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CONTINGENCY LOGICS AND PROPOSITIONAL QUANTIFICATION

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The problem of defining modal operators using contingency as a primitive is discussed by assuming what Lewis and Langford call the "Existence Postulate", an axiom which is formulated in a modal language with propositional quantifiers. It is shown that the minimal contingency logic $\mathbf{K}^{\Delta}$ *extended with the contingential counterpart of the Existence Postulate is definitionally equivalent to the deontic logic* **KD** *extended with propositional quantifiers.*

1. THE EXISTENCE POSTULATE

It is well known that in the work which gave birth to contemporary modal logic, *Symbolic Logic* (1932), Lewis and Langford introduce a controversial axiom, named by them the *Existence Postulate*, to be added to the system they hold to be the basic system of modal logic, **S2**. The formulation they give of the Existence Postulate is:

$$(EP)\ \exists p \exists q\, (\Diamond(p \,\&\, q) \,\&\, \Diamond(p \,\&\, \neg q)).$$

EP contains propositional quantifiers and so it can be formulated only in the framework of a linguistic extension of propositional modal logic allowing quantification over propositional variables. The meaning of *EP* may perhaps be grasped more easily in the equivalent form:

$$(1)\ \exists p \exists q(\Diamond(p \,\&\, q) \,\&\, \neg(p \Rightarrow q))$$

which states the existence of at least two propositions which are consistent and independent.

The formulations *EP* and (1) of the Existence Postulate are two-variable-formulas. But it is useful to remark that in every modal system including **S2** there is at least a one-variable formula which is equivalent to *EP* and (1). To begin with, let us recall that in every logic of propositional quantifiers we have at our diposal the rule:

$$(R\exists)\ \vdash A \rightarrow B \rightarrow \vdash \exists pA \rightarrow \exists pB.$$

In **S2** we have as a theorem, by the so-called Consistency Axiom $\Diamond(p \,\&\, q) \rightarrow \Diamond q$,

$$(2)\ (\Diamond(p \,\&\, q) \,\&\, \Diamond(p \,\&\, \neg q)) \rightarrow (\Diamond q \,\&\, \Diamond\neg q)$$

Moving from (2) we obtain by $(R\exists)$, *EP*, Modus Ponens and the equivalence $\exists p \exists q(\Diamond q \,\&\, \Diamond\neg q) \leftrightarrow \exists q(\Diamond q \,\&\, \Diamond\neg q)$, the simple:

$$(3)\ \exists q(\Diamond q \,\&\, \Diamond\neg q).$$

On the other hand from (3) we have, thanks to the equivalence $\vdash q \leftrightarrow (q \,\&\, T)$ (T being a truth-functional tautology) and Replacement of Proved Equivalents:

$$(4)\ \exists q\ (\Diamond(T \,\&\, q) \,\&\, \Diamond(T \,\&\, \neg q)).$$

Thus, introducing a second existential quantifier, from (4) we reach again *EP*, i.e.:

(5) $\exists p \exists q (\Diamond(p \,\&\, q) \,\&\, \Diamond(p \,\&\, \neg q))$

The simple wff $\exists q(\Diamond q \,\&\, \Diamond\neg q)$ in (3) is then equivalent to the Existence Postulate in every system having standard rules for propositional quantifiers and such as to contain **S2**: so the two formulas are equivalent in every normal modal logic. Now if we recall the definition of the contingency operator as:

(6) $\nabla A =_{Df} \Diamond A \,\&\, \Diamond\neg A$

or alternatively the definition of the non-contingency operator as:

(7) $\Delta A =_{Df}\ A \vee \neg A$

the formula (3) which is equivalent to *EP* boils down to the simple assertion:

(8) $\exists q \nabla q$

or to the equivalent wff $\exists q \neg\Delta q$.

In what follows we will call the one-variable statement (8) the *Contingency Postulate* (*CP*), which is actually the contingential counterpart of the Existence Postulate. The meaning of *CP* is then the reasonable assertion that some proposition is contingent, which means that there is at least one proposition which may be true and may be false. The negation of *CP*, $\neg$*CP*, amounts to $(q)\Delta q$, i.e. $(q)(\Diamond q \rightarrow \Box q)$; and, given that Lewis and Langford never question the law $q \rightarrow \Diamond q$, in every Lewis system $\neg$*CP* implies $(q)\ (q \rightarrow \Box q)$ and $(q)\ (q \leftrightarrow \Box q)$, i.e. the universal quantification of the collapse formula. As is well known, the intuitive meaning of the collapse formula in terms of possible world semantics is that in every model there is only one world which is accessible to the reference world. Since the Contingency Postulate is the negation of the collapse formula, its intuitive meaning is then that there are at least two possible worlds in every accessibility sphere.

To sum up, the result of assuming among the axioms the Contingency Postulate, or the equivalent Existence Postulate, is that the latter assumption turns the collapse formula into an inconsistency, while it is well known that the collapse formula is consistent with each of the standard propositional modal logics.

The Contingency Postulate is a completely intuitive claim about contingent propositions, so it is reasonable to embody it into any system of contingency logic extended with axioms for propositional quantifiers.

2. CONTINGENCY LOGICS

It has been proved that the contingency fragment of the minimal normal system **K**, which is the weakest normal contingency logic, may be axiomatized as follows by using both contingency and non-contingency operators:

$K\Delta 1\ \Delta p \leftrightarrow \Delta\neg p$

$K\Delta 2\ (\Delta p \,\&\, \Delta q) \rightarrow \Delta(p \,\&\, q)$

$K\Delta 3\ (\Delta p \,\&\, \nabla(\neg p \vee r)) \rightarrow \Delta(p \vee q)$.

$(\Delta\ \mathrm{Nec})\ \vdash A \rightarrow\ \vdash \Delta A$

Let us call the preceding system $\mathbf{K}^{\Delta}$ (see Kuhn (1995)). It is remarkable that $\mathbf{K}^{\Delta}$ turns out to be not only the contingency fragment of **K** but the contingency fragment of **KD** (i.e. of $\mathbf{K} + \Box p \rightarrow \Diamond p$). In other words adding to $\mathbf{K}^{\Delta}$ the deontic axiom **D**: $\Box p \rightarrow \Diamond p$, or the equivalent formula $\Diamond T$, yields no new theorems containing contingency operators.

The following are some useful theorems of $\mathbf{K}^{\Delta}$:

(9) $\nabla(p \,\&\, q) \rightarrow (\nabla p \vee \nabla q)$

(10) $(\nabla(p \rightarrow r) \,\&\, \nabla(p \vee q)) \rightarrow \nabla p$

(11) $(\nabla(\neg p \vee q) \,\&\, \nabla(p \,\&\, q)) \rightarrow \nabla p$.

We now extend the language of $\mathbf{K}^{\Delta}$ with suitable formation rules for propositionally quantified wffs and the axioms of $\mathbf{K}^{\Delta}$ with a set of axioms for propositional quantifiers including *CP*.

As Kit Fine remarked (see Fine (1970)), when we extend a modal system with axioms for propositional quantifiers, we have at least three choices for every modal system. If **X** is an arbitrary modal system, let us call $\mathbf{X}\pi$ the system which is **X** extended with the axioms:

*QP*1. $(p)(A(p) \rightarrow A(B))$ (where B is a **K**-formula free for p in $A(p)$)

*QP*2. $(p)(A \rightarrow B) \rightarrow ((p)A \rightarrow (p)B)$

*QP*3. $A \rightarrow (p)A$ $\qquad$ p not free in A

and the rule

UG: $A \,/\, (p)A$.

The two variants of $\mathbf{X}\pi$ identified by Fine are the following:

(i) $\mathbf{X}\pi$-: i.e. a system which is as $\mathbf{X}\pi$ with the only difference that in *QP*1 B is assumed to be a formula of the truth-functional calculus **PC** and not an arbitrary **K**-formula

(ii) $\mathbf{X}\pi$+: i.e. a system which is $\mathbf{X}\pi$ extended with the axiom:

$(QP4)\ \exists p_1\ (p_1\ \&\ (p_2)(p_2 \rightarrow\ \ (p_1 \rightarrow p_2))$

(asserting that for any world x of every model there is a proposition describing it).

In what follows we will neglect **X**π- and **X**π+, being interested in variants yielded by the presence or the absence of the Existence Postulate. More specifically, here we will consider only two different minimal normal systems with propositional quantifiers, **K**π and its extension **K**π + *EP*, which in what follows will be conventionally named **K**π°.

In a parallel way we have at our disposal two minimal contingency systems with propositional quantifiers: the first one is $\mathbf{K}^\Delta$ extended with *QP*1, *QP*2, and *UG* – which will be named here $\mathbf{K}^\Delta\pi$ – and the second one is $\mathbf{K}^\Delta\pi$ extended with the contingency axiom *CP*, which will be named here $\mathbf{K}^\Delta\pi^\circ$. Of course we have $\mathbf{K}^\Delta\pi \subseteq \mathbf{K}^\Delta\pi^\circ \subseteq \mathbf{K}\pi^\circ$. From what has been said, $\mathbf{K}^\Delta\pi^\circ$ has a high plausibility as a minimal contingency system with propositional quantifiers, due to the high plausibility of the Contingency Postulate.

Remark 0. An interesting consequence of the Contingency Postulate is the following. It is well known that among the theorems of First Order Logic we have:

$$(12)\ \ ((x)(Px \rightarrow Qx)\ \&\ \exists xPx) \rightarrow \exists x(Px\ \&\ Qx).$$

So by an obvious parallelism in logics of propositional quantification we have that:

$$(13)\ \ ((p)\ (\nabla p \rightarrow A)\ \&\ \exists p\ \nabla p) \text{ implies } \exists p(\nabla p\ \&\ A).$$

Thus if $\exists\ p\nabla p$, i.e. the Contingency Postulate, is subjoined as an additional axiom to any contingency system with propositional quantifiers, a theorem derivable from (13) is:

(14) $(p)\ (\nabla p \rightarrow A) \rightarrow \exists p\ (\nabla p\ \&\ A)$.

Something should be said about the semantical analysis of modal systems with propositional quantifiers. The completeness results which are provable for such systems may be sketched following the lines of Fine (1970). A **K**π-model is a 4-ple $M = < W, R, \Pi, v >$ where:

a) $W \neq \varnothing$

c) $R \subseteq W \times W$

d) $\Pi \subseteq P(W)$

e) v is a map from W into Π.

f) Π is closed under wffs, i.e. $\{x\ \varepsilon\ W{:}\ V^M A = 1\}\ \varepsilon\ \Pi$

The additional constraint over the accessibility relation which is required in order to validate the Existence Postulate in **K**$\pi°$ is simply:

g) $\exists x\, \exists y (xRy\ \&\ x \neq y)$.

The condition for the truth of atomic wffs is given in this way:

h_a) $V^M(p_i, x) = 1$ iff $x \in v(p_i)$, $i = 1,2...$

The truth conditions for truth-functional and modal wffs are as usual, while for quantified wffs we have the following:

h_b) $V^M((p_i)B, x) = 1$ iff $V^{M}(B, x) = 1$ for all models $M' = <W, R, \pi, v'>$ such that $v(p_j) = v(p_i)$

for all $j \neq i$, $i = 1, 2...$

A completeness proof for **K**π and **K**$\pi°$ with respect to the given semantics may be given by the standard Henkin method.

The models for the contingency logic $\mathbf{K}^{\Delta}\pi$ may be defined following the lines of Humberstone (1995). $\mathbf{K}^{\Delta}\pi$- models as defined by Humberstone are **K**π-models with the following clause replacing the standard clause for the necessity operator:

i) $V^M(\Delta\ A,\ x) = 1$ iff for all y, z such that xRy and xRz, $V^M(A, y) = V^M(A, z)$.

If a $\mathbf{K}^{\Delta}\pi$-model satisfies g) beyond the other listed conditions, it is a $\mathbf{K}^{\Delta}\pi°$-model[1].

A completeness proof of both $\mathbf{K}^{\Delta}\pi$ and $\mathbf{K}^{\Delta}\pi°$ may be given unproblematically with suitable modifications of the method of Kuhn (1995) or (endorsing the alternative semantics outlined in note 1) of Pizzi (forthcoming).

3. THE DEFINITION OF MODAL OPERATORS IN TERMS OF CONTINGENCY

A topic of contingency logic which has been only partially investigated in recent decades concerns the possibility of defining standard modal operators in terms of the contingency operator.

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[1] A different approach to contingential semantics is set out in Pizzi (forthcoming) where the accessibility relations are defined in terms of sets of possible worlds with one or two worlds (standard accessibility relations being treated as relations among singletons).

A $\mathbf{K}\pi^{\Delta}$-*model* is a 4-ple $<W, R^{\Delta}, \Pi, V>$ where:

(i) $W \neq \varnothing$

(ii) If $W^{\Delta} = \{\{x, y\}: x \in W \ \&\ y \in W\}$ (where possibly $x = y$) then

a) $R^{\Delta} \subseteq W^{\Delta} \times W^{\Delta}$

b) $\{x, l\}\ R^{\Delta}\ \{y, z\}$ iff $\{x\}\ R^{\Delta}\ \{y, z\}$ and $\{l\} R^{\Delta}\ \{y, z\}$

(iii) V is as in Fine's models with the following clause replacing the clause for necessity statements:

$V^M(\Delta\ A,\ x) = 1$ iff for every y and $z \in W$ such that $\{x\}R^{\Delta}\{y, z\}\ V^M(A,y) = V^M(A, z)$.

An important result in this field is that $\Box$ and $\Diamond$ turn out to be definable in KT^{Δ} (a contingency system equivalent to KT) thanks to the definition:

$$(15) \quad \Box A =_{Df} \Delta A \;\&\; A$$

but are not generally definable in weaker systems (see Cresswell (1988)).

However, extending contingency system with propositional quantifiers opens the door to a different approach to the problem. In order to show this point we introduce a new notion, the notion of non-contingent implication, defined as follows:

$$(16)\ A\,\Delta\!\rightarrow B =_{Df} \Delta(A \rightarrow B)$$

Notice that $\Delta(A \rightarrow B)$ equals (thanks to axiom KD1) both $\Delta\neg(A \;\&\; \neg B)$ and $\Delta(A \;\&\; \neg B)$. Some equivalences yielded by the definition in (16) and the definition of Δ (see (7)) are the following:

$$(17)\ A\,\Delta\!\rightarrow B \leftrightarrow (A \mathrel{-\!\!\ni} B \vee \neg\Diamond(A \rightarrow B))$$

$$(18)\ A\,\Delta\!\rightarrow B \leftrightarrow (A \mathrel{-\!\!\ni} B \vee \Box(A \;\&\; \neg B))$$

$$(19)\ A\,\Delta\!\rightarrow B \leftrightarrow (A \mathrel{-\!\!\ni} B \vee (\Box A \;\&\; \Box\neg B))$$

$$(20)\ A\,\Delta\!\rightarrow B \leftrightarrow ((\Diamond\neg A \vee \Diamond B) \rightarrow \Box(A \rightarrow B))$$

$$(21)\ A\,\Delta\!\rightarrow B \leftrightarrow ((\Diamond\neg A \rightarrow \Box(A \rightarrow B)) \;\&\; (\Diamond B \rightarrow \Box(A \rightarrow B)))$$

$$(22)\ A\,\Delta\!\rightarrow B \leftrightarrow (\Diamond(A \rightarrow B) \rightarrow \Box(A \rightarrow B))$$

An operator which is dual with respect to non-contingent implication turns out to be definable thanks to the following equivalences:

(23) $\neg(A \,\Delta\!\!\rightarrow \neg B) \leftrightarrow ((\Diamond \neg A \vee \Diamond \neg B) \,\&\, \neg\Box(A \rightarrow \neg B))$

(24) $\neg(A \,\Delta\!\!\rightarrow \neg B) \leftrightarrow \nabla(A \,\&\, B)$

Now a simple definition of possibility which we may add to $\mathbf{K}^{\Delta}\pi°$ by making use of the definition of noncontingent implication relies on the idea that a possible proposition is a proposition which is non-contingently implied by some contingent proposition:

(25) $\Diamond A =_{Df} \exists p(\nabla p \,\&\, \Delta(p \rightarrow A))$ (p not in A)

An equivalent definition would be of course:

(26) $\Diamond A =_{Df} \exists p(\nabla p \,\&\, \Delta(p \,\&\, \neg A))$ (p not in A)

We already know that $\mathbf{K}^{\Delta}\pi$ is included in $\mathbf{K}\pi$, so $\mathbf{K}^{\Delta}\pi°$ is included in $\mathbf{K}\pi°$. What we have now to prove in order to justify the definition is that the equivalence between $\Diamond A$ and $\exists p$ $(\nabla p \,\&\, \Delta(p \rightarrow A))$ (p not in A) is a theorem of $\mathbf{K}\pi°$ + Def ∇. In other words we have to prove:

MT1. $\exists p(\nabla p \,\&\, \Delta(p \rightarrow A))$ (p not in A) $\leftrightarrow \Diamond A$ is a theorem schema of $K\pi°$ + Def ∇.

(=>) A noteworthy theorem of **K** is

(27) $((\Diamond r \,\&\, \Diamond \neg r) \,\&\, (\Box(r \rightarrow q) \vee \Box(r \,\&\, \neg q)) \rightarrow \Diamond q$[2]

which by virtue of Def Δ and Def ∇ is equivalent to:

(28) $(\nabla r \,\&\, \Delta(r \rightarrow q)) \rightarrow \Diamond q$

[2] The proof may be given by a semantic argument, i.e. proving by *Reductio ad absurdum* that (27) is **K**-valid. Thanks to the completeness of **K** this implies that (27) is a **K**-theorem.

We then apply to (28) the rule $R\exists$ by employing a fresh variable p and then the equivalence $\exists p \Diamond q \leftrightarrow \Diamond q$. So, considering that the result which is thus obtained holds not only for q but for every arbitrary wff A which does not contain p we obtain the **K**π-theorem schema:

$$(29)\ \exists p(\nabla p \,\&\, \Delta(p \rightarrow A)) \rightarrow \Diamond A \ (p \text{ not in } A)$$

(<=) For the converse implication, first we recall the formula (13) and the following instance of it:

$$(30)\ (p)\ ((\nabla p \rightarrow s) \,\&\, \exists p \nabla p) \rightarrow \exists p(\nabla p \,\&\, s)$$

We are able then to perform the following proof:

1) $\Box q \rightarrow (p)(p \Rightarrow q)$ — **K**π

2) $(p)(p \Rightarrow q) \rightarrow (p)\ (\nabla p \rightarrow (p \Rightarrow q))$ — **K**π, PC

3) $(p)((\nabla p \rightarrow (p \Rightarrow q)) \,\&\, \exists p \nabla p) \rightarrow \exists p(\nabla p \,\&\, (p \Rightarrow q))$ — (30) $p \Rightarrow q/s$

4) $\Box q \rightarrow \exists p\ (\nabla p \,\&\, (p \Rightarrow q))$ — 1),2),3),PC, $\vdash \exists p \nabla p$

5) $\Box q \rightarrow \exists p\ (\nabla p \,\&\, \Delta(p \rightarrow q))$ — 4) $(p \Rightarrow q) \rightarrow \Delta(p \rightarrow q)$

Since 5) holds for every q, from 5) we derive by a *fortiori* the schema

$$(A)\ \Box A \rightarrow (\Diamond A \rightarrow \exists p\ (\nabla p \,\&\, \Delta(p \rightarrow A))\ (p \text{ not in } A)$$

On the other hand we may prove also the schema

$$(B)\ \neg\Box A \rightarrow (\Diamond A \rightarrow \exists p\ (\nabla p \,\&\, \Box(p \rightarrow A)):$$

1) $\Diamond q \,\&\, \Diamond \neg q \,\&\, \Box(q \to q)$ Hypothesis

2) $\exists p\, ((\Diamond p \,\&\, \Diamond \neg p) \,\&\, \Box(p \to q))$ 1), $\exists$ -Introd.

3) $(\Diamond q \,\&\, \Diamond \neg q) \to \exists p\, (\nabla p \,\&\, \Box(p \to q))$ 1), 2), $\to$ Introd.
$\vdash p \,\&\, \Box(q \to q) \leftrightarrow p$, Def ∇

4) $\neg\Box q \to (\Diamond q \to \exists p\, (\nabla p \,\&\, \Box(p \to q))$ 3), PC

From (A) and (B) we obtain then by a standard PC - argument the schema of theorem $\Diamond A \to \exists p\ (\nabla p\ \&\ (\Delta p \to A))$ (p not in A). (Q.E.D.)

As a consequence of the equivalence proved in $MT1$, we observe that every statement having the form $\Box A$ turns out to be equivalent to a statement containing only contingency operators, as one can see from the following equivalences:

(31) $\Box A \leftrightarrow (p)\,(\nabla p \to \neg(p\,\Delta \to \neg A))$ (p not in A)

(32) $\Box A \leftrightarrow (p)\,(\nabla p \to \nabla(p \,\&\, A))$ (p not in A)

(33) $\Box A \leftrightarrow (p)\,(\Delta(p \,\&\, A) \to \Delta p)$ (p not in A)

The last formula could be used to provide an especially intuitive definition of the necessity operator in terms of non-contingency.

Remark 1. Notice that in $\mathbf{K}\pi$ + Def ∇ (i.e. in $\mathbf{K}\pi^\circ$ + Def ∇ minus the Existence Postulate) the theorem schema $\Diamond A \to \exists p\, (\nabla p \,\&\, \Delta\, (p \to A))$ (p not in A) does not imply $\exists p\ \nabla p$. As is well-known, in any $\mathbf{K}\pi$-model consisting of only one world x which is an end point, every $\Diamond$-statement is false and every $\Box$-statement is true. Consequently, every Δ-statement is also true and every ∇-statement is false in such model. Thus $\Diamond A \to \exists p(\nabla p \,\&\, \Delta(p \to A))$ (p not in A) is true in such model by Duns Scotus' law, while $(p)\ \Delta p$ is true in it and $\exists p\ \nabla p$ is false in it. The implication $(\Diamond A \rightarrow \exists p\ (\nabla p\ \&\ \Delta(p \rightarrow A))\ (p$ not in $A) \rightarrow \exists p\ \nabla p$ is thus false in at least one **K**π-model, and thus cannot be a theorem of **K**π + Def ∇.

Remark 2. From the preceding remark it turns out that there is a **K**π-model in which $\exists p\ \nabla p$ is false, which means, by the soundness of **K**π, that CP is not a theorem of **K**π + Def ∇.

Remark 3. The contingency postulate CP is essential to the derivation of the equivalence proved in MT1. We may prove in fact the following metatheorem:

MT2. The formula $\Diamond A \leftrightarrow \exists p\ (\nabla p\ \&\ \Delta\ (p \rightarrow A))$ (p not in A) is not a theorem schema of **K**π + Def∇.

If the equivalence were a thesis of **K**π + Def ∇, in fact, the implication $\Diamond A \rightarrow \exists p(\nabla p\ \&\ \Delta\ (p \rightarrow A))$ (p not in A) $\rightarrow \exists p$ would be a thesis of this system. But this is not so, as was proved in Remark 1 (Q.E.D.).

An interesting consequence of MT1 is given by the following derivations in **K**π° + Def ∇:

1) $\Diamond T$	Hypothesis
2) $\exists p\ (\nabla p\ \&\ \Delta\ (p \rightarrow T))$	1, *MT*1, Modus Ponens
3) $\exists p\ \nabla p$	2), Quant. Theory
4) $\Diamond T \rightarrow \exists p\ \nabla p$	1), 3), $\rightarrow$-Introd.

On the other hand we have also:

1) $\exists p\, \nabla p$	Hypothesis
2) $(p)\ (\Delta\,(p \rightarrow T))$	$\vdash \Box(p \rightarrow T)$, UG
3) $\exists p\,(\nabla p\ \&\ \Delta\,(p \rightarrow T))$	1), 2), Quant. Theory
4) $\Diamond T$	3), MT1
5) $\exists p\, \nabla p \rightarrow \Diamond T$	1), 4), $\rightarrow$-Introd.

The equivalence $\exists p\ \nabla p \leftrightarrow \Diamond T$ is then a theorem of $\mathbf{K}\pi°$. But $\Diamond T$ is equivalent to the Deontic axiom **D**: $\Box p \rightarrow \Diamond p$. So adding the Contingency Postulate $\exists p\ \nabla p$ to $\mathbf{K}\pi$ implies having as a theorem the Deontic axiom **D**. It turns out then that $\mathbf{K}\pi°$ + Def ∇ is the same as $\mathbf{KD}\pi°$ + Def ∇.

4. EQUIVALENCE OF CONTINGENCY SYSTEMS AND MODAL SYSTEMS

What we have proved in the preceding section amounts to the result that $\mathbf{K}^{\Delta}\pi°$ + Def $\Diamond$ is contained in $\mathbf{K}\pi°$ + Def ∇, which is the same as $\mathbf{K}^{\Delta}\pi°$ + Def ∇. A remarkable fact is that the containment also holds in the other direction. In other words, $\mathbf{K}^{\Delta}\pi°$, $\mathbf{KD}\pi°$ and $\mathbf{K}\pi°$ turn out to be definitionally equivalent systems. What we are able to prove is in fact the following result:

MT3. For every A, if A is a thesis of $\mathbf{K}\pi°$ + Def ∇, A is a thesis of $\mathbf{K}^{\Delta}\pi°$ + Def $\Diamond$.

Of course the axioms for propositional quantification need not be proved in $\mathbf{K}^{\Delta}\pi°$ since they are common to both systems. So what we have to prove is (i) that all **K**-theses are theses of $\mathbf{K}^{\Delta}\pi°$ + Def $\Diamond$; (ii) that the Existence Postulate is a thesis of $\mathbf{K}^{\Delta}\pi°$ + Def $\Diamond$; (iii) that the equivalence between ∇p and $\Diamond p\ \&\ \Diamond\neg p$ is a thesis of $\mathbf{K}^{\Delta}\pi°$ + Def $\Diamond$.

(i) We recall that **K** may be axiomatized as follows:

Ax1: $(\Box p \mathbin{\&} \Box q) \rightarrow \Box(p \mathbin{\&} q)$

RNec. $\vdash A \rightarrow \vdash \Box A$

RK: $\vdash A \rightarrow B \rightarrow \vdash \Box A \rightarrow \Box B$

Assuming that Replacement of Proved Equivalents holds for both systems we may prove that Ax1, RNec., RK are all derivable in $\mathbf{K}^{\Delta}\pi^{\circ}$ + Def $\Diamond$.

(A) We begin by proving Ax1, i.e. $(\Box A \mathbin{\&} \Box B) \rightarrow \Box(A \mathbin{\&} B)$

By applying the known equivalences the antecedent of Ax1 amounts to the conjunction of the following two schemas:

(34) $(p)\ (\Delta(p \mathbin{\&} A) \rightarrow \Delta p)$ (p not in A)

(35) $(p)\ (\Delta(p \mathbin{\&} B) \rightarrow \Delta p)$ (p not in B)

So we have to prove in $\mathbf{K}^{\Delta}\pi^{\circ}$ that the joint supposition of (34) and (35) implies $(p)\ (\Delta(p \mathbin{\&} A \mathbin{\&} B) \rightarrow \Delta p)$.

Now the following steps are in order:

(36) $\Delta((p \mathbin{\&} A) \mathbin{\&} B) \rightarrow \Delta(p \mathbin{\&} A)$	(35) (-) -Elim $(p \mathbin{\&} A \,/\, p)$
(37) $\Delta(p \mathbin{\&} A) \rightarrow \Delta p$	(34), (-) -Elim
(38) $(p)\ (\Delta(p \mathbin{\&} A \mathbin{\&} B) \rightarrow \Delta p)$	(36), (37)

From (34), (35), (38), by $\rightarrow$ -Introduction and *UG* we then prove the required schema, i.e.

$((p)(\Delta (p \,\&\, A) \rightarrow \Delta p) \,\&\, (p)\ (\Delta(p \,\&\, B) \rightarrow \Delta p)) \rightarrow$
$(p)\ (\Delta(p \,\&\, A \,\&\, B) \rightarrow \Delta p)$ (p not in A and B)

(B) Rule of Necessitation (R.Nec) : $\vdash A \rightarrow \vdash \Box A$.

Given the equivalence (33) between $\Box A$ and $(p)(\Delta\ (p \,\&\, A) \rightarrow \Delta p)$ (p not in A), we have to prove in $\mathbf{K}^{\Delta}\pi^{\circ}$ + Def $\Diamond$ the following rule:

(39) $\vdash_{\mathbf{K}^{\Delta}\pi^{\circ}} A \rightarrow \vdash_{\mathbf{K}^{\Delta}\pi^{\circ}} (p)\ (\Delta(p \,\&\, A) \rightarrow \Delta p)$ (p not in A)

Let us suppose $\vdash_{\mathbf{K}^{\Delta}\pi^{\circ}} A$. We know of course $\vdash_{\mathbf{K}} (p \,\&\, A) \rightarrow p$, so, thanks to:

(40) $\vdash A \leftrightarrow (p \vee \neg p)$

we have by Replacement of Proved Equivalents:

(41) $\vdash_{\mathbf{K}} (p \,\&\, A) \leftrightarrow p$

Let us now suppose by Reductio:

(42) $A \,\&\, \exists p\ (\Delta(p \,\&\, A) \,\&\, \nabla p)$

Under this hypothesis, by applying Replacement twice in (42) in the light of (41) we would have:

(43) $\exists p(\Delta p \,\&\, \nabla p)$

which is a contradiction. Thus we conclude:

$\vdash_{\mathbf{K}^{\Delta}\pi^{\circ}} \neg(A \,\&\, \exists p\ (\Delta\ (p \,\&\, A) \,\&\, \nabla p))$,

so

(44) $\vdash_{\mathbf{K}^{\Delta}\pi^{\circ}} A \rightarrow (p)\ (\Delta(p \,\&\, A) \rightarrow \Delta p)$

and by Modus Ponens we have the derived rule $\vdash_{\mathbf{K}^{\Delta}\pi^{\circ}} A \rightarrow \vdash_{\mathbf{K}^{\Delta}\pi^{\circ}} (p)\ (\Delta(p \,\&\, A) \rightarrow \Delta p)$ (p not in A).

(C) We prove now the rule RK: $\vdash_{\mathbf{K}^{\Delta}\pi^{\circ}} A \rightarrow B \rightarrow \vdash_{\mathbf{K}^{\Delta}\pi^{\circ}} \Box A \rightarrow \Box B$.

The proof is by observing that $A \rightarrow B$ equals $(A \,\&\, B) \leftrightarrow A$, so by Replacement in the thesis $\nabla (p \,\&\, A) \leftrightarrow \nabla (p \,\&\, A)$ we have

(45) $\vdash_{\mathbf{K}^{\Delta}\pi^{\circ}} \nabla(p \,\&\, A \,\&\, B) \leftrightarrow \nabla(p \,\&\, A)$.

Let us now suppose the contingential counterpart of $\Box A$, i.e.

(46) $(p)\ (\Delta(p \,\&\, A) \rightarrow \Delta p)$ (p not in A)

We have to show that the supposition (46) implies the contingential counterpart of $\Box B$, i.e. $(p)\ (\Delta p \,\&\, B) \rightarrow \Delta p)$ (p not in B). Let us suppose by Reductio $\neg(p)\ (\Delta(p \,\&\, B) \rightarrow \Delta p)$ is i.e. $\exists p\ (\Delta(p \,\&\, B) \,\&\, \nabla p)$. Then there is a proposition p^* such that $\Delta\ (p^* \,\&\, B) \,\&\, \nabla p^*$ is true in some model. We show that, given (46), a contradiction follows from this fact. The supposition (46) implies, by contraposition and Universal Instantiation, $\nabla p^* \rightarrow \nabla(p^* \,\&\, A)$; by (45) and Replacement of Equivalents this means $\nabla p^* \rightarrow \nabla(p^* \,\&\, A \,\&\, B)$. In $\mathbf{K}^{\Delta}$ from $\nabla(p^* \,\&\, A \,\&\, \mathrm{B})$ we have $\nabla A \vee \nabla(p^* \,\&\, B)$ (see Theorem (9)), which equals $\Delta(p^* \,\&\, B) \rightarrow \nabla A$. Then from $\Delta\ (p^* \,\&\, B) \,\&\, \nabla p^*$ we would have $\Delta(p^* \,\&\, B)$ and then ∇A by transitivity. ∇A would be then true in some model. This is, however, impossible since (46), i.e. $(p)\ (\Delta(p \,\&\, A) \rightarrow \Delta p)$ implies ΔA, and so the negation of ∇A, (to grasp this point, let us simply remark that $(p)\ (\Delta(p \,\&\, A) \rightarrow \Delta p)$ implies $\Delta(A \,\&\, \neg A) \rightarrow \Delta\neg A$, but since $\Delta\bot$ is a theorem, it implies $\Delta\ \neg A$, so also ΔA). Thus, the conjunction of (46) and $\neg(p)\ (\Delta(p \,\&\, B) \rightarrow \Delta p)$ yields a contradiction: so (46) implies $(p)\ (\Delta(p \,\&\, B) \rightarrow \Delta p)$, which establishes the required result.

(ii) We have to prove in $\mathbf{K}^{\Delta}\pi^{\circ}$+ Def $\Diamond$ the Existence Postulate, namely $\exists q\ (\Diamond q \,\&\, \Diamond \neg q)$. This means to prove in the contingential language the wff $\exists q(\exists p\ (\nabla p \,\&\, \Delta\ (p \rightarrow q)) \,\&\, \exists p\ (\nabla p \,\&\, \Delta\ (p \rightarrow \neg q)))$. But the proof is straightforward if we recall that ΔT is a theorem and that we have at our disposal the Contingency Postulate $\exists p\ \nabla p$. We have as theorems then, by replacement in ΔT, both $(p \rightarrow T)$ and $\Delta\ (\perp \rightarrow \perp)$ Thus both $\exists p(\nabla p \,\&\, \Delta\ (p \rightarrow T))$ and $\exists p\ (\nabla p \,\&\, \Delta\ (p \rightarrow \neg T))$ are theorems. Thus, by existential quantification over T, $\exists q\ ((\exists p\ (\nabla p \,\&\, \Delta\ (p \rightarrow q)) \,\&\, \exists p\ (\nabla p \,\&\, \Delta\ (p \rightarrow \neg\, q)))$ is also a theorem. The equivalent two-variables Existence Postulate is then derived by following, *mutatis mutandis*, the lines set out in Section 1.

(iii) The step which concludes the proof consists in proving in $\mathbf{K}^{\Delta}\pi$ + Def $\Diamond$ the equivalence yielded by the definition $\nabla A =_{Df} \Diamond A \,\&\, \Diamond \neg A$ which belongs to $\mathbf{K}\pi$ + Def ∇.

We have now to prove in $\mathbf{K}^{\Delta}\pi$ + Def $\Diamond$ the equivalence $\nabla A \leftrightarrow (\exists p\ (\nabla p \,\&\, \Delta(p \rightarrow A)) \,\&\, \exists p\ (\nabla p \,\&\, \Delta(p \rightarrow \neg A)))$. The two directions are proved as follows:

(=>) a) Suppose by Reductio $\nabla A \,\&\, (p)\ (\nabla p \rightarrow \nabla(p \rightarrow A))$

The second conjunct implies (by elimination of the universal quantifier) $\nabla A \rightarrow \nabla(A \rightarrow A))$, so that we have $\nabla A \rightarrow \nabla T$. But since we already know $\vdash \Delta\ T$ i.e. $\vdash \neg\nabla\ T$, we would have by contraposition and Modus Ponens $\vdash \Delta\ A$: contradiction.

b) Suppose by Reductio $\nabla A \,\&\, (p)\ (\nabla p \rightarrow \nabla(p \rightarrow \neg\, A))$. Then $\nabla \neg A \rightarrow \nabla(\neg A \rightarrow \neg A))$, i.e. $\nabla A \rightarrow \nabla T$ as before: contradiction.

Now as a result of the Reductio argument based on a) and b) we have that ∇A implies both $\exists p\ (\nabla p \,\&\, \Delta(p \rightarrow A))$ and $\exists p\ (\nabla p \,\&\, \Delta\ (p \rightarrow \neg A)))$, so also their conjunction.

(<=) By converse, suppose by Reductio $\exists p\ (\nabla p \,\&\, \Delta\ (p \rightarrow A)) \,\&\, \exists p\ (\nabla p \,\&\, \Delta\ (p \rightarrow \neg A))$ and also $\neg\nabla A$, i.e. ΔA. By existential instantiation of the hypothesis we have then, for some p^* ad some

q^*, $(\nabla p^* \,\&\, \Delta\, (p^* \,\&\, \neg A) \,\&\, (\nabla q^* \,\&\, \Delta\, (q^* \,\&\, A))$. Let us suppose that there is some contingential model in which such formula is true. If, by Reductio, ΔA were also true in a world x of the model, A would be true at all accessible worlds related to x or false at all such worlds. In the latter case $\Delta\ (q^* \,\&\, A)$ would be false at x, while in the former $\Delta(p^* \,\&\, \neg A)$ would be false at x. So there is no contingential model in which the conjunction $\nabla p^* \,\&\, \Delta(p^* \,\&\, \neg A) \,\&\, \Delta A$ is true, and no contingential model in which $\nabla q^* \,\&\, \Delta(q^* \,\&\, A) \,\&\, \Delta A$ is true: so by contraposition there is no contingential model in which $\exists p\ (\nabla p \,\&\, \Delta(p \to A)) \,\&\, \exists p\ (\nabla p \,\&\, \Delta(p \to \neg A)) \,\&\, \Delta A$ is true. The conditional $(\exists p\ (\nabla p \,\&\, \Delta\ (p \to A)) \,\&\, \exists p\ (\nabla p \,\&\, \Delta\ (p \to \neg A))) \to \neg\Delta A$ is then true in every contingential model. By the completeness of $\mathbf{K}^{\Delta}\pi$ then $(\exists p\ (\nabla p \,\&\, \Delta(p \to A)) \,\&\, \exists p\ (\nabla p \,\&\, \Delta(p \to \neg A))) \to \nabla A$ is then a theorem of $\mathbf{K}^{\Delta}\pi$. (Q.E.D.)

5. FURTHER QUESTIONS

What has been showed in the preceding sections is that the Existence Postulate (or the equivalent Contingency Postulate) is essential to prove an equivalence result between the weakest normal contingency logic with propositional quantifiers and a normal modal logic with propositional quantifiers: such logic is however not the weakest normal modal logic since it contains the Deontic Logic **KD**. This fact opens the question of knowing whether the definition of modal operators in terms of contingency and propositional quantifiers can be established in systems lacking the Contingency Postulate.

In this connection it is to be noticed that there are an unlimited number of variants of the Contingency Postulate which are weaker or stronger than the Contingency Postulate itself. For instance in $\mathbf{K}^{\Delta}\pi$:

a) $\exists q(\nabla q \wedge \nabla\nabla q)$

b) $\exists q\ (\nabla q \wedge \Delta\nabla q)$

are both stronger than *CP*, while

c) $\Delta\exists q(\nabla q \vee \nabla\nabla q)$

d) $\exists q\ (\nabla q \vee \Delta\nabla q)$

are weaker than *CP* and follow from it. As a matter of fact, an analysis of such variants of the Contingency Postulate and of their relations with standard modal logics with propositional quantifiers has never been performed up to now and may be suggested as a topic for further investigations in this area.

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