A CRITICAL STUDY ON THE FOUNDATIONS OF GEOMETRY

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Uma nova maneira de abordar o estudo dos fundamentos da geometria é aqui proposta. Ela consiste essencialmente em considear o tema numa perspectiva histórica. Uma discussão crítica e comparativa dos conceitos grego e ocidental de geometria procura mostrar suas diferenças.

A new way of approaching the study of the foundations of geometry is attempted here by locating the subject in its historical perspective. A critical and comparative discussion of the Greek and the Western concepts of geometry tries to emphasize their differences.

Introduction

Transcendental Dialogue:

Euclid: "Σημεῖόν ἐστιν, οὗ μέρος οὐθέν."
Hilbert: *"Nein! Die Ebene ist ein System von Dingen, welche Punkte heissen."*

1. In this paper we intend to develop a critical study on the foundations of geometry under a broader philosophical perspective than usual. Usually geometry is understood as a science that started with the Greeks and had afterwards a "natural evolution" through many centuries, with contributions by several people, finally reaching its "higher perfection" in the hands of the Western mathematicians. We shall develop here a completely different picture. We begin by recalling some basic facts of philosophy of history and also some ideas developed in our previous works (Lintz 1977, 1988).

Any theory of history develops a certain scheme through which facts are collected and analyzed in order to explain "what happened and what will happen". A theory is better than another if it can explain more facts with a minimum of general assumptions. It is our opinion that the line of thought developed in philosophy of history starting with G. B. Vico in the 18th century and culminating with the monumental works of A. Toynbee and O. Spengler is, according to the criterion above, the most general and profound theory of history.

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It allows us to locate and explain facts better than any other we know of; hence it is the philosophy we shall use in this paper. The reason why these theories have been rejected by the majority of philosophers and scientists is easily explained by psychology: they shake the very foundation of beliefs we consider as solidly established; in particular, they hurt the pride of our Western Civilization, for they remove it from its position as the finest achievement of mankind, requiring, instead, that it be put on an equal footing with the other civilizations and historical cultures. It is about time we put aside our pride and arrogance and try to analyze the facts in a realy objective and unbiased perspective.

2. Any historical culture is an organism which exists and develops itself in *time and space*. The word *organism* is made precise in the following way. It is an object of our consideration ($\mu\acute{\alpha}\theta\eta\mu\alpha$) determined by:

a) a certain form which can be detected by us through some material media like figures is space, colors, sounds, written words or signs of any kind, which we shall call its *organogram*, expressed by a set of rules named its *syntax*;

b) a certain number of fundamental elements giving the organism its own identity and distinguishing it from other organisms which we call its *structure*;

c) a certain genetic code responsible for its evolution in time according to certain rules, its *organogen*.

Let us illustrate this concept with a few examples (cf. Lintz 1977).

Take a flower: its appearance is given by its particular shape, color, etc., which is its *organogram*; nature organizes these elements according to rules depending on its species and genus, which is its *syntax*; its internal organization, namely, its physiology – characterized by the proper functioning of its several organs - provides its *structure*; finally, its genetic code, which conditions its development from birth to death, is its *organogen*.

Similarly for a historical culture. Its *organogram* is given by all its expressive forms, namely architecture, sculpture, perhaps music and also mathematics. These are objective documents which show the existence of the historical culture in space and time. Its *syntax* is provided by the rules of succession of those forms in space and time. One of these rules is that the beginning of each historical culture is a mythological age and the end a technological age. The ***structure*** of a historical culture is given by its primitive symbols translating its feeling of the cosmos. For instance, in the Greek culture among these symbols we find a strong tendency for the finite, for the visible and plastic space; there is a *terror of the infinite and time* that explains why sculpture is its most characteristic expressive form rather than, say, music; and their mathematics comprises only geometry, the study of figures in space, together with arithmetic, the study of the integers. The Greeks have no room for the abstraction of set theory and real numbers. Finally its *organogen* is given by its law of evolution in time: its birth, youth, maturity and old age. Each of these stages emphasizes one or more particular expressive forms. In general mathematics emerges as an expressive form in the transition from maturity to old age, while architecture is usually the first great expressive form of a historical culture appearing in its youth.

The *leit-motiv* of this paper will be the consideration of geometry as an *expressive form* or, better, an *organism* attached to a historical culture. In our case we shall be dealing with geometry in Greek culture and in Western culture. By 'Western culture' we understand the historical culture born in the so-called nordic mythology and dominating today the world and perhaps a large part of the universe, due to its deep passion for the infinite, for abstract space and time. Its strong opposition to the picture of the world created by Greek civilization will be a responsible for its opposite approach to geometry.

3. The study of the syntax of the organogram of a certain organism is an important stage of the study of that organism and the discipline dealing with it shall be called *Inorganic Logic.* It corresponds, when the organogram is given by a language, to what is traditionally called *Logic* and, more recently, *Mathematical Logic.* However, there is another discipline concerned with the study of the structure and the organogen of a certain organism, which we shall call *Organic Logic.* As far as we know this discipline did not receive the same attention as *Inorganic Logic*, though it is present in all our mental activities. One of its fundamental principles is the *Principle of Analogy*, which allows, for instance, a naturalist to classify plants and animals and which is the basis for all experimental sciences. By its own nature, contrary to what happens to Inorganic Logic, it is not formalizable: it exists only in intuitive and inductive reasonings and never in deductive ones. It seems to have been already noticed by Goethe through his great and profound intuitive feelings of Nature as in (Goethe, J.W.) (for details, see Lintz 1977).

After the introduction of the fundamental notions above, we can state with more precision our aim. We intend to consider geometry as an organism attached to either Greek culture or Western culture. Consequently, what we understand by the *foundations of geometry* is not what is usually known by that name. To study geometry or its foundations is to study its organogram with its syntax, its structure and its organogen. This requires the use not only of Inorganic Logic but of Organic Logic as well. What has been done by people studying the foundations of geometry is really the study of its organogram and syntax and, moreover, *only as conceived in the West.* Clearly this is a rather limited perspective. Instead, we shall deal first with geometry as conceived by the Greek culture in its proper environment; afterwards we shall consider its study in the Western culture; and finally a parallel between both geometries will be drawn. This attitude will provide new insight into the subject with many surprising results, perhaps shaking some of our strongest convictions.

Foundations of Greek Geometry

1. The study of any manifestation of a historical culture depends on the documents available to us and one of the difficulties of the subject is the correct analysis and interpretation of those documents. The historian faces here the same type of problem as does the archaeologist who wants to reconstruct a whole sculpture from scattered pieces. The reconstruction is always *subjetive*, though he tries his best to be as objective as possible. In the case of geometry we rely on the texts that remain, to reconstruct the Greek thought on the subject. Of course, it is outside the scope of this work and also of our competence to endeavor a critical analysis of the texts. We rely here on the work of T. L. Heath and his translations of so many texts of Greek mathematicians, which are among the best available in English. The work of Heiberg and Diels will be always in the background to help us in the more obscure and debatable points.

Every analysis of the foundations of geometry starts with the question: "what does exist at the basis of geometry?" For the Greek mathematicians the answer was: *geometrical figures* conceived as entities existing in a visible space. They were realities given *a priori*, as far as possible from pure abstraction. Indeed, we have the feeling of straightness, of flatness, of spaciousness (due to our freedom of movements and felt with great intensity by a ballerina), etc. In giving form to these feelings, the Greeks explained the concepts of straight line, point and plane granting them a strong plastic, visible and finite content.

The Greeks acknowledged the existence of two kinds of mathematical entities, geometrical figures and numbers conceived as discrete units, both were completely set apart when mathematics became a rigorous discipline, in the hands of Eudoxus and Euclid. Indeed, the failure of the Pythagorean approach to geometry, leading to the "crisis of the incommensurable", originated in the idea of associating *numbers* to *segments* in the belief that "number is the origin of everything". Their theory of measure of a segment with a certain unit led to the dilemma: either the result was a rational number or they had to assume the possibility of division of a segment *ad infinitum*, creating the well known paradoxes of Zeno and others. The solution of the puzzle, is attributed to the great Eudoxus with the creation of his deep theory of magnitudes, which we know through Book V of Euclid's *Elements*. Eudoxus's fundamental idea was to eliminate numbers from geometry. *Geometrical figures* become then the primordial elements of geometry, the initial data. The measurement of length, areas and volumes is no more the business of the geometer; it belongs now to applied mathematics or logistics (λογιστική), the "art of calculation". That art was useful for engineers, architects, physicists, etc. but it was not the concern of the mathematician or philosopher. As a matter of fact, not only Plato but even Archimedes, the greatest of the Greek "applied mathematicians" considered as "ignoble and vile the business of mechanics". Certainly nature provides challenges for the mathematician but these only attain a supreme level of dignity when properly treated by rigorous geometrical methods, i.e., *more geometrico*.

Eudoxus's theory of magnitudes is the foundation of the theory of similarity of figures and their equivalence in area and volume *independently of measurements*. For the particular case of "curved" figures, like the circle, the parabola, etc. the method of exhaustion furnished the means for their comparative study relative to size (πελιχοτηs).

This point alone would be enough to repel the idea of relating the concept of real number to Greek geometry and thus to understand why it is not surprising not to find in Euclid, Archimedes and Apollonius traces of any "formulas" for "computing" the area of a triangle or any other figure. As we proceed this idea will become clearer.

2. Looking now at the *organogram* of Greek geometry we observe that it is constituted by a certain language, say Greek, plus geometrical figures taken as real entities existing in space. Space is the *inorganic space*, which for the Greek mind retains all the basic feelings of the *organic space, rather* than an abstract idea without any *visible* elements. It is hard for us, Western mathematicians, to grasp fully this idea. We shall never be able to do that, the real and deep feelings of past civilizations are lost forever! Only with great effort in trying to "think as a Greek" we shall be able to experience that feeling of the *plastic space* so magnificiently expressed not only in the Greek geometry but in the architecture of Callicrates and in the sculpture of Phidias and Praxiteles as well. This plastic space is the stage where all the drama of Greek geometry unfolds, with its figures and demonstrations.

The *syntax* of this organogram is made up of definitions, axioms, postulates and rules of inference which include not only Aristotelian logic but *geometrical constructions* as well. That is one of the most important aspects of the subject which has been completely overlooked and distorted by late Western criticism. Indeed, through the prism of our mathematical logic it is almost impossible to understand how a geometrical construction or figure can be part of rules of inference of any logical system, because it cannot be *formalized as such*. The reason is that, for Peano, Russell, Hilbert and all the other creators of "modern" logic and mathematics, a logical system is, basically, a collection of signs and the rules of inference are nothing but rules of manipulation of these signs (a point of view explicitly defended by Hilbert's formalist school). Therefore, there is no room for a geometrical figure to be part of the rules of inference. Of course, we are forgetting that signs or letters drawn on paper are, after all, geometrical figures.

For the rest of this section we shall be concerned with the analysis of the organogram of Greek geometry and its syntax. For the sake of completeness, we add a few words about its *structure* and *organogen*, which fall under organic logic. In future work we intend to focus on this area. The *structure* of Greek geometry is given by the peculiar feeling of space of Greek culture. *Organic space* is here the essential element, felt as a visible, finite and plastic object, where geometrical figures can move without deformation. The intuitive basis for the concept of *superposition of figures* is rooted in the structure of Greek geometry and consequently can only be understood under the laws of *organic logic* without any possibility of being formalized. This concept is used very often in proofs of theorems as part of the rules of inference. Only through the postulates, to be discussed below, it is possible to render this concept a formal one. But a trace of the organic will always remain behind the scene. It is this close relationship between organic logic and inorganic which is as strange to us as it is dear to the Greek mind. Finally, the *organogen* of Greek geometry is responsible for its development in time in the succession of expressive forms constituting Greek culture. As discussed elsewhere (Lintz 1977, 1988), it evolves through three stages: *primitive ornamentation*, from the beginning up to Eudoxus, *art*, from Eudoxus through Euclid, Archimedes up to Apollonius, and *posterior ornamentation*, from Apollonius up to the end of Greek culture around the 5th century A.D.

3. Let us go back to the organogram and syntax of Greek geometry. We begin with the analysis of some of the definitions given in Book I of Euclid's *Elements*. As we know it today, this book begins abruptly with the definition of point:

"Point is that which has no parts".

Our first reaction to that definition is: "it is meaningless". But let us try to understand what Euclid intended to say. The Greek word for point is σημεῖόν, which means a mark, or a visible sign. By saying that this mark *positively* has no parts, οὗ μέρος οὐθέν, as emphasized by the adverb οὐθέν, Euclid ties the concept of point with Eudoxus's definition of magnitude. More precisely, a point is not a magnitude relative to size, because it cannot have multiples and submultiples. In this way, when we talk about a point either it is given as an object by itself or as the intersection of two lines, or a line and a plane, etc. This excludes from the very beginning the possibility, which is very popular in the West, of defining a point P by a sequence of other points, i.e.

$$P = \lim_{n\to\infty} P_n$$

This expression is meaningless in Greek geometry, first of all because the point P is not given as an object by itself nor as the intersection of two figures, and second because the concept of infinity was from the start eliminated from geometry as a "dangerous concept" leading to paradoxes.

This idea of a point as an object existing in space with its own individuality is typically Greek and cannot be translated in terms of Western symbolic logic.

Let us consider now the concept of straight line:

"A straight line is a line which lies evenly with the points on itself".

Intuitively, this definition, like that of the point, is intended to express the organic concept of straight line as something that proceeds always in the same "direction" without deviating either to the "right" or to the "left", namely it is trying to explain our feeling of straightness and as such cannot be taken as a formal definition of straight line. The problem of the geometer is how to render these organic concepts into inorganic ones. This is achieved with the introduction of conveniently chosen postulates.

When Greek geometry began to be analyzed and studied by Western mathematicians its postulates were regarded only as part of the so called "logical structure of geometry". This misunderstanding had tragic consequences. It led to overlooking the fact that the postulates, besides setting the "rules of the game", where also intended to introduce the concepts of point, straight line, plane and other geometrical figures as *technical concepts*, adapted to inorganic logic and able to be used in the development of geometry as precise and rigorous tools. Otherwise, they could be only handled by organic logic which is not the right discipline to be used in the study of the organogram and syntax of Greek geometry. As a matter of fact, traditionally, when one talks about the foundations of geometry, this is usually understood as the study of the organogram and syntax of geometry, which is completely different from the study of geometry as an *organism*. The consequences of this confusion of an organism with its organogram will be discussed later. To render our ideas clearer we shall now study the postulates of Greek geometry as conceived by Euclid.

4. Starting from the assumption that point, straight line, plane and other geometrical figures had been precisely introduced as ob-

jects existing in space with their own individualities, Euclid states the following postulates:

Postulate 1. To draw a straight line from any point to any point.

Postulate 2. To produce a finite straight line continuously in a straight line.

Postulate 3. To describe a circle with any center and distance.

Postulate 4. That all right angles are equal to one another.

Postulate 5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.

Concerning postulates 1-4 we shall discuss only a few points interesting us here, directing the reader for more details to the magnificent translation of Euclid's *Elements* by Sir T.L. Heath, which is our main reference. However, in the case of postulate 5, we shall deviate from the traditional interpretation and indeed we could say that the clarification of this point is one of the reasons why we decided to write the present paper.

In postulate 1 Euclid translates in pure geometrical terms the organic feeling of going from one position in space to another in a "straight way", i.e., without going around in a crooked path. In the act of looking ahead, to another place far from oneself, is contained that "feeling of straightness", which is the deep feeling of spaciousness expressed by that postulate. As a matter of fact, the use of the verb *άγαγεῖν* in the aorist tense, infinitive mood, emphasizes the space characteristics of the line defined by two points *independent of time*. Also the uniqueness of the line is implicitly understood here, as by every mathematician after Euclid.

In particular we call attention to the adverb *συνεχὲς* attached to the verb *εκβαλεῖν*, in postulate 2, which makes it clear that the line can be extended *continuously*, without gaps. We see that the notion of continuity of the line is an essential part of that concept as a *space entity* and not as an *abstract one* and consequently it has nothing to do with Dedekind's postulate (see below). This postulate makes it also clear that the straight line is a finite object in space, not extending itself from $-\infty$ to $+\infty$ *actually*, but rather only *potentially*.

About postulate 3 we believe that what was really intended by Euclid was the following: given a point A and a straight line AB with extremities A and B, there is a circle with center A containing B in its circunference (περιφέρεια), because, as pointed out correctly by Heath (1956 vol. I, p. 199), there was no Greek word for radius. The word διαστήματι used in postulate 3 means *distance*, though not in the numerical sense of the measure of a segment but rather as something existing in space. Consequently, the definition of circle (κύκλος) as a plane figure formed by all equal segments with one extremity in A (its center) has a space content and not a metric one. Here the crucial point is the word *equal*. Intuitively, two segments are equal when they coincide with each other by a rigid motion in space. Clearly, without some care this concept is tautological, because the definition of (rigid) translation in space requires the notion of equal segment if one is to avoid the introduction of metrical concepts. To handle that, Euclid first postulates the possibility of rigid rotation of one segment over another *having one extremity in common* and then, in Proposition 2, Book I, he proves the possibility of defining equality by superposition of segments in the plane in general, i.e., without having necessarily one extremity in common. In the extant texts of the *Elements*, due to its deterioration in time by inumerable interpolations and copyists' errors, the clarity of this concept of equality of segments, which is fundamental in the logical structure of Greek geometry, is seriously jeopardized.

In the analysis of Euclid's *Elements* by Western mathematicians this assumption of rigid translation of figures in space has always been looked at as something scandalous "proper to a primitive stage of the development of mathematics", which is clearly nonsense! The introduction of group theoretical concepts as suggested by F. Klein is completely alien to Greek geometry, being meaningful only in the context of set theory. To summarize, Euclid proposes that by coupling the definition of a circle with the possibility of rigid rotations and postulate 3 we can *rigorously* define in terms of Greek geometry the concept of equality of segments, by superposition, in the general case. It is amazing how these fundamental questions have almost invariably escaped the attention of many critical analyses of Euclid's *Elements.*

Another important consequence of postulate 3 and of the concept of superposition of segments is the *equality of angles.* The concept of angle by itself would deserve a careful discussion but we shall not carry it out here. Assuming that an angle is a plane figure formed by two segments in the plane (its sides) with an extremity in common, we define the equality of angles as follows: let α and α' be angles with sides AB, AC and $A'B'$, $A'C'$, respectively. Consider a segment AD contained in AB which is smaller than AB, AC, $A'B'$, $A'C'$, i.e. AD is equal to segments contained in each side of α and α'. Draw a circle with center A and radius AD intercepting AC in a point E, according to the definition of circle. The circle's periphery contains the extremity of *any* segment with the other extremity at A and equal to AD. Since AD is smaller than AC, it is equal to a segment AE contained in AC and therefore E belongs to the circumference of the circle in question. In the same manner we can draw a circle with center A' and radius equal to a segment $A'D' = AD$, defining a point E' in $A'C'$. Now we define: α is equal to α', iff $DE = D'E'$.

In short, with the help of postulates 1, 2 and 3 it is possible to lay down the foundations of a "rigorous" theory of rigid motions in space and equality by superposition. The meaning of the word *rigorous* will became clearer as we proceed.

Postulate 4 conveys a strong and profound feeling of a visible and plastic space, the space of architecture. No one can ever conceive, say, a Doric temple without the clear idea that the columns are perpendicular to the floor and the beams are perpendicular to the columns and "consequently" are parallell to the floor. The words vertical and horizontal, related to orthogonality and parallelism, having a strong spatial content, are felt organically as such. In the same way if one makes a door with edges which are not perpendicular to each other it will not close properly: in the architectonic organic space of everyday life there is no room for non-Euclidean geometry.

Euclid's problem, as in the preceding cases, was to translate that organic architectural idea of orthogonality into a working concept for mathematicians, and we believe that his way of expressing that in postulate 4 was one of his many strokes of genius. Heath is again absolutely correct in pointing out that the equality of right angles is also equivalent to the concept of invariability of figures by translations and the homogeneity of space, a fact that reinforces the conclusions drawn above about postulates 1, 2 and 3.

Finally, we have the celebrated postulate 5, improperly known as the *postulate of parallels*, because it is equivalent to the statement that in a plane through a point not belonging to a straight line there is only one parallel to it. But the concept of *straightness* of the line is essential in the definition of parallel and the problem is: how to formulate in a technical and precise language the concept of straightness, i.e., the concept of lying "evenly without deviating from its direction", etc. That is exactly the aim of postulate 5. It expresses in a mathematical satisfactory form the intuitive ideas belonging to the *organic concept of straight line*, so that it can be used formally in the proofs of theorems, etc., and in all situations concerning the organogram of Greek mathematics.

It is hard to understand how Euclid's original intention could be interpreted in later times only as the *question of parallels*, creating for more than 2,300 years the well known orgy of proofs of the 5th postulate. What lies behind all those proofs? Let us see. Postulate 5 talks about a characteristic property of two straight lines r and s in the plane intercepted by a third straight line. It says that r will intercept s if produced in a convenient way. This expresses exactly the fact that r, when produced, remains "evenly to itself" and "does not deviate from its direction". In fact, if r could "deviate from its direction" it could behave like an arc of hyperbole and never intercept its asymptote s. Now if we want to prove that postulate we have to express in one way or another that property of the straight line of "keeping its direction" in some *formal statement* apt to be used in the proof. That is exactly what invariably happened in all the proofs of that postulate. The proofs in themselves are in general correct, but they have to use somehow, *as an hypothesis*, some assumption translating in technical terms that fundamental property of the straight line of "keeping its direction", which is not contained in the first four postulates. Otherwise, a proof in the organogram of Greek mathematics would be impossible because it is clear that one cannot prove a property of a certain object without using its definition either directly or indirectly. Certainly, all this sounds strange for a Western mathematician used to deal with abstract entities deprived of any intuitive meaning. We shall come back to this question later.

The fundamental historical mistake is the confusion of the equivalent statements of postulate 5 given by "its proofs" with the question of its independence. There is no question of independence at all in the sense understood by the symbolic logic created by Western mathematicians and philosophers. It is obvious from the very beginning that the 5th postulate cannot be a consequence of the remaining four postulates, in the organogram of Greek mathematics, because it states a property of the straight line that is not contained in them. Consequently it cannot be derived from them unless we use in one way or another a property of the straight line equivalent to that stated in postulate 5, translating the intuitive feeling of "proceeding evenly of itself", "keeping its direction", etc.

The root of the misunderstanding is the lack of historical perspective and the consequent confusion of an *organism* with its *organogram*. A straight line for a Greek mathematician is not an abstract object without any meaning, as it is the case for a Western mathematician. It is an object given initially with strong space content and characteristic properties expressed exactly by the postulates. The so called "proofs of independence" of the 5th postulate through the construction of proper models and the creation of "non-Euclidean" geometries will be one of the main points to be treated below.

To summarize, the postulates translate among other things the concept of straight line in technical terms to make it possible using it in formal proofs. Consequently the attempt to prove the 5th postulate, for instance, is cleary nonsense because in such a proof one has to use the concept of straight line, including some equivalent of that postulate, otherwise we are not using the *whole* concept of straight line. This fundamental mistake has plagued the later understanding of Euclidean and, in general, Greek mathematics. For the Greek mathematicians, the postulates are not merely conventional statements about abstract entities as conceived in the West, but rather statements about concrete objects given as initial data expressing, besides their proper definitions, some of their fundamental properties in technical terms apt to be used in formal proofs.

Geometry in the West

1. As pointed out before, in the West the concepts of space, number, magnitude, etc., are expressed through abstract forms strongly denying the visible and sensible space which is replaced by the concept of pure number represented by sets, classes, etc.: geometry becomes a chapter of set theory. Going back to the basic ideas discussed in the Introduction, this implies that the organogram of Western mathematics is formed by a certain language (metalanguage) like English, French, German, plus a collection of symbols, axioms, rules of inference of a particular set theory selected to fit the taste of the mathematician building the theory, i.e., either in the line of the logistic or of formalist or of the intuitionist school of thought. The final result is a collection of sheets of paper full of signs intelligible only to the initiated, most often without figures. However, no matter how abstract, how impartial one's point of view and how much one tries to erase any trace of subjectivity in such theories, in the background lies unavoidably the human mind with all its passions and historical roots. It is impossible to detach oneself from the profound archetypes of the human species which are the *leit-motifs* of all forms of human expression. Indeed, no matter how formal or abstract or logical one intends to become, one cannot avoid *oneself* and all of one's psychological tendencies. That may be one reason why one is sympathetic to one school of thought rather than to another.

One of the most characteristic and beautiful examples of an exposition of geometry in the Western style is the celebrated work of Hilbert (1930), which we shall take as representative of Western geometry, just as we have taken Euclid's *Elements* as representative of Greek geometry.

Hilbert starts with an abstract set X containing some particular subsets of objects called point, lines, planes, subjected to a certain number of postulates or axioms divided into five groups:

I	1-8:	Axioms of relationships.
II	1-4:	Axioms of ordering.
III	1-5:	Axioms of congruence.
IV:		Axioms of parallels.
V	1,2:	Axioms of continuity.

The rules of inference are essentially those of classical logic which were later formalized in some abstract models in terms of symbolic logic (e.g. Artin 1966). That procedure establishes the organogram and syntax of Western geometry, which are quite different from that of Geek geometry. Afterwards, Hilbert shows how one can prove theorems from the axioms and how one can develop a chain of consequences from the axioms above building a system called "Euclidean geometry". It is a nice game and good mental exercise, but at some point one gets concerned about the question of a *model* for such a theory. By that it is meant the following: starting from objects whose existence we accept, to define with them a set satisfying the axioms stated above. But what objects shall we assume to exist?

As seen before, for Euclid those objects are the geometrical figures existing in space with their own individualities. But for Hilbert the initial data or the objects which exist are *numbers*, or more generally arithmetics supposed to be a consistent system. From there one can show that the *real numbers* are a model for geometry by beginning with the set E of ordered pairs (x, y) of real numbers, called the *plane*. Each pair is called a *point* and linear combinations of them like

$$ax + by + c = 0$$

(where a, b, c are real numbers) are called *straight lines*. Of course we can generalize this by considering triples (x, y, z) forming *spaces* and even n-ples $(x_1, \ldots, x_n)$ forming the n-dimensional Euclidean space. Let us consider here only the case of pairs.

By assuming all the properties of real numbers one can prove that all axioms I-V are satisfied in the set E and therefore the original theory is not "empty", i.e. it has at least one "model". Of course, we can also argue about the existence of numbers and this would take us into the line of business inaugurated by Frege, Peano and Russell 7whose success was later jeopardized by the celebrated results of K. Gödel. But this is a *dangerous* battlefield and prudence dictates that we should stay, at least for the time being, at a safe distance from it.

Another important consequence shown by Hilbert is that from the axioms alone we can associate to each segment in a straight line a real number called its *length* and then establish a *system of coordinates* in the line as well as in the plane. Do not forget that the words point, straight line and plane continue to be understood in the abstract sense; if we draw figures on paper, to help our reasoning, that is necessary only from the *psychological* point of view and not from the *logical point* of view. For instance, by a *segment AB in a straight line r* we understand the abstract set of points $P \in r$ such that $A \leq P \leq B$ where the relation $\leq$ is given by the ordering axioms II, 1-4 and its pictorial representation on paper is only a psychological "help" and (perhaps) "guide" to our thoughts. In this way we establish a one-to-one correspondence between our original set X and the set E of Cartesian geometry defined by the pairs of real numbers. This allows us to talk about the coordinates of a point, the equation of a straight line, etc. In particular we can introduce the idea of distance in X, which thus becomes a metric space and soon we are dealing with more general sets called topological spaces. But let us leave these general considerations and return to the analysis of the concepts introduced above.

2. The fundamental question is: by considering the concepts of point, line and plane as understood by Euclid, is it possible to show that they satisfy the axioms I-V stated above? We intend to argue that the answer is *no* and that the main obstacle to a positive answer is the postulate of continuity.

First of all, to show that a certain object *satisfies* certain conditions we have to know the intrinsic properties of that object, namely, its *definition*.

Secondly, in what consists the *act of verifying* that an object *satisfies* certain properties? Is it logical or psychological? Here we have one of the typical situations where we cannot separate the organic from the inorganic. The first stage of this process and, as a matter of fact, of almost every process of knowledge, is *organic*. Indeed, to verify whether an object A satisfies a property p we have to use the Principle of Analogy. For example, suppose we say that a certain ball is red. Our conclusion is based on the fact that we look at the ball and compare it in our mind, *by analogy*, with our concept of red, a pure organic attitude, impossible to formalize in a system of inorganic logic. In a second stage we express our organic knowledge in a language or a formal system, achieving its reduction to inorganic knowledge. Thus, after concluding that the ball A is red, we express that in ordinary language or in a formal language: let E be the set of balls in the world and let $E(p)$ be the set of red balls, then "$A \in E \Rightarrow A \in E(p) \Leftrightarrow A$ is red". In this way our primitive observation that A is red is reduced to a statement in propositional logic.

Having this in mind, let us consider a segment of straight line AB with extremities A and B. For Euclid, the points A and B as well the segment AB are realities existing in space; for Hilbert they are abstract concepts, perhaps having as a model the set of real numbers between two real numbers, A and B. Then our original question reduces to: is it possible to establish a one-to-one correspondence between the points of the Euclidean straight line and the set of real numbers? Let us begin with an analysis of the concept of *length of a segment*.

3. The Pythagoreans looked at this question in the following way: let us consider a segment AB and let us select a segment u as unit. Since "number is the origin of everything" it must be possible to associate a number to every segment. One possibility is that by applying the segment u to AB it will fit in there a number m of times (m being an integer). In this case we say that the length of AB is m, as measured in terms of the unit u:

$$l(AB) = m.$$

Another possibility is that u does not fit exactly an integer number of times in AB. In this case we subdivide u in smaller units u_1, say, $u_1 = 10^{-1}u$, and so on, up to $u_n = 10^{-n}u$, until we eventually have integer number of segments u_n covering AB:

$$l(AB) = u + 10^{-1}u + \ldots + 10^{-n}u = mu_n.$$

It was assumed that these are the only possible results of the operation of measuring a segment, for otherwise we would have to proceed indefinitely and we could never associate a number to AB, contrary to the basic philosophical position of the Pythagoreans. Recall that number for a Greek mathematician was a non-negative *integer* i.e. a *collection of units*, a definition attributed to Thales.

As we know, this point of view was seriously shaken by the discovery that no number could be associated to the diagonal of the square if we take one of its side as the unit. That is to say, there are segments which cannot be measured, in the sense described above, with a common unit, i.e. they are "incommensurable".

This discovery produced a terrible crisis in Greek geometry. Only after almost two centuries a way out was found by the great genius of Eudoxus of Cnidos. His solution was to *abandon the idea of associating numbers to geometrical figures* and to develop a theory of magnitudes which is independent of any process of measurement. This led to the separation of number – as an independent concept whose study belonged to *arithmetic* – and *geometrical figure* – whose study belonged to *geometry*. His theory of magnitudes, exposed in Book

V of Euclid's *Elements*, and his principle of exhaustion remained the back-bone of the whole of geometry. It excluded any relationship with measurement of lengths, areas and volumes, which were left to *logistics*, the analogue of our applied mathematics. The question of measurement of lengths, areas and volumes became the business of the engineer, the architect and the practical man, having nothing to do with the considerations of the geometer. This point of view has been emphasized by Plato, Archimedes and most of the great mathematicians of Greece. That was the origin of the establishment of the concept of geometrical figure as the basic element, given "a priori".

What happened in the West? What concepts were taken as primitive, i.e., as given "a priori"? Following Hilbert we should take the concept of number to build a model for geometry. An abstract set X, with a certain structure defined by certain particular objects called point, straight line and plane subjected to a system of axioms or postulates, is initially given. By using, in particular, the axioms of congruence and continuity we can attach to an interval AB, understood here as an abstract object, a real number called its length, $m(AB)$. Following Dedekind, (1872) in order to define real numbers we assume the existence of the rational numbers and consider the set R of all pairs of classes of rational numbers (AB) such that:

D_1) $A \neq \emptyset$, $B \neq \emptyset$ and $A \cap B = \emptyset$.

D_2) Every rational number belongs either to A or to B.

D_3) If $r \in A$ and $s \in B \Rightarrow r < s$.

Each pair of sets A, B satisfying the above conditions is a "Dedekind cut" and defines a real number. In so far as measurements are made with real numbers, the 'theory of measure' is, on this account, reduced to a correspondence between abstract sets without any contact with "reality". In "practice", however, the engineer, the architect, the merchant, the physicist, etc. have the same attitude as in Greece, Egypt or anywhere else: they deal with concrete objects and not with abstract forms. In the design of mechanisms, electric circuits, public buildings, railroads, etc., *all numbers are rational.* For the engineer, there is no such a thing as real number: for him, $\sqrt{2} = 1.414\ldots$ and the dots just mean that if he needs more accuracy in the results he can get further decimals. In short, our applied mathematics is what the Greeks called logistics, an empirical collection of rules and experimental data accumulated through ages and transmitted from generation to generation and from civilization to civilization. It is this accumulation of empirical knowledge that gives the impression of constant progress if we do not distinguish carefully the *empirical* from the *symbolic*. From the empirical point of view the engineers who built the pyramids, or a Doric temple or a Gothic cathedral faced similar technical problems, e.g. the stability of the structure, the resistance of materials, etc. But, as *expressive forms*, a pyramid, a Doric temple and a Gothic cathedral are three distinct symbolic representations of space belonging three distinct historical cultures.

At some point in our mathematical education, our teachers would solemnly announce: "today we intend to show you how to establish a one-to-one corresponce between the points of a straight line and the set of real numbers". The embarassing question here is: what shall we understand by *straight line*? If it is Hilbert's abstract concept, disregarding any model, we shall have exactly the situation described above: a correspondence between two abstract sets and nothing is gained from an "intuitive representation" in space. But if by straight line we understand an object represented by a model taken from analytic geometry then there is nothing to prove because this model is already the set of real numbers itself. The only alternative left to our teacher, and that is what he probably had in view from the very beginning, is that by straight line we should understand the Greek or Euclidean line as an object in space given "a priori". It is here that our troubles begin.

The only way left was that of the Pythagoreans. Accordingly a Western mathematician would proceed as follows. First of all we have to analyze the question of applying a unit segment u over a straight line r a certain number of times. That depends on the homogeneity of space and on the rigid motion of bodies. To define that, we must assume the rigidity of the unity u and we clearly get involved in a vicious circle. A honest and diligent Western mathematician should conclude that the procedure used by the Pythagoreans is logically unattainable and could only be accepted from an intuitive and practical point of view. But then we should stop short and declare impossible any attempt of establish a one-to-one correspondence between the Greek straight line and the set of real numbers. But, suppose a peace-maker comes along and says: "Let us close our eyes for the time being to that vicious circle and let us assume that in some way or another we have the possibility of associating with any real number a uniquely defined point in the Greek straight line". However, such a peace proposal is not free of difficulties. Consider all pairs of classes of rational numbers (A, B) satisfying conditions D_1, D_2, D_3 above, defining a Dedekind cut. For each pair (A, B) *Dedekind's continuity postulate* says: there is a "point" $\alpha \in r$ corresponding to (A, B).

The first difficulty is that the definition of α as the "supremum of A", or as the "infimum of B" requires a great logical sophistication, much more than originally thought, as shown by H. Weyl (1917, pp. 19-20, p. 71).

Secondly – and that is the main question – from the point of view of Greek geometry the point α ($\sigma\eta\mu\varepsilon\hat{\iota}o\nu$) is an object with proper identity existing in space and is either given from the beginning, as by the phrase "let α be a point in the line r", or it is given by the intersection of two geometrical figures, as by the phrase "let α be the intersection of r with the straight line s". Consequently, the definition of the "point" α, as indicated above, by a pair of classes (A, B) is completely meaningless and could never be accepted by a Greek mathematician "as rigorously defined". As a matter of fact, the reason why the celebrated method of Hippias for the squaring of the circle by using the *quadratrix* was never accepted by Greek mathematicians is exactly motivated by the fact that a certain point is defined by a "limit process" (cf. Heath 1981, vol. I, p. 226).

Therefore, from the Greek point of view, there is in general *no point* corresponding to a Dedekind cut. This leads to the conclusion that *it is logically impossible from the point of view of Western mathematics to establish a one-to-one correspondence between the set of real numbers and the Euclidean straight line*. That can only be done in the so called "applications of geometry" in everyday life.

To summarize, it can be shown that, under the proper perspective of Greek mathematics, and trying to reason as a Greek geometer, we can, in a satisfactory way, show that, with the exception of axioms V of continuity, the remaining axioms are valid for Greek geometry. Indeed, even axiom V, 1, "Archimedes' axiom", is true for Greek geometry, being introduced in Book V as proposition 1, which really depends on the definition of comparable magnitudes given in Book V. As argued before, the method of exhaustion is the real substitute for the axioms of continuity whenever they are needed in Greek geometry. All this will be studied in detail in a forthcoming paper.

Geometry and Logic

1. Traditionally when one talks about mathematics people immediately get the idea of something precise, where one can "prove what one says", under an appropriate formalism. In particular, geometry has always been a model of logical reasoning where we can establish facts by deduction from initial premisses in such a way that its validity is beyond doubt.

The investigations about the act of thinking in general, with its principles and laws, has in Aristotle one of the most celebrated pioneers. In the West, logic developed under the influence of Aristotle, and of scholastic philosophy, but what we call mathematical logic today is quite remote from the ideas of the Stagirite, just as we cannot say that contemporary geometry is closely related to Greek geometry. Of course, analogously to what happens in mathematics, an act of thinking has, from the *organic point of view*, the same background for any man, in any historical culture. However, the *form* in which one expresses one's ideas about logic is different in different historical cultures and this has an influence on the foundations of geometry, as we intend to show in this last part.

Mathematical logic, or inorganic logic in our terminology, became established in the West at the beginning of this century as a result of the works of Frege, Peano, Russell, founders of the logistic school, Hilbert, founder of the formalistic school and Brouwer, founder of the intuitionistic school. Whatever might be their differences, they are all concerned with the *organogram* of mathematics, in particular with its *syntax*. Mathematics and in particular geometry as an *organism* with historical and cultural dimensions cannot be studied with mathematical logic alone, but requires also the help of organic logic. Since the organogram of Greek geometry is different from the organogram of Western geometry, the "logical structure" of the former is essentially different from that of the latter and consequently we cannot study critically Greek geometry by using the "logical structure" of Western geometry and vice-versa.

With those remarks in mind let us analyze the logical structures of geometry in Greece and in the West.

2. We have seen that the organogram of Greek geometry is formed by words of the Greek language and by geometrical figures taken as initial data existing in space with their own individuality. The syntax of this organogram, i.e. its rules of inference, is formed by Aristotelian logic as understood in Euclid's time and by geometrical constructions, as we shall exemplify in a moment. For the Western mathematician this looks like a very strange and incomprehensible attitude because he is used to the organogram of Western geometry which, as a chapter of set theory, is formed by a certain language, say English, plus a collection of symbols and rules of inference given by mathematical logic of some particular school. Definitely, geometrical figures as such are not part of the business. They are not needed at all as independent entities lying in space. Rather, they are taken as *names*, like point, straight line, plane, of particular objects attached to some abstract set and subjected to certain axioms.

Let us consider an example. It is very common to refer to Proposition 1 in Book I of Euclid's *Elements* as defective because it fails to use the postulate of continuity. This proposition says that given a segment AB there exists an equilateral triangle with AB as one of its sides. The proof depends on the fact that two circles with same radius AB and centers respectively at A and at B intersect each other at a certain point C. The reaction of a Western mathematician is the following: to prove the existence of the point C we have to use the postulate of continuity. But, as seen before, that is impossible because segments, circles, etc. are understood in the Greek sense with their intrinsic spatial content. The only way to handle the situation is to translate everything in terms of Western geometry and "to solve" the question in terms of Hilbert's *Grundlagen*. This attitude would be similar to the following one: a Western architect visiting Athens concludes that the structure of the Parthenon is not strong enough and decides to rebuild the whole thing with reinforced concrete, materials of "better quality", etc. The final result would be in its appearance exactly like the original, but with a little difference: it would not be a piece of Doric architecture anymore, which is a unique product of a given historical culture. That is exactly what we do when we try do "adjust" Euclid's proof to a Western standard of "rigor". It is amazing how such crystal clear facts are completely overlooked in the usual criticism of the foundations of geometry.

Now let us look at that same proposition from the point of view of Greek geometry. To be strictly logical and precise, it is permitted to use in the proof only the syntax of that geometry. Therefore, the rules of inference include Aristotelian logic and geometrical constructions.

Hence, it is logically admissible to accept the existence of the point C as a consequence of the properties of the circle as a figure in the plane. The geometrical figures of the two circles as described *include* the point C. However, this line of reasoning is inadmissible in a statement like: "in a triangle one side is smaller than the sum of the other two". In this case we are not talking about a *particular* triangle but rather about the genus triangle and then a proof without using data from a particular figure becomes necessary. As a matter of fact, in his time Euclid had been criticized for his "excess of rigor", as someone who tried to prove what is "evident even for an ass".

To summarize, the logical characteristics of geometry in Greece and in the West are:

In Greece, geometry is an organism such that: a) its *organogram* is formed by an ordinary language, most often ionic, with a *syntax* given by Aristotelian logic plus geometrical constructions as objects existing in space; b) its *structure* is formed by the concept of geometrical figure as objects given a priori with proper individuality and *space* is considered as a real entity, finite and with no relationship with *time*. *Number* is understood as natural number, i.e., a collection of units.

In the West, geometry is an organism such that: a) its *organogram* is formed by an ordinary language, for instance, English and a *syntax* given by some logical theory, say formal logic as conceived by Russell and the symbols and axioms of set theory; b) its *structure* is formed by abstract concepts named point, straight line, plane, etc. attached to some theory of sets; space is infinite, not visual and strongly connected with time, disguised in the form of sequences of points, limit processes, etc. *Number* is the abstract concept of real number.

3. In the light of these differences, let us return now to the question of *continuity*. As we saw before, in the West the idea of continuity of the straight line is expressed through Dedekind's cuts and it is meaningful only if straight line is understood in the abstract sense of Hilbert. It is impossible to apply that concept to the Euclidean straight line and the question arises: how did the Greek geometers consider the idea of continuity of the straight line? This question has a great importance for the foundations of geometry and also for the history of mathematics mainly because it is still an open question. Indeed, what we know about Greek mathematics, through documents which survived the destructive action of both time and man, is only a small part of what was done by the Greek geometers. Hence we find here and there some facts giving a faint idea of the use of continuity in geometry. One of these uses is related to the question of the existence of the fourth proportional.

Following Eudoxus and Euclid, suppose that we are given three magnitudes a, b, c, where a and b are comparable and attempt to find a magnitude d comparable with c, called the fourth proportional of a, b, c, such that $a/b = c/d$. Clearly this is not possible for all classes of magnitudes. For instance, it is not true in general for numbers (natural numbers) like 2, 3 and 5. However, the existence of the fourth proportional is always true for classes of magnitudes which are capable of "changing in size continuously", which is actually the fundamental hypothesis in the classical proof by De Morgan as reproduced by Heath (1956) following Prop. 18, Book V.

The question is to clarify the meaning of "changing in size continuously". For the case of the straight line we find in Postulate 2 the word συνεχὲς in the neuter, which is equivalent to the adverb συνεχῶς meaning *continuously, without gap*. That is, it is assumed that a straight line can be increased in size continuously, as the result of the combination of συνεχὲς with the verb ἐκβαλεῖν which is the aorist tense mood of the infinitive ἐκβάλλω meaning "to throw" or "to cut out". Therefore, from Postulate 2, taking in consideration its formulation in Greek, we have to assume that the continuity of the straight line was a datum *a priori*, with spatial significance. Of course, a more technical approach to the idea of continuity had to be provided and this is achieved by the existence of the fourth proportional coupled with the principle of exhaustion.

According to Book VI, Prop. 12 of Euclid's *Elements*, the fourth proportional exists for the case of straight lines and by Prop. 1 of the same book the result can also be extended to polygons: if a, b are polygons and c is a straight line, then there is a straight line d which is the fourth proportional of a, b, c. Afterwards, by using the method of exhaustion it is possible to extend the result to figures A having the property: there are two polygons P and Q with Q contained in A and P containing A such that the figure, difference $P - Q$, can be made smaller than a given square B. That is, of course, quite similar to the definition of Peano-Jordan measure but only formally. Indeed, to say that $P - Q$ is smaller than B means that $P - Q$ is equivalent to a square B' which fits inside B without any numerical content. On the contrary, the Peano-Jordan measure relies heavily upon the concept of real number, an abstraction incompatible with Greek thought.

Assuming as well defined the concept of a "continuously changing" magnitude, the Greek geometers used very often the principle of the existence of the fourth proportional in the following context: suppose that we have to prove that, for magnitudes a, b, c, d it is true that

$$\frac{a}{b} = \frac{c}{d} \,. \qquad (1)$$

Now, if (1) is false, we have either $>$ or $<$. If it is $>$ then there is a magnitude $b' > b$ such that

$$\frac{a}{b'} = \frac{c}{d} \,. \qquad (2)$$

Then we try to find a contradiction as a consequence of (2) and similarly for $<$. It is clear that this type of reasoning, namely the existence of b', depends on the "changing with continuity" of the magnitude b and this technique is used very often by the Greek geometers, as for instance in Book XII, Prop. 2 of the *Elements*, without any comments. This leads us to think that it must have been a well known result in Greece, just as in the West, whenever we use the idea of continuity in one way or another, we do not bother to recall its definition. Despite the fact that so far we have been unable to find any work by a Greek geometer dealing specifically with that issue, we are convinced that they did it, in some way or another.

4. Let us inquire now about non-Euclidean geometries. It hard to find an area of mathematics where the misinterpretation of facts has gone so far and so deep. For about 23 centuries the "problem of parallels" got the attention of mathematicians both in Greece and in the West. The great comedy (or tragedy) has been: *it was never really a problem and its "solution", found in the 19th century, has never been a solution!* We devote this section to the clarification of this statement.

The word *parallel* is itself Greek, of course, and it means "side by side". For instance, in architecture we say that two beams run parallel to each other, i.e., side by side. This concept is deeply connected with the principle of analogy of organic logic. When we draw parallels on a sheet of paper we start with one line and we imitate that line by analogy, drawing other lines parallel to it. Without this organic notion of parallels architecture cannot exist. That is why from the very beginning the notion of parallels was associated with three other notions:

(1) two straight lines in the plane without a common point;

(2) two straight lines in the plane with the same direction;

(3) two straight lines in the plane with constant distance from each other.

Due to the fact that an excellent historical-critical study of the relationship among the three notions above with the concept of parallels is given by Heath (1956) when discussing def. 23 of Book I of the *Elements*, we only focus here on those aspects of the question which are relevant to our point of view in this paper, strongly recommending the reader to examine Heath's work.

Euclid takes (1) as the definition of parallels in the more precise statement of def. 23, Book I, where he ues the idea of "producing a straight line indefinitely". The word ἀπειρον, which means generally "without limits" and was already used by the philosopher Anaximander in relation to the Universe, is connected in geometry with the idea of some magnitude which can be increased *as much as needed* but never *infinitely* in the sense of *actuality*, but rather only in the sense of *potentiality*. For a Greek geometer it is meaningless to say that the straight line is infinite. From the very beginning they realized that to avoid trouble and paradoxes it was better to get rid of the idea of *actual infinity*. That is why this notion was banned from geometry and, as a matter of fact, from the whole Universe as felt by the Greek soul.

The idea of using (2) or (3) as a definition of parallels was rejected, among others, by Aristotle as leading very easily to a *petitio principii* and consequently it was discarded by Euclid (cf. Heath 1956, vol. II, p. 190). However, from the organic or intuitive point of view (2) and (3) are very appealing and indeed are the "everyday" definitions of parallels. Suppose, for instance, that a company decides to build a railroad in South America from South to North in a "straight line". One of the rails could follow exactly a meridian, the other running "parallel" to it. Of course, parallel here means "keeping the same distance from each other along the way". The notion of distance here is very clear for the engineers and workers: it is given by the *modulus* which is an iron bar used to fix the rails at equal distance before nailing them to the beams. Therefore, irrespective of the curvature of the earth, for the railroad builders the geometry to be used is the Euclidean and not the Riemannian, i.e., if the rail is taken as a straight line the other rail is also a straight line parallel to it.

Another example: two airplanes are to fly from South to North keeping the "same direction". Clearly this means that they are supposed to fly "parallel to each other", i.e., they should *not* follow two meridians; otherwise they would collide at the north pole. To safely circumnavigate the earth they have to keep parallel to each other in the sense of never deviating themselves from a "fixed direction". Here we realize that the idea of direction is an organic concept and cannot be reduced to the formalism of inorganic logic. This does not mean of course that with suitable conventions we could not formalize this concept in an abstract formalism. Indeed, with the help of algebraic topology, differential geometry, etc., the concept of orientation of a manifold, for example, can be introduced in abstract terms, but never in terms of Greek geometry. The organic intuition of space is present in our "visualization of the orientation", like the *rule of three fingers* in the "geometrical" definition of the cross product of two vectors as used in the notion of momentum of a force, etc.

Considering all those facts, Euclid had to introduce the notion of parallel in geometry by preserving in the background the concept of equal distance, same direction and of not having a point in common, and also, as a fundamental assumption, to preserve the definition of straight line. All that is achieved by a stroke of genius, with the introduction of postulate 5. As argued before, this postulate is only meaningful when understood as a technical procedure for connecting the definition of straight line as something which "proceeds evenly without deviating from its direction" with the notion of parallel. Therefore, any attempt to "prove" this postulate has to use some alternative definition of straight line and parallel. But this is the same thing as attempting to substitute that postulate by something equivalent to it, while preserving the concepts of straight line and parallels as entities and properties existing in space and given *a priori* with their own individuality and not as abstract entities belonging to some set theory.

Consequently this postulate is *clearly* not provable by using *only* the other four postulates, because we cannot prove something about some object without using its essential properties known *a priori*. Now, "to proceed evenly without deviating from its direction" is an essential property of the straight line and indeed if we analyze all "proofs" of postulate 5 they invariably use along the way some assumption translating in one way or another that essential property of the straight line which is not present in the first four postulates!

So, regarding postulate 5 we either provide a proof, by using some equivalent hypothesis to include the concept of straight line and parallel and in this case the proof itself is correct, as the classical case of Ptolemy's proof; or we do *not* use any equivalent assumption and in this case the postulate is not a consequence of the other four postulates by the reasons explained above. That is what we meant before by the statement that there was no "problem" of the independence of postulate 5 with respect to the remaining four. Let us look now at the "solutions" proposed.

In the 19th century, geometry has to be understood in the Western sense and consequently the words point, straight line and plane do not have any spatial content: they refer to abstract concepts. A straight line, for instance can be "curved", like meridians on the surface of a sphere. For the Greek geometers a spherical triangle was *not* a triangle in the Euclidean sense but just a figure drawn on the sphere whose sides were arc of meridians. As a matter of fact, there existed a whole branch of geometry called *Sphaerica*, dealing with figures drawn on a sphere, of great importance to astronomy, connected with the names of Menelaus, Ptolemy and others. But they never considered that as a "new geometry".

When Gauss, Bolyai and Lobachewsky started their celebrated research in the theory of parallels, set theory did not exist and the concepts of point, straight line and plane were not yet clearly conceived as abstract entities in the sense of Hilbert; on the contrary, the Euclidean tradition was still very strongly rooted in their minds, even though the Greek concept of space had already been forgotten through the action of time. The real problem of Gauss, for instance, when he got disturbed by imagining a "non-Euclidean geometry", was his fear of contradiction with the usual (Euclidean?) concepts of points, straight and plane and the reaction of the public, which led that great genius to hide his discovery for the time being.

Bolyai and Lobachewsky were less concerned with *plebis opinio* and decided to prove theorems until the eventual finding of a contradiction resulting from the substitution of the fifth postulate by something else, as for example: through a point P outside a straight line r in the plane we have two families of straight lines formed by those lines which intersect r from one side and by those which do not intersect r from the other side; the boundary lines of both families were named parallels to r. Of course, no contradiction was found with "Euclidean geometry" because the concept of straight line used by them was *not* the Euclidean one in the Greek sense, namely "proceeding evenly ...". If they had decided to use the Greek concept of straight line, they would have to introduce this concept through some hypothesis equivalent to the fifth postulate, and in this case they would have, for sure, reached a contradiction.

After Beltrami, Poincaré and others, a series of models of non-Euclidean geometries have been built, but in all of them the concept of straight line is not the Euclidean one and consequently they do *not* solve the "problem" of the fifth postulate. All they do is to present *sets with certain structures* which obey statements similar to the postulates 1-4 of Euclid's *Elements* but not the fifth. To make this point clearer let us analyze briefly one of those models. Let us call "plane" the interior of a disc D, and straight lines, all segments inside D with extremities in its boundary; "points" will be the "usual ones". The notion of distance of two points P, Q is defined by the log of the absolute value of the double ratio of P, Q and the points A, B defined by the intersections of the line r through P, Q with the boundary of D. (For details, see Hilbert 1930 or Klein 1928). Now taking a "straight line" r in this model and a point P outside r we have two "straight lines" intersecting r in the boundary of D (at infinity) at points A and B and hence they are "parallel to r". But, of course, this model does *not* show that postulate 5 is independent of the other because the concept of straight line used there is not the same as Euclid's. All it proves is: if a set E with *abstract concepts* named point, straight line and plane satisfies *formally* postulates 1-4 to the *Elements*, then there is another model E' which satisfies 1-4 but not 5. Therefore postulate 5 is independent of postulates 1-4 of model E. "Elementary, my dear Watson!". And this, of course, has nothing to do with geometry in the Greek sense. As a matter of fact, a "small being" living in our previous model in the disc D perhaps would have a feeling that the segments AP and PB together would form a "straight line" and therefore in his intuitive perception of facts in his universe maybe the fifth postulate would be true after all.

Finally we come to the question: in the real world of our perceptions is the Euclidean geometry *experimentally* verified or not? Here we are dealing with organic logic and consequently everything depends on our intuitive feeling of point, straight line, plane and space. As seen before, for the architect and the engineer the geometry to be used is definitely the Euclidean one. Indeed, we doubt if any one of us would ever buy an apartment in a building knowing that the architect used "non-Euclidean geometry" in its design. What tragic consequences would result if an engineer were to build a railroad assuming as "straight lines" the meridians on Earth! To be sure, for the physicist who assumes that a straight line is the distance between two points P and Q as defined by a beam of light from P to Q, the four dimensional Riemannian manifold is the natural model to use. Of course, that is only an *abstract object* that, although confirmed by experience, has no claim *to be* the real world. The latter is organic and its existences is felt by our intuition through our senses. Here more than ever the distinction between the organic and the inorganic is fundamental (see Lintz 1989 for details).

As is well-know, Kant raised the question: is the concept of space synthetic *a priori* or analytic? (Kant, 1926). The crucial point here is to clarify the meaning of the word *space*. In his criticism of Kant, Gauss makes it clear that he rejected the assumption that Euclidean geometry is the geometry of the real world. But the disagreement between the two thinkers was simply that they were talking about different concepts. Gauss intended to show that other *geometrical models* for the world could be proposed besides the Euclidean one. We see clearly that he was talking about *inorganic space* as a model or normal representation of our intuition of the space of real world. Kant, on the other hand, was talking about *organic space*. One might go as far as suggesting that the Kantian distinction between "synthetic *a priori*" and "analytic" corresponds to the distinction between inorganic and organic logic. Consequently, when Kant says that space is a *synthetic a priori* concept, he is absolutely correct because he is talking, perhaps without a clear idea of it, about *organic space*. If our assumption is correct, then Kant's ideas about space have been systematically misundestood until our days. It is time to redeem this unfortunate mistake.

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