$

e) For all $A \subseteq L$, $\tau(\tau(A)) = \tau(A)$.

Proof : a) If $x \in \tau(A)$, there is $n \geq 0$ such that $x \in \tau_n(A)$. By [c 2], $\tau_n(A \cup \{x\}) = \tau_n(A)$. It follows from 6.7 that for all $l > n$

$$\tau_l(A \cup \{x\}) = \tau_l(\tau_n(A \cup \{x\})) = \tau_l(\tau_n(A)) = \tau_l(A),$$

and so $\tau(A \cup \{x\}) = \tau(A)$. The converse is clear.

b) If the right-hand side of the equivalence holds, then (a) yields

$$\tau(A \cup \{x\}) = \tau(A \cup \{x\} \cup \{y\}) = \tau(A \cup \{y\}),$$

while the converse is obvious.

c) We have already observed that $\tau(A)$ is a filter. If $(x * y) \in \tau(A)$, since $\tau(A) \subseteq \tau(A \cup \{x\})$ and this last set is a filter, we get $y \in \tau(A \cup \{x\})$. Similarly, $x \in \tau(A \cup \{y\})$ and equality follows from (b). Conversely, if $\tau(A \cup \{x\}) = \tau(A \cup \{y\})$, since the sequence $\tau_n(A)$ is increasing, (b) yields $n \geq 0$ such that $x \in \tau_n(A \cup \{y\})$ and $y \in \tau_n(A \cup \{x\})$. Consequently, $\tau_n(A \cup \{x\}) = \tau_n(A \cup \{y\})$ and $(x * y) \in \tau_{n+1}(A) \subseteq \tau(A)$.

d) By induction on $n \geq 0$, we prove

Fact 1 *For all $n \geq 0$ and all $B \subseteq L$*

$$B \subseteq \tau(A) \quad \textit{implies} \quad \tau_n(B) \subseteq \tau(A).$$

Proof By induction on $n \geq 0$. For $n = 0$, since $\tau(A)$ is a filter, it must contain $\tau_0(B)$ (6.2). Assume the result true for $n \geq 0$. We then have

Fact 2 *For all $D \subseteq L$, $D \subseteq \tau(A)$ implies $\mu_{n+1}(D) \subseteq \tau(A)$.*

Proof If $\tau_n(D \cup \{x\}) = \tau_n(D \cup \{y\})$, then $D \subseteq \tau(A) \subseteq \tau(A \cup \{x\})$ implies $D \cup \{y\} \subseteq \tau(A \cup \{y\})$. By induction, we get

$$x \in \tau_n(D \cup \{y\}) \subseteq \tau(A \cup \{y\}).$$

Similarly, one shows that $y \in \tau_n(D \cup \{x\}) \subseteq \tau(A \cup \{x\})$. By (b), $\tau(A \cup \{x\}) = \tau(A \cup \{y\})$, and so (c) yields $(x * y) \in \tau(A)$. Since $\tau(A)$ is a filter, we get $\mu_{n+1}(D) \subseteq \tau(A)$, ending the proof of the Fact 2.

By Fact 2, if $B \subseteq \tau(A)$, then $\sigma_k^{n+1}(B) \subseteq \tau(A)$, for all $k \geq 0$. Hence, $\tau_{n+1}(B) \subseteq \tau(A)$, completing the induction step and the proof of Fact 1. Item (d) is now clear, while (e) is a consequence of (d). □

By Proposition 6.8, τ_L is a T-operator on L. From 6.7.(b) comes

Corollary 6.9 *If $\{B_i : i \in I\}$ is a right-directed family of subsets of a wEa L and $B = \bigcup_{i \in I} B_i$, then*

$$\tau(B) = \bigcup_{i \in I} \tau(B_i).$$

In particular, for all $A \subseteq L$

$$[compactness] \qquad \tau(A) = \bigcup_{\alpha \in 2^A_\omega} \tau(\alpha),$$

where 2^A_ω is the set of finite subsets of A.

We now prove

Proposition 6.10 *If L is a wEa and $\beta \in T_{op}(L)$, then $\tau \leq \beta$.*

Proof : By induction on $n \geq 0$, it will be verified that for all A, $B \subseteq L$,

$$B \subseteq \beta(A) \text{ implies } \tau_n(B) \subseteq \beta(A). \tag{I}$$

For $n = 0$ there is nothing to prove because $\beta(A)$ is a filter. Assume the result true for $n \geq 0$; we then have

Fact $B \subseteq \beta(A)$ *implies* $\mu_{n+1}(B) \subseteq \beta(A)$.

Proof Since $\beta(A)$ is a filter, it is enough to verify that if x, $y \in L$ are such that $\tau_n(B \cup \{x\}) = \tau_n(B \cup \{y\})$, then $(x * y) \in \beta(A)$. The induction hypothesis guarantees that

$x \in \tau_n(B \cup \{y\}) \subseteq \beta(A \cup \{y\})$ and $y \in \tau_n(B \cup \{x\}) \subseteq \beta(A \cup \{x\})$,

and so $\beta(A \cup \{x\}) = \beta(A \cup \{y\})$; but then $(x * y) \in \beta(A)$, as desired.

The Fact implies that if $B \subseteq \beta(A)$, then $\sigma_k^{n+1}(B) \subseteq \beta(A)$, for all $k \geq 0$. Hence, $\tau_{n+1}(B) \subseteq \beta(A)$, completing the induction step. □

Our next result describes some of the basic properties of T-operators.

Proposition 6.11 *Let L be a wEa and let β, $\gamma \in T_{op}(L)$. Let B_i, $i \in I$, be a family in 2^L and let β_i, $i \in I$, be a family in $T_{op}(L)$.*

a) $\gamma \leq \beta$ iff $\gamma \circ \beta = \beta$ iff $\beta \circ \gamma = \beta$.

b) Write $Fix(\beta) = \{A \in 2^L : \beta(A) = A\}$ for the set of fixed points of β in L. Then, $\gamma \leq \beta$ implies $Fix(\beta) \subseteq Fix(\gamma)$.

c) If $D = \bigcap_{i \in I} \beta(B_i)$, then $D = \beta(D)$.

Proof : a) if $\gamma \leq \beta$ then for $A \subseteq L$, we have

$$\begin{cases} \beta(A) \subseteq \gamma(\beta(A)) \subseteq \beta(\beta(A)) = \beta(A) \\ \qquad\qquad\qquad \text{and} \\ \beta(A) \subseteq \beta(\gamma(A)) \subseteq \beta(\beta(A)) = \beta(A), \end{cases}$$

verifying the stated identities. The converses are left to the reader.

b) For $A \in Fix(\beta)$, (a) yields $\gamma(A) = \gamma(\beta(A)) = \beta(A) = A$, as needed.

c) Since $D \subseteq \beta(B_i)$, we have $\beta(D) \subseteq \beta(\beta(B_i) = \beta(B_i)$, $i \in I$. Hence, $\beta(D) \subseteq D$ and equality follows. □

Definition 6.12 *Let L be a wEa. A subset A of L is a* **$*$-filter** *if $A \in Fix(\tau)$. Write $S(L)$ for the set of* **proper** *$*$-filters on L.*

Proposition 6.11.(b) yields

Corollary 6.13 *If $\beta \in T_{op}(L)$, then all fixed points of β are $*$-filters on L.*

Our next order of business is to give concrete examples of the $*$-filters. We start with

Proposition 6.14 *In the Lindenbaum algebra $\mathcal{L}$ of* **MEC**, *the $*$-filters correspond to its theories. In particular, for all $x \in \mathcal{L}$, $x^{\rightarrow}$ is a $*$-filter.*

Proof : It follows from 3.4 that $\mathcal{L}$ is a wEa and from 3.6 that the operation

$$U \subseteq \mathcal{L} \mapsto U^t \text{ (the theory generated by } U)$$

is a T-operator on $\mathcal{L}$ (it is easy established that any theory in $\mathcal{L}$ is a filter). By 6.10, for all $U \subseteq \mathcal{L}$, we have $\tau(U) \subseteq U^t$. To prove the reverse containment, we proceed by induction on the length of proof trees to verify that for all $U, V \subseteq L$

$$U \subseteq \tau(V) \quad \text{and} \quad U \vdash x \quad \text{implies} \quad x \in \tau(V). \tag{I}$$

We may as well suppose that $x \notin U$. The following possibilities arise :

(i) x comes from an application of the elimination rule. Then, there are proofs of strictly smaller length that $U \vdash (x * y)$ and $U \vdash y$. By induction, $(x * y), y \in \tau(V)$; since $\tau(V)$ is a filter, we get $x \in \tau(V)$, as needed.

(ii) x comes from an application of the introduction rule. Then, we have $x = (a * b)$ and there are proofs of strictly smaller length of $U, a \vdash b$ and $U, b \vdash a$. By induction, $a \in \tau(V \cup \{b\})$ and $b \in \tau(V \cup \{a\})$, that is, $\tau(V \cup \{a\}) = \tau(V \cup \{b\})$. But then $x = (a * b) \in \tau(V)$, completing the proof. □

If H is a Heyting algebra (see section 2), consider the structure $H_{eq} = \langle H, \leq, \top, \leftrightarrow \rangle$. Then,

Remark 6.15 Let L be a subset of a Ha H, that contains $\top$ and is closed under $\leftrightarrow$. Then, with the operations induced by

H_{eq}, L is a wEa. If $\bot \in L$, then L is a wEan. In particular, $\langle H, \leq, \top, \leftrightarrow \rangle$ is a wEan.

Proposition 6.16 *Let H be a Heyting algebra.*

a) The following are equivalent for $F \subseteq H$:

(1) *F is a filter in H_{eq};*

(2) *F is a (lattice-theoretic) filter in H.*

b) A subset of H_{eq} is a $*$-filter iff it is a filter.

Proof : a) Clearly, (2) implies (1). For the converse, it is enough to verify that a filter F is the wEa H_{eq} is closed under meets. For x, $y \in F$, note that

$$y \leq (x \to y) = (x \leftrightarrow (x \wedge y)).$$

Thus, $(x \leftrightarrow (x \wedge y)) \in F$. But then, since $x \in F$ and F is a wEa-filter, we get $(x \wedge y) \in F$, as needed.

b) It is easily established that in any wEa L the operation $A \mapsto [A]$ is inflationary, increasing and idempotent and obviously satisfies $[T\ 1]$. Since $[A] \subseteq \tau(A)$, for all $A \subseteq L$, it follows from 6.10 that to prove equality it suffices to show that the operation of "filter generated by" satisfies $[T\ 2]$. If F is a filter in H_{eq} – which by (a) is a filter in H –, and x, $y \in H$ are such that $[F \cup \{x\}] = [F \cup \{y\}]$. Since F is closed under meets, by 2.1 there are a, $b \in F$ such that

$$y \geq a \wedge x \quad \text{and} \quad x \geq b \wedge y.$$

Hence, $x \wedge (a \wedge b) = y \wedge (a \wedge b)$, and 2.3.(b) yields $(a \wedge b) \leq (x * y)$. But then, $(x * y) \in F$, verifying $[T\ 2]$ and ending the proof. □

Definition 6.17 *A wEa (wEan) L is an* **equivalence algebra (Ea)** *if for all $x \in L$, $x^{\to}$ is a $*$-filter in L. An* **equivalence algebra with negation (Ean)** *is an wEan which is an Ea.*

If L and R are Ea's, a **Ea-morphism**, $L \xrightarrow{f} R$, *is a wEa-morphism. In case L and R are Ean's, f is required to take $\bot$ to $\bot$. Write* **EA** *and* **EAn** *for the categories of equivalence algebras and equivalence algebras with negation, respectively.*

From 6.14 and 6.16 we get

Corollary 6.18 *a) The Lindenbaum algebra of* **MEC** *is an Ea.*

b) The Lindenbaum algebra of **MECn** *is an Ean.*

c) If H is a Ha, then $H_{eq} = \langle H, \leq, \top, \leftrightarrow \rangle$ is an Ean.

With section 4 as a model, we can define assignments and valuations of **MEC** and **MECn** in wEa and wEan's, respectively. Moreover, as before, any assignment of the propositional parameters can be extended to a valuation. Consequently, the proof of Proposition 6.14 leads to

Theorem 6.19 (*Completeness of* **MEC** *and* **MECn**.)

a) If L is a wEa, there is a bijective correspondence between $Hom(\mathcal{L}, L)$ and valuations of **MEC** *in L. A similar result holds for $\mathcal{L}_n$ and* **MECn**.

b) If L is an Ea and $\Gamma \cup \{\varphi\} \subseteq$ **MEC** *is such that $\Gamma \vdash \varphi$, then for all valuations v of* **MEC** *in L and all $*$-filters F in L,*

$$v(\Gamma) \subseteq F \quad \textit{implies} \quad v(\varphi) \in F,$$

where $v(\Gamma) = \{v(\psi) : \psi \in \Gamma\}$. A similar result holds for **MECn**.

c) Let L be an Ea (Ean) and $\Gamma \cup \{\sigma, \varphi\}$ be a set of formulas in **MEC**. *If v is a L-interpretation of* **MEC** *(resp.,* **MECn**), *then*

(1) $\varphi \vdash \sigma$ *implies* $v(\varphi) \leq v(\sigma)$.

(2) *If* $\Gamma \vdash \sigma$ *and* $v(\psi) = \top$, *for all* $\psi \in \Gamma$, *then* $v(\sigma) = \top$.

(3) *If* $\vdash \varphi$, *then* $v(\varphi) = \top$.

d) $\mathcal{L}$ and $\mathcal{L}_n$ are, respectively, the free Ea and the free Ean on the set of propositional parameters that constitute their basic alphabet.

f) (Completeness Theorem) Let $\Gamma \cup \{\varphi\}$ be a set of formulas in **MEC** *or* **MECn**. *The following are equivalent :*

(1) $\Gamma \vdash \varphi$;

(2) *For all Ea's (resp., Ean's) L, for all valuations v in L and all $*$-filters F in L*

$$v(\Gamma) \subseteq F \quad \textit{implies} \quad v(\varphi) \in F.$$

Our next theme is the behavior of T-operators under inverse image by wEa-morphisms.

Remark 6.20 If $L \xrightarrow{f} R$ is a wEa-morphism, then the inverse image of a filter in R is a filter in L. If f is a wEan-morphism, then inverse image by f takes proper filters in R to proper filters in L.

Proposition 6.21 *Let $L \xrightarrow{f} R$ be a wEa-morphism. For $\beta \in T_{op}(R)$ and $A \subseteq L$, set*

$$\alpha(A) = f^{-1}(\beta(f(A))).$$

Then, $\alpha \in T_{op}(L)$.

Proof: It is clear that α is inflationary and increasing. Moreover, by 6.20, $\alpha(A)$ is a filter in L. To verify that α is idempotent, we have for $A \subseteq L$ and recalling that $f(f^{-1}(B)) \subseteq B$ $(B \subseteq R)$,

$$\alpha(\alpha(A)) = f^{-1}\ \beta\ f\ \alpha(A) = f^{-1}\ \beta\ f\ f^{-1}\ \beta\ f(A)$$
$$\subseteq f^{-1}\ \beta\ \beta\ f(A) = f^{-1}\ \beta\ f(A) = \alpha(A),$$

where composition is written by superposition for ease of reading. Hence, $\alpha(A) = \alpha(\alpha(A))$, as desired. It remains to verify that α

satisfies [T 3] in 6.3. For $A \cup \{x\} \cup \{y\} \subseteq L$, assume that $\alpha(A \cup \{x\}) = \alpha(A \cup \{x\})$. Now observe that

$$f(x) \in \beta(f(A) \cup \{f(y)\}) \quad \text{and} \quad f(y) \in \beta(f(A) \cup \{f(x)\}).$$

which follow from $f(A \cup \{y\}) = f(A) \cup \{f(y)\}$ and the equality $\alpha(A \cup \{x\}) = \alpha(A \cup \{y\})$. Thus, $f(x) * f(y) = f(x * y) \in \beta(f(A))$ and $(x * y) \in \alpha(A)$, ending the proof. □

If $L \xrightarrow{f} R$ is a wEa-morphism and $\beta \in T_{op}(R)$ write $f^*\beta$ for the T-operator α defined in 6.21.

Corollary 6.22 *Let $\xrightarrow{f} R$ be a wEa-morphism.*

a) The inverse image of a $*$-filter in R is a $*$-filter in L. If f is a wEan-morphism, then inverse image by f takes $S(R)$ into $S(L)$.

b) If R is an Ea (Ean) and f is an embedding, then L is an Ea (resp., Ean).

c) If H is a Ha and $L \subseteq H$ is such that $\top \in L$ ($\bot \in L$) and L is closed under $\leftrightarrow$, then, with the structure induced by H_{eq}, L is an Ea (resp., Ean).

Proof : For (a), let B be a $*$-filter in R. We check that $f^{-1}(B)$ is a fixed point of $f^*\tau_R$ (6.21), and then 6.13 will guarantee that $f^{-1}(B)$ is a $*$-filter. We have

$$f^*\tau_R(f^{-1}(B)) = f^{-1}\ \tau_R\ f\ f^{-1}(B) \subseteq f^{-1}\tau_R(B) = f^{-1}(B),$$

and so $f^{-1}(B) \in Fix(f^*\tau_R)$, as claimed. The remaining statement in (a) is a consequence of 6.20. For (b), note that $f^{-1}(f(x)^{\rightarrow}) = x^{\rightarrow}$, for all $x \in L$, and the conclusion follows from (a). Item (c) is immediate from (b). □

By 6.16.(a), if $L \xrightarrow{f} R$ is an embedding of *Heyting algebras*, then $\tau_L = f^*\tau_R$. For wEa's in general we pose

Open Problem 6.23 *Suppose that* $L \xrightarrow{f} R$ *is a wEa-embedding. Is it true that* $\tau_L = f^*\tau_R$ *?*

In the next section it will be seen that every Ea or Ean can be embedded in a cHa in such a way that the answer to 6.23 is affirmative.

We end this section with a discussion of the concept of **cokernel** of wEa-morphisms.

Definition 6.24 *Let* $L \xrightarrow{f} R$ *be a wEa-morphism. Define*

$$coker\ f = f^{-1}(\tau_R(\{\top\})) = f^*\tau_R(\{\top\}).$$

When $\{\top\}$ is a $*$-filter in R – in particular if R is an Ea –, coker f takes the familiar form $\{x \in L : f(x) = \top\}$. The following is a straightforward consequence the previous results :

Corollary 6.25 *If* $L \xrightarrow{f} R$ *is a wEa-morphism, then*

a) coker f *is a* $*$*-filter in* L.

b) If $\{\top\}$ *is a* $*$*-filter in* R*, then* f *is injective iff coker* $f = \{\top\}$.

7 Basic Properties of Negation

We now describe some of the properties of negation in a wEan which satisfies axioms [$*$ 5], [$*$ 6] and [$*$ 7] in Lemma 3.4, as well as [$*$ 8] in Corollary 8.6, namely

[$*$ 5] : $x * y = ((x * y) * y) * y$,

[$*$ 6] : $x \leq y \leq z$ implies $x * z \leq x * y$,

[$*$ 7] : $(x * y) * z \leq [x * z] * [(y * z) * z]$,

$[*\ 8] : [(x * z) * (x * z)] * z \leq (x * y) * z,$

referred to as **special wEa's**. The most fundamental of these properties is item (c) in Proposition 7.4. It will be shown in 8.6 that all Ea's and Ean's are special wEa's.

Remark 7.1 It follows immediately from [E 6] that a special wEan satisfies the *contra positive law*

[neg 2] For all $x, y \in L$, $x \leq y$ implies $(y * \bot) \leq (x * \bot)$.

If L is a special wEa, define

$$A = \{x \in L : \text{For all } u \in L, \quad x = (xu)u\},$$

called the set of **associative** elements of L. It is clear that $\top \in A$. If A has negation, then $\bot \in A$. The name for A is justified by

Lemma 7.2 *Let L be a special wEa. Then,*

$$\textit{For all } x, z \in A \textit{ and all } y \in L, \quad (xy)z = x(yz).$$

Proof : a) From [$*$ 7] and Lemma 5.3.(d) comes

$$(xy)z \leq ((xz)z)\ (yz) = x(yz).$$

Similarly, one proves that $(yz)x \leq (xy)z$ and equality follows. □

Open Problem 7.1 *Is A a sub-algebra of L ? Or equivalently, is A closed under $*$?*

Another important property of associative elements is that [$*$ 7] and [$*$ 8] become equalities.

Lemma 7.3 *Let L be a special wEa and let t an associative element of L. Then, for all $x, y \in L$*

$$\begin{aligned}[(x * t) * (y * t)] * t = (x * y) * t &= (x * t) * [(y * t) * t] \\ &= (y * t) * [(x * t) * t].\end{aligned}$$

Proof : Applying [$*$ 7] and [$*$ 8], we get

$$(xt)\ [(yt)t] \leq [(xt)\ (yt)][(t\ (xt))\ (xt)] = [(xt)\ (yt)]\ t \leq (xy)\ t,$$

a relation that, together with the inequalities in [$*$ 7], [$*$ 8] and Lemma 5.3.(d), yields the desired conclusion. □

Proposition 7.4 *Let L be a special wEan. For all x, y, $z \in L$*

a) $x \leq \neg\neg x$ *and* $\neg x = \neg\neg\neg x$; $x \leq y$ *implies* $\neg\neg x \leq \neg\neg y$.

b) $x * y \leq \neg x * \neg y = \neg\neg x * \neg\neg y$.

c) $\neg(x * y) = \neg x * \neg\neg y = \neg y * \neg\neg x = \neg(\neg x * \neg y)$.

d) $\neg\neg(x * y) = \neg\neg x * \neg\neg y$.

e) $\neg x = \neg[y * (x * y)]$; $[y * (x * y)] \leq \neg\neg x$.

f) $\neg\neg x = \neg y * \neg(x * y)$.

g) $\neg(x * (y * z)) = \neg((x * y) * z)$.

h) $\neg\neg x * (\neg\neg y * \neg\neg z) = (\neg\neg x * \neg\neg y) * \neg\neg z$.

i) $x * y = \bot$ *iff* $x * \neg\neg y = \bot$ *iff* $\neg x = \neg\neg y$.

Proof : a) comes from Lemma 5.3.(b), [$*$ 4] and **[neg 2]** (Remark 7.1). Item (b) follows from Lemma 5.3.(c) and (a). Item (c) is just a restatement of Proposition 7.3, with $\bot = t$. For (d), we get, using (c)

$$\neg\neg(x * y) = \neg(\neg x * \neg\neg y) = \neg\neg x * \neg\neg(\neg\neg y) = \neg\neg x * \neg\neg y,$$

as desired. For (e), first note that by **[neg 2]** (Remark 7.1) and Lemma 5.3.(b),

$$\neg[y * (x * y)] \leq \neg x.$$

For the reverse inequality, we have, by 5.3.(b) and (c)

$$\neg x \leq \neg\neg y * (\neg x * \neg\neg y) = \neg\neg y * \neg(x * y) = \neg[y * (x * y)],$$

completing the verification of the first part of (e); the second follows immediately from (a). For (f), the preceding results yield

$\neg\neg x = \neg\neg[y * (x * y] = \neg[\neg\neg y * \neg(x * y)] =$

$\neg(\neg\neg y) * \neg\neg[\neg(x * y)] = \neg y * \neg(x * y),$

as needed. For (g), first note that it is enough to verify that

$$\neg[x * (y * z)] \leq \neg[(x * y) * z]. \tag{I}$$

In fact, we have, applying (I) in succession

$\neg[(x * y) * z] = \neg[z * (y * x)] \leq \neg[(z * y) * x] = \neg[x * (y * z)].$

To prove (I), compute as follows, recalling [∗ 7], (c), (d) and (f) :

$\neg[x * (y * z)] = \neg x * \neg\neg(y * z) = \neg x * (\neg\neg y * \neg\neg z)$

$\leq (\neg x * \neg\neg y) * [(\neg\neg z * \neg x) * \neg x] = (\neg x * \neg\neg y) * \neg\neg z$

$= (\neg x * y) * \neg\neg z = \neg[(x * y) * z],$

completing the proof of (g). For (h), we have, using (d) and (g) :

$$\begin{aligned} \neg\neg x * (\neg\neg y * \neg\neg z) &= \neg\neg x * \neg\neg(y * z) = \neg\neg[x * (y * z)] \\ &= \neg\neg[(x * y) * z] = \neg\neg(x * y) * \neg\neg z \\ &= (\neg\neg x * \neg\neg y) * \neg\neg z. \end{aligned}$$

To verify (i), note that if $x * y = \bot$, then $(x * y) * \bot = \top$. Thus, by [∗ 7], $\neg x * \neg\neg y = \top$, that is, $\neg x = \neg\neg y$. On the other hand, if this equation is true, then

$$\bot = \neg x * \neg y = \neg\neg x * \neg\neg y = \neg\neg(x * y),$$

and so (a) implies $x * y = \bot$, ending the proof. □

8 The Embedding Theorem. Applications

The ∗-filters on a wEa L were constructed so as to have the following separation property (3.7) :

Lemma 8.1 *Let L be a wEa and let $a, b \in L$. Let A be a proper $*$-filter on L. If $(a * b) \notin A$, then there is a proper $*$-filter B, containing A, that separates a and b.*

Proof : Since A is a $*$-filter, we cannot have $\tau(A \cup \{a\}) = \tau(A \cup \{b\})$, otherwise $(a * b) \in \tau(A) = A$. Hence, either $\tau(A \cup \{a\})$ or $\tau(A \cup \{b\})$ separates a and b. □

Let L be a wEa. Recall (6.12) that $S(L)$ is the set of all proper $*$-filters on L. For $x \in L$, set

$$S_x = \{F \in S(L) : x \in F\}.$$

We take $\mathcal{B} = \{S_x : x \in L\}$ as a sub-basis for a topology $\Omega(L)$ on $S(L)$, that is, for all $U \subseteq S(L)$

$U \in \Omega(L)$ iff There is $K \subseteq 2^A_\omega$ such that $U = \bigcup_{k \in K} \bigcap_{x \in k} S_x$.

Thus, the empty set together with the finite intersections of elements in $\mathcal{B}$ constitute a basis for $\Omega(L)$. Note that $S_\top = S(L)$; if L has a least element $\bot$, then $S_\bot = \emptyset$. It is well-known that $\Omega(L)$ is a *complete Heyting algebra* (*cHa*) (see section 2). We are now in a position to generalize Theorem 3.9 to all equivalence algebras.

Theorem 8.2 *If L is an equivalence algebra, the map*

$$\sigma : L \longrightarrow \Omega(L), \text{ given by } a \mapsto S_a,$$

is an embedding of L in $\Omega(L)$, satisfying

a) If L has a least element $\bot$, then σ takes $\bot$ to $\bot$ in $\Omega(L)$.

b) If $F \in S(L)$, then the filter G generated by F in $\Omega(L)$ is a proper filter and

$$\text{For all } a \in L, \quad a \in F \quad \text{iff} \quad S_a \in G,$$

that is, $F = \sigma^{-1}(G)$.

Proof : Write Ω for $\Omega(L)$. The definition of filter (5.4) yields $x \leq y$ implies $S_x \subseteq S_y$. For the converse, note that since $x^{\rightarrow}$ is a $*$-filter and $x^{\rightarrow} \in S_x$, it follows that $S_x \subseteq S_y$ implies $y \in x^{\rightarrow}$.

For $x, y \in L$, Lemma 5.5.(a) yields

$$S_x \cap S_{x*y} = S_y \cap S_{x*y}.$$

Thus, $S_{x*y} \subseteq (S_x \equiv S_y)$ in the cHa Ω. Since the intersection of finite subsets of $\mathcal{B}$ is a basis for Ω and intersection distributes over arbitrary joins in Ω (it is a cHa), to show that

$$S_{x*y} = (S_x \equiv S_y) \quad (\text{in } \Omega) \tag{1}$$

it is enough to verify that if $\{t_1, \ldots, t_n\} \subseteq L$ and $V = \bigcap_{i=1}^{n} S_{t_i}$, then

$$V \cap S_x = V \cap S_y \quad \text{implies} \quad V \subseteq S_{x*y}.$$

Assume that there is $F \in V$ such that $(x * y) \notin F$. By 8.1, there is $G \in S(L)$ satisfying $F \subseteq G$ and separating a and b, that is,

$$\text{either} \quad (x \in G \text{ and } y \notin G) \quad \text{or} \quad (y \in G \text{ and } x \notin G). \tag{2}$$

Since $F \subseteq G$, we have $t_i \in G$, $1 \leq i \leq n$, that is, $G \in V$; but then, the alternatives in (2) imply that $S_x \cap V \neq S_y \cap V$, a contradiction. Therefore, (1) is true and σ is an embedding. Item (a) is clear.

For (b), if F be a proper $*$-filter in L, then

$$G = \{U \in \Omega : \text{There is a finite } K \subseteq F \text{ such that } \textstyle\bigcap_{t \in K} S_t \subseteq U\},$$

is the filter generated by $\sigma(F) = \{S_t : t \in F\}$ in Ω. Since $F \in S_t$, $t \in F$, $\sigma(F)$ has the finite intersection property. Thus, G is a proper filter in Ω. Clearly, $\sigma(F) \subseteq G$. Conversely, if $S_a \in G$, then there is a finite $A \subseteq F$ such that $\bigcap_{t \in A} S_t \subseteq S_a$; thus $F \in S_a$, i.e., $a \in F$, ending the proof. □

Theorem 8.2 has a number of important consequences. Here is a sample.

Corollary 8.3 *If L is an Ea, the embedding $\sigma : L \longrightarrow \Omega(L)$ satisfies $\sigma^* \tau_{\Omega(L)} = \tau_L$. In particular, the $*$-filters in L are precisely the inverse image of the filters in $\Omega(L)$ by σ.*

Proof : By 6.10, it is enough to show that $\sigma^* \tau_{\Omega(L)} \leq \tau_L$. Recall (6.16) that for $B \subseteq \Omega(L)$, $\tau_{\Omega(L)}(B) = [B]$, the filter generated by B. For $A \subseteq L$, since $\tau_L(A)$ is a $*$-filter, 8.2.(b) yields

$$\tau_L(A) = \sigma^{-1}([\sigma(\tau_L(A))]) = \sigma^* \tau_{\Omega(L)},$$

as claimed. □

Corollary 8.4 *a) An equivalence algebra with $\bot$ is an equivalence algebra with negation.*

b) Every equivalence algebra can be embedded in an equivalence algebra with negation.

Proof : (b) is clear. For (a), we give two proofs. Let L be an Ea with $\bot$.

<u>First Proof</u> : By 8.2, L is isomorphic to a sub-algebra K of Ω, with $\bot$ $(= \emptyset) \in K$. Since K is an Ean (6.22.(c)), so is L.

<u>Second Proof</u> : Suppose that for some $a \in L$, $b = a * (a * \bot) \neq \bot$. Then, $b^{\rightarrow}$ is a proper $*$-filter in L; moreover, $(a * \bot)$ cannot be in $b^{\rightarrow}$, for otherwise it would not be proper. By 8.1, there is a proper $*$-filter F in L, containing $b^{\rightarrow}$, such that $a \in F$. Then, from $b \in F$ and $a \in F$, we get $\bot \in F$, a contradiction. Thus, for all $a \in L$, $\bot = (a * \bot) * a$, and L is an Ean. □

Corollary 8.5 *Let u be an element of an Ea L. Then, $u^{\rightarrow}$ is an Ean. In particular, for all $a \geq u$, $u = a * (a * u)$.*

Proof : Immediate from Corollary 8.4.(a), once it is remarked that $u^{\rightarrow}$ is a sub-algebra of L, with $\bot = u$. □

Corollary 8.6 *Every Ea satisfies axioms* $[* \ i]$, $5 \leq i \leq 7$ *in 3.4, as well as the following rules :*

$[* \ 8]$: $[(x * z) * (x * z)] * z \leq (x * y) * z$.

$[* \ 9]$: $x \leq y$ *iff* $y * (x * y) \leq x * y$ *iff* $x = y * (x * y)$.

Proof : It is left to the reader to check that the aforementioned rules hold in H_{eq}, for any Heyting algebra H. The result then follows from 8.2.(a). □

9 Ea-quotients

Let L be a wEa and let F be a proper filter on L. For x, $y \in L$, define

$$x \, \theta_F \, y \quad \text{iff} \quad (x * y) \in F.$$

Lemma 9.1 θ_F *is a congruence on* L.

Proof : It must be verified that θ_F is an equivalence relation, such that for all x, y, t, $z \in L$

$$x \, \theta_F \, t \ \text{ and } \ y \, \theta_F \, z \quad \text{implies} \quad (x * y) \, \theta_F \, (t * z). \qquad \text{(I)}$$

Clearly, θ_F is reflexive and symmetric, while its transitivity follows from the fact that F is a filter. Hence, θ_F is an equivalence relation on L. To show that it is a congruence with respect to $*$, we may apply 5.5.(c) to get (I), ending the proof. □

If F is a proper filter in a wEa L and x, $y \in L$,

* Write x/F for the equivalence class of x with respect to θ_F;

* Write $L/F = \{x/F : x \in L\}$ for the set of equivalence classes of elements of L by θ_F;

$*$ Write $\pi_L : L \longrightarrow L/F$ for the canonical quotient map, $x \mapsto x/L$;

$*$ Define an operation $*$ on L/F by

$$x/F * y/F = (x * y)/F,$$

which is independent of representatives by 9.1. It is clear that the structure $\langle L/F, *, \top/F \rangle$ satisfies axioms [$*$ 1], [$*$ 2] and [$*$ 3] in 5.1.

Now suppose F is a $*$-filter on the wEa L and define, for x, $y \in L$,

[po] $\quad x/F \leq y/F \quad$ iff $\quad y \in \tau(F \cup \{x\})$.

Proposition 9.2 *If F is a $*$-filter on a wEa (wEan) L, then*

a) The relation defined in [po] *is independent of representatives and constitutes a partial order in L/F whose largest element in $\top/F$ and, whenever L has $\bot$, has $\bot/F$ as its least element.*

b) $L/F = \langle L/F, \leq, \top, * \rangle$ is a wEa (resp., wEan) and the quotient map π_F is a wEa-morphism (resp., wEan-morphism).

Proof : a) Suppose (xt), $(yz) \in F$ and $y \in \tau(F \cup \{x\})$. Then,

$$(yz),\, y \in \tau(F \cup \{x\}) = \tau(F \cup \{t\}),$$

and so $z \in \tau(F \cup \{t\})$ (5.5.(a)). Clearly, $\leq$ is reflexive on L/F. Since F is a $*$-filter

$$x \in \tau(F \cup \{y\}) \text{ and } y \in \tau(F \cup \{x\})$$

implies $xy \in F$, showing that $\leq$ is antisymmetric in L/F. For transitivity, we have, note that (6.8.(a))

$$y \in \tau(F \cup \{z\}) \quad \text{implies} \quad \tau(F \cup \{y\}) \subseteq \tau(F \cup \{z\}),$$

and so $x \in \tau(F \cup \{y\})$ and $y \in \tau(F \cup \{z\})$ yields $x \in \tau(F \cup \{z\})$, verifying transitivity. Since

$$\tau(F \cup \{\top\}) = \tau(F) = F \quad \text{and} \quad \tau(F \cup \{\bot\}) = \tau(\{\bot\}) = L,$$

it follows that $\bot/F$ (when L has $\bot$) and $\top/F$ are, respectively, the least and largest elements of L/F in the partial order $\leq$.

b) It remains to verify [$*$ 4] for L/F. Assume that $a/F \leq (x * y)/F$ and $a/F \leq (t * z)/F$. Set $G = \tau(F \cup \{a\})$. Hence, xy, $tz \in G$, which implies, since G is a filter, that $(xt) * (yz) \in G$, as needed. It is clear that π_F is a wEa-morphism or a wEan-morphism, when L is a wEan. □

Before proving that if L is an Ea or an Ean, the same is true for the quotient of L be a $*$-filter, we recall some basic facts about quotients of Heyting algebras. This will also lead to a version of the fundamental theorem for morphisms of Ea's and Ean's.

If H is a Heyting algebra and G is a filter on H, recall that the congruence θ_G defined by G on H may be described by

$$a \; \theta_G \; b \quad \text{iff} \quad \text{there is } t \in G \text{ such that } a \wedge t = b \wedge t.$$

The quotient H/G is a Heyting algebra and the canonical quotient map,

$$\pi_G : H \longrightarrow H/G, \; a \mapsto a/G,$$

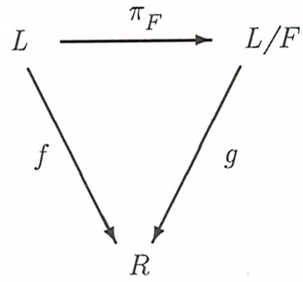
is a morphism of Heyting algebras, that is, it preserves $\bot$, $\top$ and the operations meet ($\wedge$), join ($\vee$), negation ($\neg$), implication ($\rightarrow$) and equivalence ($\equiv$). In particular, π_G is a morphism of Ean's. The next result is stated for equivalence algebras, but **is valid *verbatim* for Ean's.**

Theorem 9.3 *Let L be an Ea and let F be a proper $*$-filter in L. With notation as in 8.2, let G be the (proper) filter generated by $\sigma(F)$ in $\Omega(L)$. Then,*

a) The map $\alpha : L/F \longrightarrow \Omega(L)/G$, defined by $\alpha(a/F) = S_a/G$, is a wEa-embedding.

b) $L/F = \langle L/F, \leq, *, \top, * \rangle$ is an Ea.

c) If $L \xrightarrow{f} R$ *is a Ea-morphism such that* $F \subseteq$ *coker* f*, then there is a unique Ea-morphism* $L/F \xrightarrow{g} R$*, such that the following diagram commutes :*



Moreover, g *is an embedding iff* $F =$ *coker* f*.*

Proof : We identify L its image via the embedding $\sigma : L \longrightarrow \Omega(L)$. Under this identification, if $A \subseteq L$, write $[A]_H$ for the filter generated by A in $\Omega(L)$. Note that for all $B \subseteq \Omega(L)$, $B \cap L$ stands for $\sigma^{-1}(B)$. Write τ for the T-operator τ_L on L.

a) Since $F \subseteq G$, it is clear that α is well-defined and that $\alpha(\top/F) = \top/G$. To check preservation of $*$, let $x, y \in L$. Then,

$$\begin{aligned}\alpha(x/F * y/F) &= \alpha((x * y)/F) = S_{x*y}/G = (S_x \equiv S_y)/G \\ &= (S_x/G \equiv S_y/G) = [\alpha(x/F) \equiv \alpha(y/F)],\end{aligned}$$

as needed. Now we observe

Fact *For all* $x \in L$, $[\tau(F \cup \{x\})]_H = [F \cup \{x\}]_H$.

Proof Clearly, $[F \cup \{x\}]_H \subseteq [\tau(F \cup \{x\})]_H$. For the reverse inclusion :

$*$ By 6.16.(a), every filter in $\Omega(L)$ is an $*$-filter;

$*$ 6.22.(a) guarantees that $I = L \cap [F \cup \{x\}]_H$ is a $*$-filter on L.

Note that $F \cup \{x\} \subseteq I$. Hence, $\tau(F \cup \{x\}) \subseteq \tau(I) = I$. It follows that $[\tau(F \cup \{x\})]_H \subseteq [F \cup \{x\}]_H$, ending the proof of the Fact.

Now let $x, y \in L$ be such that $x/F \leq y/F$. Then, the Fact yields

$$y \in \tau(F \cup \{x\}) \subseteq [\tau(F \cup \{x\})]_H = [F \cup \{x\}]_H,$$

and so there is $u \in F$ such that $S_x \cap S_u \subseteq S_y$. Hence,

$$S_x \cap S_y \cap S_u = S_x \cap S_u,$$

that is, $S_x/G \leq S_y/G$, showing that α is a wEa-morphism. Conversely, suppose that $S_x/G \leq S_y/G$ in $\Omega(L)$. Hence, there is $V \in G$, such that $S_x \cap V \subseteq S_y$. Since G is the filter generated by F, there are $a_1, \ldots, a_n$ in F, satisfying $\bigcap_{i=1}^n S_{a_i} \subseteq V$. Therefore

$$S_x \cap \bigcap_{i=1}^n S_{a_i} \subseteq S_y.$$

This last inequality implies that $y \in [F \cup \{x\}]_H$, and so, by the Fact and 8.2.(b), $y \in \tau(F \cup \{x\})$, and $x/F \leq y/F$. This completes the proof that α is a wEa-embedding. Item (b) follows immediately from 6.22.(b).

c) Uniqueness is clear. For $x \in L$, set $g(x/F) = f(x)$; since $\{\top\}$ is a $*$-filter in R (it is an Ea) and $F \subseteq coker\ f$, for all $x, y \in L$

$$(x * y) \in F \quad \Rightarrow \quad f(x * y) = \top \quad \text{iff} \quad f(x) = f(y), \tag{1}$$

and g is well defined. If $F =$ coker f, then the first implication in (1) is an equivalence, and so g will be injective iff coker $f = F$. Clearly, g preserves $*$. If $x/F \leq y/F$, then $y \in \tau(F \cup \{x\})$. It must be verified that $f(x) \leq f(y)$. Consider the $*$-filter $I = f(x)^{\rightarrow} \subseteq R$; by 6.22.(a), $G = f^{-1}(I)$ is a $*$-filter on L. Since $\{f(x), \top\} \subseteq I$, we conclude that $F \cup \{x\} \subseteq coker\ f \cup \{x\} \subseteq G$. Thus,

$$\tau(F \cup \{x\}) \subseteq \tau(G) = G,$$

whence $y \in G$, that is, $f(x) \leq f(y)$, ending the proof. □

When L is an Ea or an Ean and F is a $*$-filter on L, the structure L/F constructed above is called the **quotient** of L by F. It comes with a canonical quotient morphism, $\pi_F : L \longrightarrow L/F$.

10 A non-constructive Embedding Theorem

In this section we describe a non-constructive way of embedding an equivalence algebra in a complete Heyting algebra, making use of the concept of irreducible filter, as in section 1 of chapter II in [Ras74].

Definition 10.1 *A proper filter F in a wEa L is* **irreducible** *if for all filters G_1, G_2 in L*

$$F = G_1 \cap G_2 \quad \textit{implies} \quad F = G_1 \textit{ or } F = G_2.$$

Write $\mathcal{I}(L)$ for the set of irreducible $*$-filters in L. For $x \in L$, set

$$I_x = \{F \in \mathcal{I}(L) : x \in F\}.$$

Note that $I_\top = \mathcal{I}(L)$, while if L is an wEan, $I_\bot = \emptyset$.

Proposition 10.2 *Let L be a wEa, F a proper $*$-filter in L and let a, b be elements of L.*

a) If $a \notin F$, then there is an irreducible $*$-filter G such that $F \subseteq G$ and $a \notin G$.

b) If $(a * b) \notin F$, then there is an irreducible $*$-filter G containing F and separating a and b.

Proof : a) Let $V = \{G \in S(L) : F \subseteq G$ and $a \notin G\}$, partially ordered by inclusion. Clearly, V is non-empty and all chains in

V have an upper bound. By Zorn's Lemma, V has a maximal element G, with $F \subseteq G$ and $a \notin G$. To show that G is irreducible, assume that $G = H_1 \cap H_2$, for filters H_1, H_2 in L. Since $a \notin G$, a must be outside one of the H_i's, say $a \notin H_1$. But then, $H_1 \in V$ and so the maximality of G and $G \subseteq H_1$ imply that $G = H_1$.

b) By Lemma 8.1, there is a proper $*$-filter K, containing F and satisfying the alternative in the statement. If $a \in K$ and $b \notin K$, (a) yields an irreducible $*$-filter G, containing K, satisfying the same condition. The other possibility is handled similarly and the proof is complete. □

Irreducibility generalizes primeness in distributive lattices :

Lemma 10.3 *In a distributive lattice A with top element, a filter F is irreducible iff it is prime, that is*

For all $a, b \in L$, $(a \vee b) \in L$ implies $a \in F$ or $b \in F$.

Proof : (1) $\Rightarrow$ (2) : Suppose that neither a nor b are in F. Define

$$G_1 = \{x \in A : x \geq a \wedge z, \text{ for some } z \in F\}.$$

It is straightforward to check that G_1 is a filter in A, that is, it satisfies

$*$ $\top \in G_1$;

$*$ $x \in G_1$ and $y \geq x$ implies $y \in G_1$;

$*$ $x, y \in G_1$ implies $x \wedge y \in G_1$.

It is clear that $a \in G_1$ and that $F \subseteq G_1$. Thus, $F \neq G_1$. Similarly, we may define

$$G_2 = \{x \in A : x \geq b \wedge z, \text{ for some } z \in F\},$$

to get $b \in G_2 \setminus F$, with $F \subseteq G_2$. We now show that $G_1 \cap G_2 = F$, a contradiction that will end the proof of (1) $\Rightarrow$ (2). For $x \in G_1 \cap G_2$, there are $t, z \in F$ such that

$$x \geq a \wedge t \quad \text{and} \quad x \geq b \wedge z.$$

Let $c = t \wedge z$ ($\in F$); the inequalities above imply that x is larger than $(a \wedge c)$ and $(b \wedge c)$. Thus,

$$x \geq (a \wedge c) \vee (b \wedge c) = c \wedge (a \vee b) \in F,$$

and so $x \in F$, as claimed.

(2) $\Rightarrow$ (1) : Suppose that $F = G_1 \cap G_2$, with $F \neq G_i$, $i = 1, 2$. Select $a \in G_1$ and $b \in G_2$, both outside F. Since $a, b \leq a \vee b$, we conclude that $a \vee b \in G_1 \cap G_2 = F$ a contradiction since neither a nor b are in F. □

Remark 10.4 In spite 10.3, there is an important difference between prime filters in distributive lattices and irreducible filters in Ea's : prime filters are functorial and irreducible filters are not.

We take $\{\emptyset\} \cup \{I_x : x \in L\}$ as a sub-basis for a topology on $\mathcal{I}(L)$; let $\boldsymbol{\Omega_{ir}(L)}$ **be the cHa of opens of this topology**. The proof of Theorem 8.2, with Proposition 10.2 in place of Lemma 8.1, can be adapted to yield

Theorem 10.5 *Let L be an equivalence algebra. Then, the map*

$$h : L \longrightarrow \Omega_{ir}(L), \text{ given by } x \mapsto I_x,$$

is an Ea-embedding of L into $\Omega_{ir}(L)$. Moreover, if L has a least element $\perp$, then h takes $\perp$ to $\perp$ in $\Omega_{ir}(L)$.

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