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ON 'ALMOST ALL' AND SOME PRESUPPOSITIONS

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We examine some issues concerning a logical system for the precise treatment of assertions involving "almost all" objects and analyse its underlying ideas. We concentrate on the usage of ultrafilters for capturing the intended meaning of "almost universal" assertions by analysing its underlying presuppositions and some basic intuitions. We first reassess the ultrafilter proposal, suggesting an alternative interpretation, and then analyse a few questions, trying to overcome some objections against using ultrafilters.

"The truth is rarely simple,
and never pure" (Oscar Wilde).

1. INTRODUCTION

In this paper we discuss, trying to explain and justify, some fundamental issues in the precise treatment of assertions involving "almost all" objects. We shall focus mainly on the usage of ultrafilters for capturing the intended meaning of "almost universal" assertions by analysing its underlying presuppositions and some basic intuitions.

Arguments involving assertions about "almost all" (or "typical") objects occur often, not only in ordinary language, but also in some branches of science. A precise treatment of such arguments has been a basic motivation for ultrafilter logic, a logical system with generalised quantifiers for "almost all", also handling "generic" objects.

The meaning of "almost all objects have a given property" can be given either directly or by means of the set of exceptions, i. e. those objects failing to have this property. Prior proposals relied on the idea that "almost all objects have a given property" is intended to mean that the set of exceptions is "small" (Carnielli & Sette (1994); Carnielli & Veloso (1997)). Later, the idea of "very small" set of exceptions was found to be more intuitively appealing (Veloso (1998)). In any case, a precise formulation hinges on some ideas concerning (very) small or large sets. By relying on some apparently reasonable criteria for subsets of a universe to be considered (very) small, one is led to the precise concept of ultrafilter, as capturing the dual notion of (very) large subsets (Sette *et al.* (1999)).

This approach leads to a well-behaved logic with interesting mathematical properties. But, what about its underlying presuppositions? Are they really reasonable?

Probably, no one acquainted with ultrafilters would doubt their power or versatility[1]. But, even though one often uses the metaphor of (ultra)filters as congregating the (very) large sets, this does not necessarily mean that ultrafilters are natural or intuitive[2]. In particular, one may cast some doubt on the assertion that ultrafilters are a *natural* way to capture the idea of "almost all".

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[1] Ultrafilters have many applications in Mathematics. In Algebra and Topology, they are the conerstone of Stone's duality (in particular in the representation theorem: every (abstract) Boolean algebra is isomorphic to a Boolean algebra of sets) (Halmos (1972)). In Logic they underlie the idea of ultraproducts and ultrapowers (Bell & Slomson (1971)).

[2] The existence of some ultrafilters is often established by appealing

In this paper we reassess the ultrafilter proposal, suggesting an alternative interpretation, and analyse some questions. The main alleged objections to using ultrafilters we shall address are as follows.

1. Using ultrafilters lacks intuitive justification.
2. Ultrafilters involve a certain degree of arbitrariness.
3. Ultrafilters are not interesting for finite situations.

We shall attempt to circumvent the first objection by means of an analysis of the underlying ideas. We shall then argue that the other two can be overcome by a proper understanding of the roles of models and theories. Besides considering these objections, we also recall some ideas about our logic and present a novel result concerning its expressive power.

The structure of this paper is as follows. In the next section, we shall attempt to circumvent the first objection (lack of intuitive justification for using ultrafilters) by means of an alternative interpretation, suggested by reassessing the ultrafilter proposal and analysing the underlying notions. Then, in section 3, we briefly recall some basic ideas about ultrafilter logic, an extension of classical first-order logic by a new quantifier for precise reasoning about "almost all". We then proceed to section 4, where we examine the contention that an ultrafilter embodies a certain degree of arbitrariness, arguing that, even though this may be the case for a specific ultrafilter, this is largely dissolved when one considers theories. In section 5, we examine the expressive power of ultrafilter logic, showing that it is a proper extension of classical first-order logic, by means of a novel result characterising those formulae from which the new quantifier can be eliminated. Then, the third contention (that finite situations trivialise ultrafilters) is taken up in section 6, where we argue that it is practically dissolved when one considers, and reasons with, theories, in lieu of a

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to the Axiom of Choice (Bell & Slomson (1971); Chang & Keisler (1973)).

particular model. Finally, section 7 presents some concluding remarks.

2. WHY ULTRAFILTERS?

We will first indicate how one can arrive at ultrafilters as capturing the intuitive idea of “almost all”. The approach is based on explicating “almost all objects have a given property” as the set of objects failing to have this property is “negligible”.

The prior idea of understanding “almost all objects” in terms of (very) large sets might suggest a quantitative notion, such as viewing as (very) small the unlikely sets (those with (very) low probability). But, we wish instead a qualitative account, dealing with properties of a topological rather than metrical nature.

Towards this goal, one might start from the notion of “having about the same size” and introduce some reasonable properties of this relation $\simeq$ between sets.

It is tempting to consider that we have an equivalence relation. Indeed, reflexivity and symmetry seem reasonable. But, what about transitivity: are we prepared to accept that the extremes X_0 and X_n of a long chain $X_0 \simeq X_1 \simeq \ldots \simeq X_n$ are still sets with about the same size?

Actually, a notion such as “having about the same size” does not seem to be such a good starting point. This is so because one is naturally led to think that sets with the same size should have about the same size. In other words, this is a non-local notion. The notion we are seeking is, in contrast, a local notion.

As an example, consider two sets, one consisting of a horse and an ox, and another one consisting of a horse and a dog. These sets, which have the same size, may be just as important to a conservationist. But, the former may be more important to a farmer, whereas the latter might be preferred by an English gentleman, keen on fox hunting. So, sets with the same size may not be equally important.

For another example, consider sets with different sizes, say, a set with twenty birds and another one with a couple of elephants. The Zoo director is likely to consider them equally important. But, an ornithologist would rank the former as more important, whereas a truck driver in charge of transporting them would probably give more attention to the latter. So, a smaller set may be more important than a larger set, or just as important.

Thus, our intuitive notion is not only local but also relative to the situation or intended application. We might say that we really have a family of notions and we attempt to describe some of their common properties.

In view of these considerations, we shall prefer to use names like "almost as important as" for our basic comparison between subsets of a given universe V, which we shall denote by $\approx$.

Also, instead of assuming at the outset that we have an equivalence relation, we shall put forward some more basic – and hopefully more palatable – postulates. (This enterprise is somewhat reminiscent of that of "reverse mathematics", with an important difference[3]).

In this section we shall be resorting to two kinds of arguments, namely

- intuitive arguments (based mainly on common sense and ordinary understanding), to try to justify the acceptance of the proposed postulates, as well as

- (simple) mathematical proofs, to derive some properties from our postulates.

[3] "The fundamental question in reverse mathematics is to determine which set existence axioms are required to prove particular theorems of mathematics" (Solomon (1999), p. 45). Here, instead of locating familiar axioms, we will be suggesting some new postulates, whence the need for justifying their acceptance on intuitive grounds.

One might, and perhaps should, keep them separate, but we prefer to employ a structure that helps seeing the dependence of the properties on the postulates.

If we understand negligible as "fit to be neglected or discarded" (Webster (1970))[4], it appears reasonable to say that two sets are almost as important when their difference is negligible. The difference, being the part where they differ, is the so called symmetric difference

$$X \Delta Y := (X - Y) \cup (Y - X).$$

Equivalently, we may say that two sets are almost as important when the part in either one but not in both of them is negligible[5].

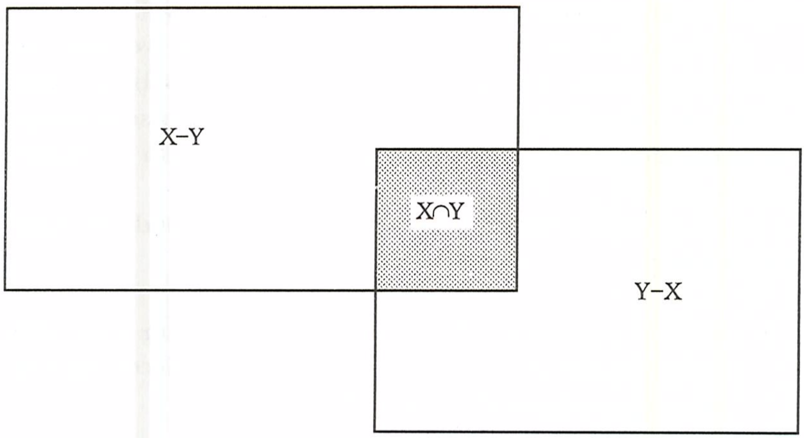

Figure 1: Symmetric difference of sets

[4] "Something that is negligible is so small or unimportant that is not worth considering or worrying about" (Collins (1987)).

[5] Indeed, we have $(X - Y) \cup (Y - X) = (X \cup Y) - (X \cap Y)$.

We are thus led to formulate our first postulate, explicating 'almost as important as' in terms of the family $\mathcal{N}$ of negligible sets.

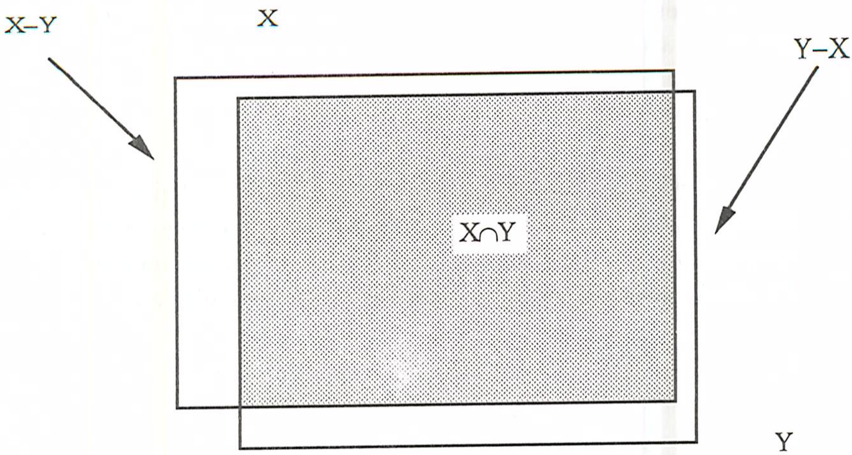

Figure 2: Sets with about the same importance

P1. For sets *X* and *Y*, *X* is almost as important as *Y* iff their symmetric difference is negligible.

[Δ] $X \approx Y \Leftrightarrow X \Delta Y \in \mathcal{N}$

As an immediate consequence of this postulate, the negligible sets can be described as those almost as important as the empty set, which appears intuitively reasonable.

C0: A set *N* is negligible iff *N* is almost as important as the empty set $\varnothing$.

$$X \in \mathcal{N} \Leftrightarrow X \approx \varnothing \qquad (\mathcal{N})$$

Another immediate, and also intuitively reasonable, consequence of this postulate is that when two sets are almost as important so are their complements (relative to the universe V).
C1: For sets X and Y, X is almost as important as Y iff their complements are so (i. e. X^C is almost as important as X^C)[6].

$$X \approx Y \Leftrightarrow X^C \approx Y^C \qquad (^C)$$

One would probably say that a subset of a negligible set is (even more) negligible, which is the content of our second postulate, about the behaviour of the family $\mathcal{N}$ of negligible sets under inclusion.
P2. Each subset X of a negligible set $N \in \mathcal{N}$ is negligible.

$$[\subseteq] \qquad X \subseteq N \in \mathcal{N} \Rightarrow X \in \mathcal{N}$$

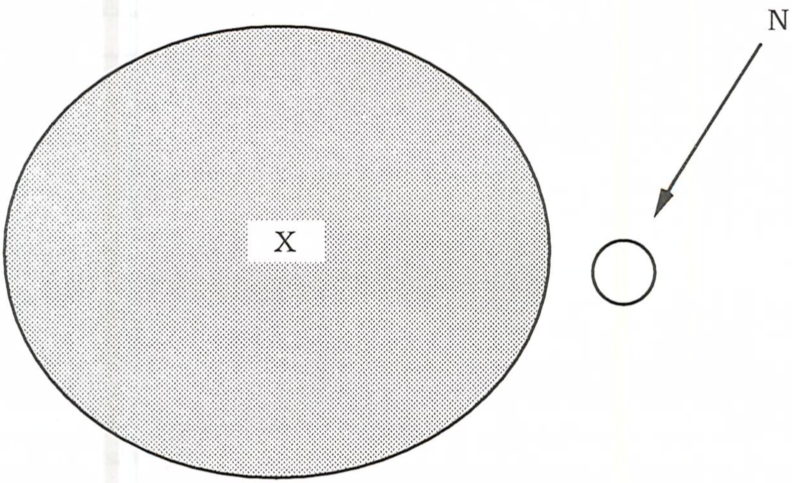

Figure 3: Union with a negligible set

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[6] This follows immediately from P1, since $X^c \Delta Y^c = X \Delta Y$ (as $X^c - Y^c = Y - X$).

One would probably agree that the addition of a negligible set should have negligible impact, leaving a set almost as important as before. This intuitively acceptable assertion is an immediate consequence of our two postulates.
C2: For each set X, the union $X \cup N$ of X with a negligible set $N \in \mathcal{N}$ is almost as important as X (i. e. $X \cup N$ is almost as important as X)[7].

$$N \in \mathcal{N} \Rightarrow X \cup N \approx X \qquad (+)$$

Now, in view of our intuitive ideas concerning negligible and almost as important, it seems reasonable to accept that a set almost as important as a negligible set is negligible, as well. This gives our third postulate, about the behaviour of the family $\mathcal{N}$ of negligible sets under being almost as important.
P3. Each set X almost as important as a negligible set $N \in \mathcal{N}$ is negligible.

$$[\approx] \qquad X \approx N \in \mathcal{N} \Rightarrow X \in \mathcal{N}$$

As a consequence, we have the closure of the family $\mathcal{N}$ of negligible sets under union.
C3: If sets X and Y are negligible (i. e. $X \in \mathcal{N}$ and $Y \in \mathcal{N}$), then so is their union $X \cup Y$ negligible (i. e. $X \cup Y \in \mathcal{N}$)[8].

$$X \in \mathcal{N} \,\&\, Y \in \mathcal{N} \Rightarrow X \cup Y \in \mathcal{N} \qquad (\cup)$$

One may or may not find this property intuitively reasonable. Notice, however, that it is an inescapable conclusion, once one has accepted the preceding postulates[9].

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[7] This follows from P1 and P2, since $(X \cup N) \Delta X = N - X \subseteq N$.

[8] Consequence C2 gives $X \cup Y \approx X$, whence P3 yields $X \cup Y \in \mathcal{N}$.

[9] Another consequence is that our comparison $\approx$ turns out to be tran-

Our intuitive ideas concerning negligible suggest that the empty set is (most) negligible, which we shall accept as our fourth postulate.

P4. The empty set $\varnothing$ is negligible.

$$[\in] \qquad \varnothing \in \mathcal{N}$$

An immediate consequence of this postulate, to which it is equivalent (in view of P2), is the existence of negligible sets. Indeed, what would be the point of considering such sets if there were none?

C4: There exist negligible sets.

$$\mathcal{N} \neq \varnothing \qquad (\neq)$$

It seems reasonable to accept that the universe is not negligible. Otherwise, in view of P2, every subset would be negligible, which would trivialise the notion. This idea gives us our fifth postulate.

P5. The universe V is not negligible.

$$[\notin] \qquad V \notin \mathcal{N}$$

An immediate consequence of this postulate, equivalent to it by P2 (as noted above), is the existence of non-negligible sets.

C5: There exist non-negligible sets.

$$\mathcal{N} \neq \wp(V) \qquad (\subset)$$

Now, we have already agreed that the empty set is (most) negligible. Dually, the universe is (least) negligible, i. e. (most) important. Our intuition suggests that a subset is very important (worth considering or worrying about) when its complement is

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sitive (because $X - Z \subseteq (X - Y) \cup (Y - Z)$).

negligible. Such considerations give our sixth postulate, explicating the family $\mathcal{W}$ of very important subsets (those carrying considerable weight) in terms of the family $\mathcal{N}$ of negligible sets.

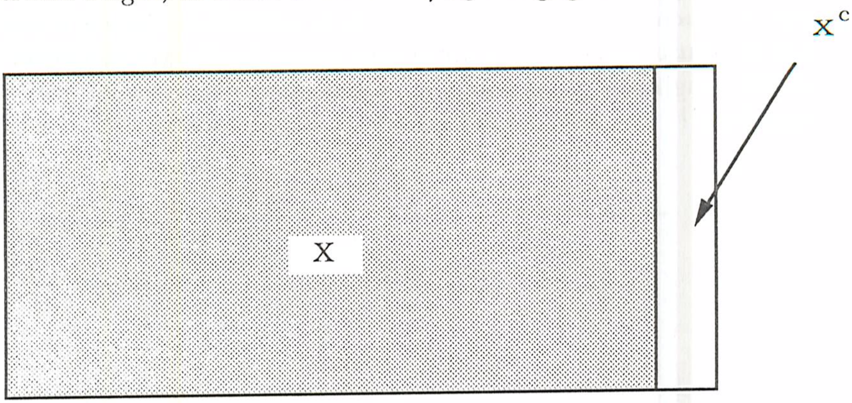

Figure 4: Subset with negligible complement

P6. A subset $H \subseteq V$ is very important iff its complement H^C is negligible.

[$\mathcal{W}$] $$H \in \mathcal{W} \Leftrightarrow H^C \in \mathcal{N}$$

As a consequence of our postulates so far, we can characterise the very important subsets as those almost as important as the universe, which appears intuitively reasonable.

C6: A subset $H \subseteq V$ is very important iff H is almost as important as the universe[10].

$$H \in \mathcal{W} \Leftrightarrow H \approx V \qquad (\mathcal{N})$$

[10] This follows immediately from P6 and C1.

We now come to our final postulate, which is probably the least intuitively acceptable one (and with more profound impact). The underlying idea is that the universe is so important (i. e. carries so much weight) that any attempt to cover it by finitely many subsets must employ a very important subset (one carrying considerable weight, or equivalently, almost as important as the entire universe)[11].

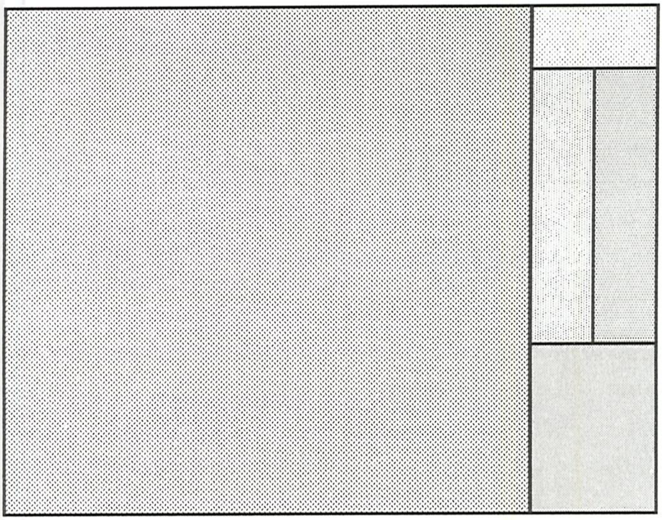

Figure 5: Finite cover of the universe

P7. Any finite cover of the universe V must have a very important subset.

[11] Over an infinite universe, one may regard the finite subsets as not carrying considerable weight. Another example where this postulate holds is provided by considering as carrying considerable weight only subsets with elephants.

[V] $$V = X_1 \cup \ldots \cup X_n \Rightarrow X_k \in \mathcal{W}, \text{ for some } k$$

As a consequence of our postulates, we have that a subset or its complement must be very important.
C7: If a subset $X \subseteq V$ is not very important then its complement X^c is very important[12].

$$X \notin \mathcal{W} \Rightarrow X^c \in \mathcal{W} \quad \text{(p)}$$

Summarising, in virtue of our postulates, the family $\mathcal{W}$ of very important subsets (those carrying considerable weight) has the following properties[13]

- non-empty: $\mathcal{W} \neq \varnothing$;
- proper: $\mathcal{W} \neq \wp(V)$;
- upward closed: $Y \in \mathcal{W}$, whenever $X \in \mathcal{W}$ and $X \subseteq Y$;
- closed under intersection: $X \cap Y \in \mathcal{W}$, whenever $X \in \mathcal{W}$ and $Y \in \mathcal{W}$;
- prime: $X^c \in \mathcal{W}$, whenever $X \notin \mathcal{W}$.

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12 This follows immediately from P7, since $X \cup X^c = V$. (Notice that it requires only the simpler instance of P7 concerning covers by two subsets, namely if $X_1 \notin \mathcal{W}$ and $X_2 \notin \mathcal{W}$ then $X_1 \cup X_2 \neq V$.)

13 The last property is C7, and the first four follow, by P6, from the corresponding dual properties of family $\mathcal{N}$, of negligible subsets (namely C5, C4, P2, and C3).

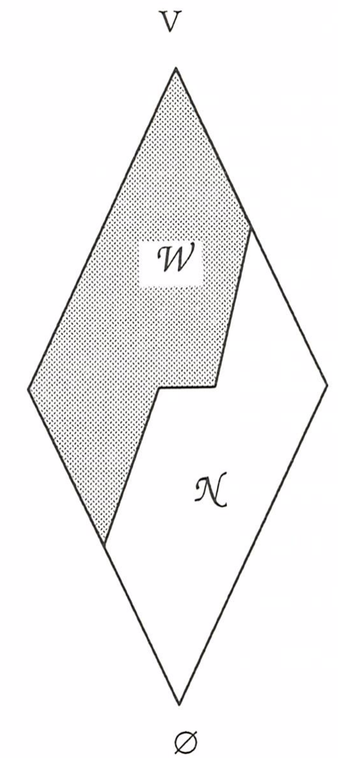

Figure 6: Negligible and very important subsets of the universe

Thus, the family

$$\mathcal{W} = \{ X \subseteq V : H^c \in \mathcal{N} \}$$

of very important subsets is[14]

[14] A similar hierarchy of families of subsets can be used for variations of the logic of almost all (Grácio (1999)): upward closed families to capture 'many' (a sizeable portion) and filters for 'plausible' (a reasonably large set of evidences).

- a non-empty, proper, upward closed family of subsets of V,

being also

- closed under intersection (thus, a proper filter on V),

and, moreover, is

- prime (thus, a maximal proper filter on V).

Hence, the family $\mathcal{W}$ of very important subsets is a proper ultrafilter over the universe V[15].

Summing up, the acceptance of some simple (and arguably reasonable) postulates about subsets of a universe being

- almost as important,
- negligible,
- very important,

leads to the conclusion the family $\mathcal{W}$ of very important subsets is a proper ultrafilter over the universe.

Conversely, each proper ultrafilter on a universe gives rise to a family of subsets satisfying our seven postulates[16].

Therefore, we can see that

- ultrafilters, and
- families of subsets satisfying our seven postulates

turn out to be equivalent formulations of the same basic ideas.

3. ULTRAFILTERS FOR "ALMOST ALL"

We shall now briefly recall some basic concepts of ultrafilter logic, an extension of classical first-order logic for reasoning about "almost all"[17].

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15 The dual family $\mathcal{N}$ of negligible subsets is seen to be a maximal proper ideal.

16 By using formula [Δ] (in postulate P1) to introduce relation $\approx$ and formula [$\mathcal{W}$] (in postulate P6) to introduce family $\mathcal{N}$ we can see that each ultrafilter gives rise to a model of our seven postulates.

17 More details concerning ultrafilter logic can be found in (Carnielli & Veloso (1997); Veloso & Carnielli (1997); Sette *et al.* (1999); Veloso (1998), (1999)).

Under the light of the preceding section, the interpretation of

- "almost all objects have a given property"

as

- "the set of objects failing to have this property is negligible"

can be seen to amount to

- "the set of objects having this property belongs to a given ultrafilter".

Our logic for almost all adds to classical first-order logic a generalised quantifier ∇, with intended interpretation "almost all objects" and whose behaviour can be seen to be intermediate between $\forall$ and $\exists$[18]. We now briefly examine this logic: its syntax, semantics and axiomatics.

The syntax of our logic is obtained by extending the usual first-order syntax by the new quantifier. We extend the usual first-order syntax by adding the new quantifier ∇ together with a new (variable-binding) formation rule giving almost universal formulae, namely those of the form $\nabla v\varphi$[19].

The semantics for our logic is obtained by extending the usual first-order definition of satisfaction to the new quantifier[20].

For this purpose, we resort to ultrafilter structures, an ultrafilter structure $\mathfrak{M}^{\mathcal{U}}$ being the expansion of a first-order structure $\mathfrak{M}$ by an ultrafilter $\mathcal{U}$ over its universe. We then extend the usual Tarskian definition of satisfaction to almost universal formulae, so as to capture the above interpretation: an almost universal for-

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[18] But with different properties. For instance, whereas $\forall x$ and $\forall y$ commute (and likewise for $\exists$), in general $\nabla x \nabla y\ x < y$ and $\nabla y \nabla x x < y$ are not equivalent.

[19] Notice that iterated applications of ∇ are allowed. More precisely, we have the new formation rule: for each variable v, if φ is a formula then so is $\nabla v\ \varphi$ a formula (where variable v is bound).

[20] So, the propositional connectives as well as the usual quantifiers $\forall$ and $\exists$ will keep their familiar interpretations.

mula $\nabla v\varphi$ is satisfied iff the extension of φ belongs to the given ultrafilter[21].

We set up a deductive system for our logic by adding a set A^{∇} of schemata, coding properties of ultrafilters[22], to a calculus for classical first-order logic[23] .

In this deductive system, we have substitutivity of equivalents[24]. Also, we can see that, within equivalence, the new quantifier ∇ provably

- commutes[25] with negation $\neg$,

and

- distributes[26] over the binary propositional connectives $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$.

We also have prenex normal forms[27].

This system provides a sound and complete deductive calculus for reasoning about assertions involving "almost all": a sentence τ is derivable from a set Γ iff τ holds in all ultrafilter models of Γ[28].

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21 More precisely, given an ultrafilter structure $\mathfrak{M}^{\mathcal{U}} = (\mathfrak{M}, \mathcal{U})$, for a formula $\nabla y\psi(\underline{x}, y)$, we define $\mathfrak{M}^{\mathcal{U}} \models^{\nabla} \nabla y\psi(\underline{x},y)\ [\underline{a}]$ iff the set $\{b \in M : \mathfrak{M}^{\mathcal{U}} \models^{\nabla} \psi(\underline{x},y)\ [\underline{a},b]\}$ belongs to the ultrafilter $\mathcal{U}$. For a purely first-order formula $\theta(\underline{v})$ (without ∇), $\mathfrak{M}^{\mathcal{U}} \models^{\nabla} \theta(\underline{v})\ [\underline{m}]$ iff $\mathfrak{M} \models \theta(\underline{v})\ [\underline{m}]$ (as indicated in the preceding remark).

22 These schemata code the properties: the empty set is not in an ultrafilter, ultrafilters are closed under intersection, and a set or its complement is in an ultrafilter. (Closure under supersets follows from these properties.)

23 We will then have $\Sigma \vdash^{\nabla} \varphi$ iff $\Sigma \cup A^{\nabla} \vdash \varphi$ (which yields monotonicity).

24 We have $\Sigma \vdash^{\nabla} \nabla v\, \psi \leftrightarrow \nabla v\, \theta$ whenever $\Sigma \vdash^{\nabla} \psi \leftrightarrow \theta$.

25 More precisely, $\vdash^{\nabla} \neg \nabla v\, \varphi \leftrightarrow \nabla v\, \neg\varphi$.

26 For instance, $\vdash^{\nabla} \nabla v(\psi \wedge \theta) \leftrightarrow (\nabla v\, \psi v \wedge \nabla v\, \theta)$ and $\vdash^{\nabla} \neg \nabla v(\psi \vee \theta) \leftrightarrow (\nabla v\, \psi v \vee \nabla v\, \theta)$.

27 Every formula is provably equivalent to one consisting of a prefix of quantifiers ($\forall$, $\exists$ and ∇) followed by a quantifier-free matrix.

28 More precisely, $\Gamma \vdash^{\nabla} \tau$ iff $\Gamma \models^{\nabla} \tau$.

Our ultrafilter logic is a proper (as we will see later) and conservative extension of classical first-order logic[29], with which it shares some metamathematical properties, such as compactness and Löwenheim-Skolem properties[30].

4. ARBITRARINESS?

We shall now examine the contention that an ultrafilter embodies a certain degree of arbitrariness. The idea is that, even though this may be the case for a given ultrafilter, this is largely dissolved when one considers, and reasons with, theories.

As an example, let us consider the set $\mathbb{N}$ of the natural numbers and the two (infinite) subsets of $\mathbb{N}$:

- the set E of the even numbers,

and

- its complement E^C (the set of odd numbers).

These two infinite subsets have the same cardinality and appear to be equally important, but exactly one of them must be negligible. Now, considering either one of them as negligible, and the other one as very important, appears to be somewhat arbitrary.

Sometimes, a context (given by an application) may remove this feeling of arbitrariness (rendering it only apparent). For instance, the set P of prime numbers may be rightfully deemed very important by a number-theorist working on problems of Cryptography.

Now, let us return to the general case, where ultrafilters may appear to involve some arbitrariness.

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[29] For classical formulae (without ∇), our ∇-axioms add no extra deductive power, i. e. for such Σ and φ without ∇, we have $\Sigma \vdash^{\nabla} \varphi$ iff $\Sigma \vdash \varphi$.

[30] The apparent conflict with Lindström's results (Lindström (1966)) is explained because we are using a non-standard notion of model (due to the ultrafilters). This feature may confer to ultrafilter logic some independent model-theoretic interest.

First, we recall an example of a proper filter over an infinite universe V. It is the so-called Fréchet filter consisting of the cofinite subsets of V (i. e. those with finite complement)[31].

Now, consider the Fréchet filter $\mathcal{F}$ over the universe of $\mathbb{N}$ of the naturals. Given any infinite subset Z of $\mathbb{N}$, we can see that the family $\mathcal{F}\cup\{Z\}$ has the finite intersection property (fip)[32]. Hence, it can be extended to a proper ultrafilter[33] $\mathcal{U}_Z$ over $\mathbb{N}$, i.e. $\mathcal{F}\cup\{Z\}\subseteq\mathcal{U}_Z$.

So, we have several proper ultrafilters over $\mathbb{N}$, for instance

- an ultrafilter $\mathcal{U}_E$, with $E\in\mathcal{U}_Z$ (hence $E^C\notin\mathcal{U}_E$), and
- an ultrafilter $\mathcal{U}_{E^c}$, with $E^C\in\mathcal{U}_{E^c}$ (hence $E\notin\mathcal{U}_{E^c}$).

The set E of the even numbers is very important in the former case, but not in the latter, when the set E^C of the odd numbers is very important.

We thus have ultrafilter models where the assertion "almost all numbers are even" holds, as well as models where it fails to hold. Hence, in a theory having such models the assertion "almost all numbers are even" is left undecided: neither it nor its negation is provable[34].

In this sense, the alleged degree of arbitrariness, that may be attached to a given model, is largely dissolved when one considers theories.

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31 The dual family consisting of the finite subsets is clearly a proper ideal.

32 For any finite family of sets $X_1, \ldots, X_n$ in $\mathcal{F}\cup\{Z\}$, we have $X_1\cap\ldots\cap X_n\neq\varnothing$ (otherwise Z would be finite).

33 It can be extended to a proper filter, and then, by resorting to Zorn's Lemma (or, equivalently, to the Axiom of Choice), to a proper ultrafilter.

34 Notice that the set of even naturals is definable in the structure $\mathfrak{N}=(\mathbb{N},+)$ of the naturals with addition, and its theory $Th(\mathfrak{N})$ is decidable (Presburger's Theorem, see, e. g. (Enderton (1972), p. 188)).

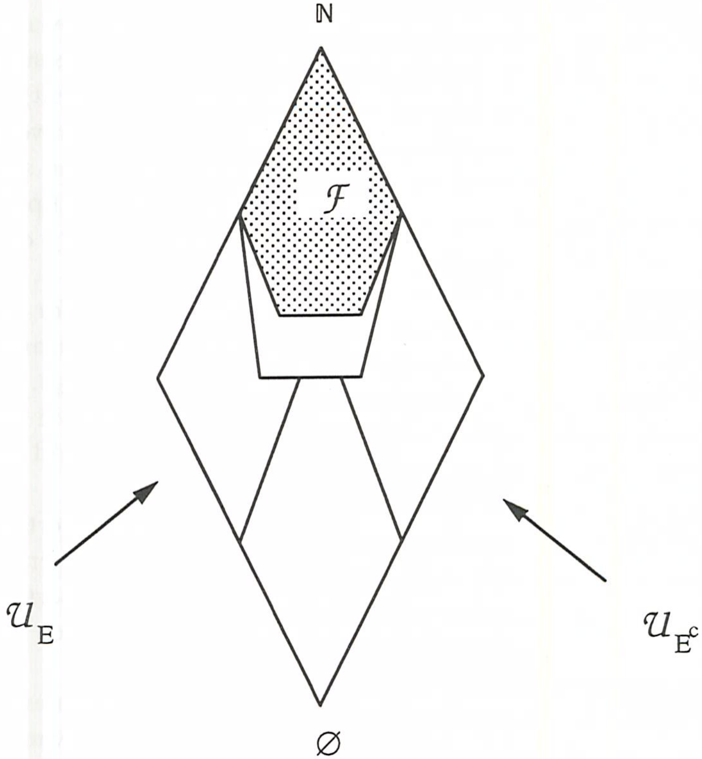

Figure 7: Filters over the naturals

This fact is corroborated by a result of ultrafilter logic: in the absence of almost universal information, the only almost universal consequences are the universal ones[35]. This expresses the fact that the universe is the only set that can be guaranteed to be in every ultrafilter.

A perhaps pertinent analogy is with probability. Probability theory is more concerned with obtaining some probabilities from others, by means of properties, than with assigning probabilities to particular events. Particular distributions may assign different

[35] More precisely, for Σ and φ without ∇, $\Sigma \vdash^{\nabla} \nabla v\, \varphi$ iff $\Sigma \vdash \forall v\, \varphi$.

probabilities to evens and odds, but, without information about distributions, one would know very little about their probabilities.

Even in a theory with almost universal information expressing that the cofinite sets are very important (i. e. a finite set has almost no number)[36], the assertion "almost all numbers are even" is still left undecided.

5. EXPRESSIVE POWER

We shall now briefly consider the expressive power of our ultrafilter logic, showing that it is a proper extension of classical first-order logic.

As a motivating example, imagine a (consistent) first-order theory Σ expressing information concerning flying birds. Assume that

- we know that some birds fly (i. e. $\Sigma \vdash \exists v F(v)$);
- we do not know that all birds fly (i. e. $\Sigma \nvdash \forall v F(v)$).

Now, consider the almost universal assertion

- almost all birds fly, i. e. $\nabla v F(v)$.

Can this almost universal assertion be expressed by a purely first-order logic sentence (without ∇)? One is likely to lean toward a negative answer.

In view of the result mentioned at the end of the preceding section, we know that we cannot derive the assertion $\nabla v F(v)$ from the information in Σ[37]. (But, notice that our question does not concern truth, or derivability, of this assertion, but rather its expressibility in simpler terms.)

Now, assume that we have some other predicates, such as W (for 'has wings'), K (for 'has beaks'), and D (for 'is a biped'), as well as perhaps further (consistent) classical information, such as

- all birds have beaks (i. e. $\Sigma \vdash \forall v K(v)$),

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[36] This can be expressed by the assertion $\forall y \nabla x \neg x = y$ (see section 6).

[37] Indeed, $\Sigma \nvdash^{\nabla} \nabla v F(v)$ since $\Sigma \nvdash^{\nabla} \forall v F(v)$.

- every bird is a biped (i. e. $\Sigma \vdash \forall v D(v)$),
- flying birds have wings (i. e. $\Sigma \vdash \forall v[F(v) \rightarrow W(v)]$);
- but we still do not know that all birds fly (i. e. $\Sigma \nvdash \forall v F(v)$)[38].

Can we now express the almost universal assertion $\nabla v F(v)$ by an equivalent sentence without ∇? The question seems to have become more difficult, but it still has a negative answer. Moreover, the reason for this negative answer rests entirely on classical first-order reasoning, namely

$$\Sigma \nvdash [\exists v F(v) \rightarrow \forall v F(v)].$$

We will now indicate why the only almost universal assertions $\nabla v\varphi$ (where φ has no ∇) that can be expressed without ∇ are the trivial ones (such that $\exists v\varphi \rightarrow \forall v\varphi$ can be derived).

Considering Σ and φ without ∇ we will show that

- if $\Sigma \vdash^{\nabla} [\nabla v\varphi \leftrightarrow \theta]$, for some θ without ∇,
- then $\Sigma \vdash [\exists v\varphi \rightarrow \forall v\varphi]$.

Towards this goal, we first note that, for θ without ∇,

- if $\Sigma \vdash^{\nabla} [\nabla v\varphi \rightarrow \theta]$,
- then $\Sigma \vdash [\exists v\varphi \rightarrow \theta]$,

by reasoning with models[39].

We then also have, for θ without ∇,

- if $\Sigma \vdash^{\nabla} [\theta \rightarrow \nabla v\varphi]$,
- then $\Sigma \vdash [\theta \rightarrow \forall v\varphi]$,

by duality[40].

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[38] Or, even more strongly, that we know that not all birds fly (i. e. $\Sigma \vdash \neg\forall v F(v)$), as long as Σ is consistent.

[39] Otherwise, we would have a model $\mathfrak{M}$ of Σ such that $\mathfrak{M} \nvDash \theta$ where the extension of φ is nonempty, and thus can be extended to a proper ultrafilter $\mathcal{U}$. This gives an ultrafilter model $\mathfrak{M}^{\mathcal{U}} = (\mathfrak{M}, \mathcal{U})$ of Σ such that $\mathfrak{M}^{\mathcal{U}} \vDash^{\nabla} \nabla v\,\varphi$.

[40] Since $\vdash^{\nabla} \neg\nabla v\,\varphi \leftrightarrow \nabla v\,\neg\varphi$.

Finally, by combining these two implications, we can conclude the announced necessary[41] condition for eliminating ∇.

Now, returning to our example, we see that this necessary condition fails[42]. Thus, the almost universal assertion $\nabla v F(v)$ cannot be expressed without ∇.

Hence, in ultrafilter logic we have formulae that cannot be expressed within classical first-order logic, showing that the former is a proper extension of the latter.

6. FINITENESS?

We shall now examine the contention that ultrafilters are not interesting for finite situations. Again, the idea is that, even though this may be the case for a particular (finite) structure, this is practically dissolved when one considers theories.

Clearly, over a finite universe V, every ultrafilter is finite. Moreover, as it is well known, such an ultrafilter must be generated[43] by a singleton[44]. So, such an ultrafilter consists of all the subsets including some element of V, i. e. it must be of the form

$$g^{\mathcal{U}} = \{ X \subseteq V : g \in X \}$$

for some element $g \in V$ (its generator).

For ultrafilter $g^{\mathcal{U}}$ generated by $g \in V$, we have

- almost all objects have a property $\varphi(v)$, i. e. $\nabla v \varphi(v)$

iff[45]

[41] It is also sufficient: if $\Sigma \vdash [\exists v\, \varphi \rightarrow \forall v\, \varphi]$, then we can take θ as $\exists v\, \varphi$ (or $\forall v\, \varphi$).

[42] Because $\Sigma \vdash \exists v v\, F(v)$ but $\Sigma \nvdash \forall v\, F(v)$.

[43] The ultrafilter generated by subset $G \subseteq V$ consists of all the subsets $X \subseteq V$ including the generator G.

[44] In view of postulate P7, because each finite universe has a finite cover by all its singletons.

[45] Indeed, we have the following equivalences

- the generator g this property, i. e. $\varphi(g)$.

This equivalence provides a crucial test for property $\varphi(v)$, reducing almost all to element g. So, such an element g may be termed archetypal[46] or generic[47] for this property (Carnielli & Veloso (1997); Veloso (1998)).

Notice that this equivalence implies that "almost all objects are equal to the element g", which may sound exaggerated, but is really intended to mean that the singleton $\{g\}$ is very important.

One must, however, bear in mind that we are making these considerations in the presence of a given ultrafilter. It is in this context that

- the almost universal assertion $\nabla v \varphi(v)$

reduces to

- the simpler assertion $\varphi(g)$.

In a given finite ultrafilter model, we have its ultrafilter, so we can assume to know its generator.

Let us now consider several finite ultrafilter structures, each one with its own ultrafilter. We know that each such ultrafilter has a generator, but we may not have access to it.

A similar situation occurs with theories. Even though we may know that each ultrafilter has a generator, we may unable to identify it.

- almost all objects have a property $\varphi(v)$

iff

- the set $[\varphi(v)]$ of objects having this property is in the ultrafilter $_g\mathcal{U}$.

iff

- the generator g is in the set $[\varphi(v)]$ of objects having this property

iff

- the generator g this property, i. e. $\varphi(g)$.

[46] For a typical element g, $\nabla v\, \varphi(v)$ would follow from $\varphi(g)$.

[47] The term 'generic' has been used in other contexts, such as (Fine (1985)), for a similar, but not quite the same, idea

For instance, reconsider the case of the flying birds, now with information, such as

- almost all birds fly (i. e. $\nabla vF(v)$),
- Mother Goose does fly (i. e. $F(m)$),
- Woody, a woodpecker, flies (i. e. $F(p)$);
- Sam, a penguin, does not fly (i. e. $\neg F(s)$).

Assuming that the universe B of birds is finite, we can be sure that each ultrafilter will have a generator, i. e. we know that

$$\exists y \nabla x x = y.$$

But, we do not know its identity. In fact, we have very little information concerning the generator. Besides the fact that it cannot be the non-flying Sam, for all we know the generator might be

- either Woody;
- or Mother Goose;
- or even some unnamed bird.

Thus, if Tweety is (the name of) a bird, the theory will not decide whether or not it flies.

Having names for all the possible objects in a finite universe will not change the situation. For, even though the theory guarantees the existence of an object (a generator), it still may happen that all we can know is that it is one of the objects in the finite universe, without being able to decide the disjunction and identify it.

For a universe with three objects, consider theory Δ involving three constants, say s (for 'solid'), l (for 'liquid'), and k (for 'gaseous'), with axioms stating that these three constants are pairwise distinct and exhaust the entire universe[48]. We will then have

[48] For instance, $\neg s = l$, $\neg s = k$, $\neg l = k$, and $\forall x[x = s \vee x = l \vee x = k]$.

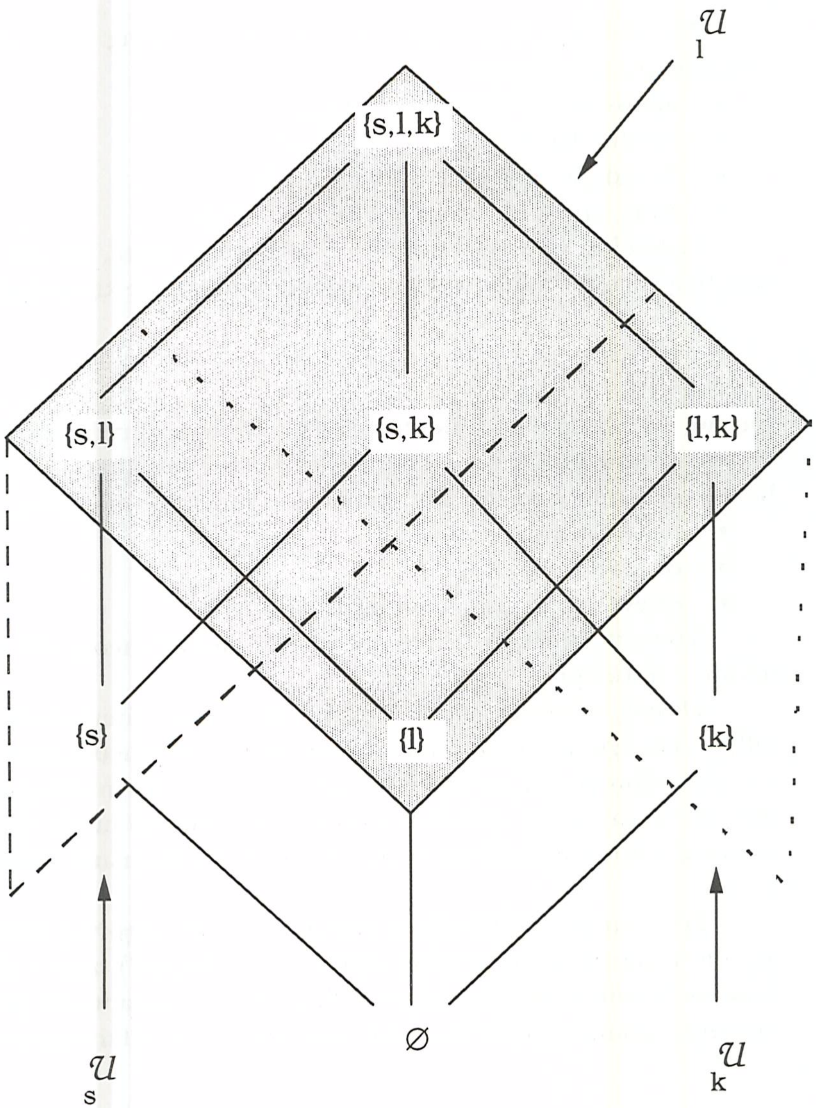

Figure 8: Ultrafilters over a three-element universe

$$\Delta \vdash^{\nabla} \exists y \nabla x x = y,$$

whence, also

$$\Delta \vdash^{\nabla} [\nabla x x = s \vee \nabla x x = l \vee \nabla x x = k\,];$$

but, theory Δ will not decide this disjunction[49], which is to be expected (there are no grounds to deem an element much more important than the others).

Indeed, over the universe $\{s,l,k\}$, we have three ultrafilters, namely

- ultrafilters ${}_s\mathcal{U}$, ${}_l\mathcal{U}$, and ${}_k\mathcal{U}$,

with respective generators s, l, and k.

So, theory Δ has three ultrafilter models[50], namely

- ${}_s\mathfrak{T}$, ${}_l\mathfrak{T}$, and ${}_k\mathfrak{T}$,

expanding the three-element universe $\{s,l,k\}$, respectively, with the ultrafilters ${}_s\mathcal{U}$, ${}_l\mathcal{U}$, and ${}_k\mathcal{U}$.

In this sense, the contention that ultrafilters over a finite universe are not interesting (because then they are generated by an element) is dissolved when one considers theories. Even though in each particular finite structure, its ultrafilter has a generating element, this does not mean that a theory will be able to pinpoint such a generator (for all its models).

7. CONCLUSION

We have examined, trying to clarify and justify, some issues underlying a logical system for the precise treatment of assertions

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49 More precisely, $\Delta \nvdash^{\nabla} \nabla x x = s$, $\Delta \nvdash^{\nabla} \nabla x x = l$, and $\Delta \nvdash^{\nabla} \nabla x x = k$.

50 Each one of these models decides the disjunction in favour of the generator of its ultrafilter, e. g. ${}^s\mathfrak{T} \vDash^{\nabla} \nabla x x = s$, but ${}^s\mathfrak{T} \vDash^{\nabla} \neg \nabla x x = l$ and ${}^s\mathfrak{T} \vDash^{\nabla} \neg \nabla x x = k$.

involving "almost all" objects. We have concentrated on the usage of ultrafilters for capturing the intended meaning of "almost universal" assertions, trying to expose and explain some basic intuitions.

We have addressed three objections to using ultrafilters, namely

1. lack of intuitive justification,
2. degree of arbitrariness,
3. trivialisation in finite situations.

We have attempted to circumvent the first objection by means of an analysis of the underlying concepts, and then argued that the other two can be overcome by a proper understanding of the roles of models and theories.

We have approached the first alleged objection (lack of intuitive justification for using ultrafilters) by analysing the ultrafilter proposal. Instead of trying to justify directly the usage of ultrafilters, we have suggested an alternative interpretation, based on local and relative notions concerning sets (the relation of 'having about the same importance', as well as the properties of being 'negligible' and 'very important'). An analysis of our intuitive understanding of these basic notions has suggested some reasonable postulates[51], from which we can rigorously derive the characteristic 'properties of ultrafilters.

Clearly, the alternative interpretation suggested has little, if any, impact on ultrafilter logic in so far as mathematical logic is concerned[52]. It may, though, affect its acceptance as well as its possible applications.

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[51] These postulates are, admittedly, not all equally acceptable on intuitive grounds.

[52] In other words, this interpretation has negligible impact on the mathematical results, but it may affect the search for results and their applications.

We have then briefly recalled some basic ideas about ultrafilter logic, a logical system for the precise reasoning about "almost all". We have sketched its syntax, semantics and axiomatics, indicating that we have a proper, conservative extension of classical first-order logic, with which it shares some properties, such a simple sound and complete deductive calculus, as well as compactness and Löwenheim-Skolem properties.

We have then examined the second alleged objection (arbitrariness in connection with ultrafilters). Indeed, this may be the case with a particular ultrafilter (since it must have either a set or else its complement). We have, however, argued that this is largely dissolved when one considers a theory (with models for both alternatives).

Before embarking on considerations about the third contention, we have examined the expressive power of our ultrafilter logic, showing that it is a proper extension of classical first-order logic. We have established the result that an almost universal prefix can be eliminated from a pure matrix only when this matrix is trivial (with existence implying universality). This result shows that we have a proper extension, and it also yields that, even in finite situations, almost universal sentences express more than classical first-order sentences.

Finally, we have taken up the third alleged objection to ultrafilters (trivialisation in finite situations). Indeed, this may be so with a particular ultrafilter (since, being over a finite universe, it must have a generator, providing a crucial test for the almost universal assertions). Again, this objection is dissolved when one considers a theory (with several ultrafilter models). Then, the mere existence of a generator (established by the theory) does not identify it (and the crucial test is lost).

We shall now briefly mention some other aspects of ultrafilter logic and comment on possible applications (Carnielli &

Veloso (1997); Veloso & Carnielli (1997); Sette *et al.* (1999); Veloso (1998)).

For reasoning about "typical" or "archetypal" objects, ideal individuals can be introduced by means of almost all, and internalised as constants, thereby producing conservative extensions where one can reason about generic objects as intended. For instance, from "almost all swans are white" one can conclude that "typical swans will be white" and "a non-white swan cannot be typical". So, ultrafilter logic provides a logical system for reasoning about assertions with "almost all", as well as "typical" or "archetypal" objects, over a given universe.

More interesting situations, however, require such assertions relative to several universes, involving "almost all birds", "almost all penguins", and "typical eagle", for instance. Finer analyses show that relativisation fails to capture the intended meaning[53], thus indicating the need for distinct notions of negligible subsets. Towards this aim, one introduces a many-sorted version of ultrafilter logic, with notions of negligible subsets relative to the universes, which shares many properties, such as supporting generic reasoning, with the original version. Moreover, some situations require comparing distinct notions of negligible subsets over some universes. Many-sorted ultrafilter logic can handle such comparisons by means of appropriate transfer assertions.

Thus, ultrafilter logic provides a logical system for precise reasoning about assertions involving "almost all" as well as "typical" or "archetypal" objects.

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[53] This problem appears to be related to the so-called "Confirmation Paradox" in Philosophy of Science (Hempel (1965)). Each flying eagle is considered as confirmatory evidence in favour of "Eagles fly", whereas a non-flying non-eagle is not felt so, even though "Eagles are fliers" and "Non-fliers are non-eagles" are logically equivalent (Carnielli & Veloso (1997); Veloso (1998)).

As a logic with generalised quantifiers, ultrafilter logic is connected to such extensions of first-order logic (Barwise & Feferman (1985); Keisler (1970)). It is also related, though to a lesser extent, to the tradition of analysis and formalisation of language[54].

One of the original motivations for extending first-order logic was capturing "common-sense" reasoning (Schlechta (1995)), thereby providing an alternative to non-monotonic logic (Carnielli & Sette (1994)). Ultrafilter logic is thus related to default logic (Reiter (1980)) and its variants[55] (as well as, even though to a lesser extent, to belief revision[56]). Indeed, they do have a large intersection in so far as applications are concerned, as indicated by benchmark examples. They turn out to be, however, quite different logical systems, both technically[57] and in terms of intended interpretation[58] (Carnielli & Veloso (1997); Veloso (1998)).

We finally comment on some perspectives and directions for further work, specifically some interesting connections with fuzzy logic and inductive and empirical reasoning, which suggest the

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[54] As in (Frege (1879); Tarski (1936); Church (1956)), for instance.

[55] See, e. g. [Antoniou (1997); Brewka (1991); Lukaszewicz (1990); Marek & Truszczynski (1993)).

[56] See (Gärdenfors (1988); Makinson & Gärdenfors (1991)), for instance.

[57] Concerning technical aspects, ultrafilter logic is monotonic and a conservative extension of classical first-order logic, in sharp contrast to the non-monotonic nature of the approaches via defaults.

[58] Concerning intended interpretation, one can perhaps phrase the difference in terms of positive and negative views (Sette *et al.* (1999)). Our approach favours a positive view, in the sense that we wish to express assertions involving 'almost all' and 'typical' explicitly. The default approach takes a negative view in the sense of interpreting such assertions as "in the absence of information to the contrary".

possibility of other applications for our ultrafilter logic. The basic idea is exploiting the expressive and deductive powers of ultrafilter logic.

A first possible application is to the realm of imprecise reasoning, in the spirit of fuzzy logic (Turner (1984)). Some common ground is indicated by the basic intuitions of 'almost all', 'negligible', 'very important' and 'about as important as'. For instance, a fuzzy concept, such as 'very tall' might be explicated as: a very tall person is a person that is taller than almost everybody (else)[59]. Such approach may provide alternative qualitative foundations for (versions of) fuzzy logic.

Another possible application could be to the area of inductive reasoning, as in empirical experiments and tests. This arises from the observation that, whereas laws of pure mathematics may be of the form "All M's are N's", one can argue that empirical laws (as in natural sciences) can be regarded as assertions of the – more cautious – form "Almost all M's are N's". Here, the expressive power of ∇ may be helpful[60].

The possible applications outlined above suggest another interesting avenue: the weakening of some mathematical concepts. For instance, the idea of 'almost dense' is close to that of

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59 Another example is provided by a non-standard model of the naturals with a Fréchet ultrafilter (one excluding all the finite subsets). Then, the standard naturals would be "very short", whereas the non-standard numbers would be "very high" (and numbers in the same copy would be "about as high").

60 The case of program testing may be illustrative: one tests the behaviour of a program for a (small) set of data and then hopes to argue that the program will exhibit this behaviour in general. Here, the rationale is that the set of test data is "representative" in that it covers the possible execution paths. This may be considered as an example of an inductive jump: from fairly small experimental evidence to an almost universal conclusion.

'almost coverage', which was found useful in expressing connections between sorts (Veloso (1998)). Along similar lines, concepts such as 'almost equal' or 'almost disjoint' might be useful. The basic idea is weakening some universal quantifiers to almost all. For instance, one might consider the notion of 'almost partition' as a set of blocks almost covering the universe where intersecting blocks would have almost the same elements. Analogously, 'almost equivalence' could be obtained. Similar weakening of some mathematical ideas might be of interest[61].

To conclude, we have discussed and tried to explain some fundamental issues in the precise treatment of assertions involving "almost all" objects. We hope that our attempted rational reconstruction of some basic intuitions has contributed to clarify, and justify, the usage of ultrafilters as capturing the intended meaning of "almost universal" assertions. We trust that our analysis of the contentions concerning (apparent) degree of arbitrariness and trivialisation in finite situations has helped to dispel some possible misconceptions about ultrafilter logic. Logics such as this one appear to merit further investigation, both as logical systems and in connection with other fields, which may suggest interesting applications or variants of these ideas.

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[61] Note that we are not proposing a programme; one can expect that the weakening of only some ideas – typically those with qualitative flavour – will be of interest.

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