**BOOK REVIEW**

Pasquale Frascolla, *Wittgenstein's Philosophy of Mathematics.* (London & New York, Routledge, 1994), pp. viii + 189.
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Almost half Wittgenstein's surviving writings deal with the philosophy of mathematics, and this speaks for itself as to the importance he attached to this area of philosophy.[1] Wittgenstein also repeatedly stresses the important connections between his ideas in this field and other areas of philosophy (Cf. Wittgenstein (1953), p. 232, (1976) pp. 31 & 111, (1978), p. 162, and Rhees (1984), p. 340). Yet commentators have not exactly shared Wittgenstein's own estimate of the value of his work in this area. In contrast with the ever-increasing number of books on every other aspect of his philosophy, Wittgenstein's philosophy of mathematics has been the subject of only a handful of books of any substance, notably those by Crispin Wright (Wright (1980)) and Stuart Shanker (Shanker (1987)).

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[1] Indeed, he is actually reported to have said that he regarded his work in this area as his most important. Cf. Rush Rhees's report cited in Monk (1990), p. 466.

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There are several reasons for this neglect. Partly, no doubt, it is because philosophy of mathematics is regarded as something of a specialist's subject, requiring specialised knowledge, as does philosophy of physics, or philosophy of biology. And up to a point this is obviously correct. Yet Wittgenstein's discussions of mathematics do not require very much in the way of mathematical or logical sophistication.[2] A much more influential reason for the lack of interest in Wittgenstein's ideas in this area is the extreme and implausible nature of many of the views he puts forward, e.g., his radical conventionalism about necessity, and his views on consistency proofs, both ideas which can seem, as Crispin Wright aptly put it, not so much like challenges to established views, as "good sense outraged" (Wright (1980), p. 295), and have been treated accordingly.

This has without doubt also been a major factor contributing to the fact that, with the exception of the recent upsurge of interest in Wittgenstein's discussions of rule-following, in the wake of Kripke's monograph (Kripke (1982)), and other equally important, earlier contributions by Fogelin (Fogelin (1976)) and Wright (Wright (1980), Wittgenstein has been almost totally ignored by mainstream philosophers of mathematics. It is very striking that in the "Opinionated Introduction" to their recent anthology, in which they survey the principal approaches to the philosophy of mathematics in the twentieth century, Aspray and Kitcher do not so much as mention Wittgenstein, despite the fact that they draw attention to

[2] As he himself is reported to have remarked, "I know nothing about [the foundations of mathematics] ...I practically know only the first volume of *Principia Mathematica*" (Wittgenstein (1976) p. 14), and McGuinness tells us that "over the years Wittgenstein acquired something like a second-year undergraduate's knowledge of Pure Mathematics" (McGuinness (1988), p. 62).

the existence of a "maverick tradition" headed by Imré Lakatos, which has many important features in common with the ideas of the later Wittgenstein. (Aspray and Kitcher (1988), pp. 17ff)[3]

Yet, far from being a reason for neglecting this part of Wittgenstein's work, the extreme and apparently implausible nature of his ideas in this area provides a strong reason for trying to fathom what led Wittgenstein to put forward such ideas, especially since, as he himself emphasizes, they may very well be connected in unexpected ways with his other apparently more palatable views. Here it is worth recalling the wise words of Thomas Kuhn who proposed the following maxim for students of the history of philosophy: "When reading the work of an important thinker, look first for the apparent absurdities in the text and ask yourself how a sensible person could have written them. When you find an answer ... when those passages make sense, then you may find that more central passages, ones that you previously thought you understood, have changed their meanings." (Kuhn (1977), p. xii). For this reason, even if, in the end, Wittgenstein's vision of mathematics does turn out to be indefensible, we would still do well to make

[3] Notably their common hostility to formalization in mathematics, and their resulting emphasis on the diversity of the concept of proof. Needless to say, there are also fundamental differences, such as Wittgenstein's lack of interest in the history of mathematics, although there seems nothing incompatible in his general approach with such an interest, and, above all, Wittgenstein's complete rejection of the idea that mathematics is *descriptive* of anything, an idea which Lakatos never questions. (Cf. Sluga (1982), p. 122 for some brief but illuminating remarks on the points of contact between Lakatos and Wittgenstein, and the possibility of a fruitful synthesis of their approaches.)

sure we understand exactly what he was saying and why he was driven to say it.

Pasquale Frascolla's excellent new book greatly advances this task. It should be said straightaway that this is an exceptionally important contribution to Wittgenstein studies, and will remain basic for future work on Wittgenstein's philosophy of mathematics for a considerable time to come. Frascolla describes his aim as that of providing a more systematic and less "rhapsodic" exposition of the central ideas of his philosophy of mathematics than Wittgenstein himself gave (p. vi), and in this he succeeds admirably. He discusses all Wittgenstein's principal ideas and their interrelationships with exemplary clarity, and documents his interpretations meticulously with references to Wittgenstein's texts. Unlike earlier commentators, such as Wright, he does not subject Wittgenstein's ideas to detailed critical scrutiny, although he does point out internal tensions in Wittgenstein's thought. Nor does he discuss their implications for contemporary debates in the philosophy of language and mathematics, as Wright and Shanker do extensively. As a result he is able to present a much more detailed and comprehensive picture of Wittgenstein's philosophy of mathematics than either Wright or Shanker.

An especially important feature of Frascolla's book is that it gives the first detailed account of the *development* of Wittgenstein's philosophy of mathematics. Although he does cite material from the middle period, and, very occasionally, the *Tractatus*, Wright's focus is firmly on Wittgenstein's later philosophy of mathematics, and he offers no account of how these ideas emerged from the earlier stages of Wittgenstein's thought. Shanker does address the issue, but Frascolla's discussion is much more detailed. Frascolla divides Wittgenstein's work on mathematics into three phases: the *Tractatus*, the intermediate phase (1929-33), and the later phase (1934-44), and devotes a chapter to each. I'll comment on some of the most important aspects of each in turn, and then conclude by briefly discussing a more general issue raised by Frascolla's book.

Frascolla's discussion of the *Tractatus* philosophy of mathematics is perhaps the most important part of his book. Its discussion of mathematics is among the most obscure and least understood parts of the *Tractatus*, and those commentators who have not dismissed it completely as too sketchy to make sense of at all have invariably claimed that it is vitiated by elementary lapses in rigour. Frascolla's brilliant analysis establishes once and for all that such negative verdicts are completely unjustified, and lifts discussion of this part of the *Tractatus* onto a new level of sophistication and precision, providing an object lesson in how careful attention to the smallest detail of Wittgenstein's text can unlock the secrets of even the most cryptic and obscure passages of the *Tractatus*, and thus filling a major gap in the literature.

Among the many illuminating aspects of Frascolla's discussion of the *Tractatus* his meticulous reconstruction of Wittgenstein's account of arithmetic by working out in detail the implications of the definition of number as the exponent of an operation at 6.021 is of particular interest. According to Frascolla, Wittgenstein's view should be understood as implying that 'arithmetic deals ...with formal properties of linguistic expressions generated by processes of iteration and composition of logical operations' (p.39), and this amounts to a 'reduction of arithmetic to the general theory of operations' (p.36). One important fact about Wittgenstein's account of number which Frascolla's discussion makes clear is that it anticipates, by over a decade, the central idea of the treatment of numbers in

Church's λ-calculus. (p.176 n26)[4]. An especially interesting aspect of Frascolla's discussion of the *Tractatus* is his claim that the 'reduction of the arithmetical primitives to the notion of application of a logical operation' (p.11) implied by Wittgenstein's definition of number is tantamount to endorsing a version of logicism (pp. 25, 26 ,37, 38, 43, 131, 152-3). This challenging claim deserves a much fuller discussion than it can be given here, and I will limit myself to suggesting two reasons for a certain scepticism about it. Firstly, there is the evidence of the extra propositions which Wittgenstein added to Ramsey's copy of the *Tractatus* in 1923 (Cf. Lewy (1967)), which strongly suggest that Wittgenstein himself did not regard his definition of number as having implications of this kind. There Wittgenstein tells us that 'the fundamental idea of math[ematics] is the idea of *calculus* represented here by the idea of *operation*', and that 'Number is *the* fundamental idea of calculus and must be introduced as such'. Putting these two statements together we get the conclusion that number is a fundamental idea of mathematics, which suggests that Wittgenstein did not see his *Tractatus* definition as amounting to a *reduction* of the concept of number to more fundamental, logical, concepts. This conclusion appears to be confirmed by Wittgenstein's further statement to Ramsey that 'The beginning of logic presupposes *calculation* and so number'. For, if logic, in whatever sense precisely, *presupposes* number then this seems to be just the reverse

[4]This has also been pointed out by Odifreddi (Cf. Odifreddi (1989), p.84). I owe this reference to Rodolfo Ertola. Frascolla has since substantiated this claim in rigorous detail in Frascolla (forthcoming), a version of which was read at the XIth Brazilian Logic Conference, held in Salvador, Bahia in May 1996. Cf. also Frascolla (1997) for further discussion of this point.

of logicism in any shape or form, whose central claim is that apparently purely mathematical concepts, above all that of number, can be analysed in purely logical terms.[5]

Secondly, if Wittgenstein were indeed proposing a version of logicism in the *Tractatus* then we would expect him to deny that there is any essential difference between the propositions of logic and those of mathematics. Yet we find him doing just the opposite. Logical and mathematical propositions are indeed both classified as 'pseudo-propositions', but it is clear that Wittgenstein regards them as importantly different kinds of pseudo-proposition. Logic consists of tautologies, which are truth-functions of elementary propositions, and although they are senseless [*sinnlos*] they are not nonsensical [*unsinnig*], whereas the equations of mathematics, and, in particular, arithmetic, are not truth-functions and are nonsensical. This being the case, the possibility of any kind of reduction of mathematics to logic, as envisaged by logicism, seems to be ruled out. In fact, Frascolla himself is well aware of this point. One of the passages in the *Tractatus* discussion of mathematics which has baffled many commentators is his claim at 6.22 that the logic of the world which is shown by logic in tautologies is shown in equations by mathematics. Max Black, for example, comments 'It is hard to see how what is shown in equations can be assimilated in this way to what is shown in tautologies' (Black (1964), p.341). Yet, as Frascolla makes clear (pp. 20-2), the mystery disappears once we realise that Wittgenstein is not claiming that logic and mathematics show the very same thing,

[5]Frascolla does not discuss these additional "Ramsey" propositions in his book, however he has done so in Frascolla (1997), in which he replies to Wrigley (1997), and argues that, far from counting against his logicist reading, they support it.

but rather distinct dimensions of the logic of the world. However, it is not clear why Frascolla does not see this as an objection to his logicist reading.

One topic which Frascolla does not discuss is what account of real numbers is implicit in the *Tractatus*. Reasonably enough, since Wittgenstein says absolutely nothing about this. However attempts have been made to outline a theory of real numbers that is compatible with the general approach of the *Tractatus* (Cf. Kaufmann (1930), p. 117, Weinberg (1936), p. 99 and Cuter (1995), pp. 129-137). This question is of particular interest because of its bearing on the question of the degree of continuity between Wittgenstein's philosophy of mathematics before and after 1929. (Cf. Wrigley (1993) for discussion of some other aspects of this issue.) In his middle period writings Wittgenstein discusses the nature of real numbers extensively. The question naturally arises of the source of these views, in particular, whether they owe anything to the *Tractatus*. Apart from the fundamental difference that after 1929 Wittgenstein rejected the concept of the completed infinite, whereas in the *Tractatus* he had accepted it, the central role given to the concept of rule in Wittgenstein's post-1929 ideas about real numbers seems very much in accord with the main ideas of the *Tractatus* and, hence, suggests that that may well be where their origin is to be found. It would be of great interest to know Frascolla's views on this issue.

Turning to his discussion of Wittgenstein's philosophy of mathematics after 1929, Frascolla's central claim about the evolution of Wittgenstein's ideas is that his application of a verificationist theory of meaning to mathematics, which is the distinctive idea of his middle period philosophy of mathematics, is gradually undermined by the development of his views on rule-following and the consequent rejection of the idea of unacknowledged necessary connections, a thesis which is fundamental to his later views on mathematics.

In his middle period, according to Frascolla, Wittgenstein's verificationist theory of meaning for mathematics takes the form of requiring, at least for a substantial class of mathematical propositions, that understanding a mathematical proposition consists in knowing a decision procedure for it (pp. 58-59). Frascolla argues that many of Wittgenstein's specific views result from this underlying principle, and that it is this mathematical version of verificationism which allows Wittgenstein to admit that 'we are not all-seeing in grammar' and 'unacknowledged internal connections are accepted, though only in the ... sense in which lack of knowledge can be made good, in principle, by the application of a general method of calculation' (p. 70). However, as Frascolla also makes clear, even in the middle period, Wittgenstein recognizes that such a verificationist approach cannot be applied universally (pp. 63-64, 67-68), and that there is also a domain of mathematics for which 'the principle *esse est percipi* holds true ... [where] we are all-seeing simply because there are no unacknowledged necessary connections' (p. 70, cf. p. 101). Frascolla shows how Wittgenstein's specific views during the middle period can be explained as the results of this basic perspective. For example, his notorious *laissez faire* attitude towards the idea of hidden contradictions stems directly from his insistence that within the region of mathematics for which verificationism applies if a contradiction exists there must be a method of finding it (p. 100), and that once we have stepped beyond the limits of Wittgenstein's mathematical verificationism the very idea of an unacknowledged hidden contradiction is nonsensical (p. 101). Similarly, Frascolla explains how Wittgenstein's views on induction (pp. 79ff), and on generality (pp. 72ff) all result from this same basic perspective.

With the later phase of Wittgenstein's philosophy of mathematics the rejection of unacknowledged internal connections becomes completely general, as a result of Wittgenstein's 'purely linguistic [theory of the] nature of necessity' (p. 111), a theory which Frascolla argues is the *result* of his analysis of rule-following being taken to its logical conclusion. (Cf. Wright (1986a) and (1986b) for a different perspective on the relation between Wittgenstein's discussions of rules and his later view of necessity.). In his discussion of the later phase of Wittgenstein's philosophy of mathematics, Frascolla is on relatively familiar ground, exploring the connections between his rejection of the idea of unacknowledged necessary connections and his views on such topics as proof (pp. 128ff), and finitism (p. 144). One novelty is that Frascolla defends a version of the "communitarian" view of rule-following (p. 120), challenging what has become virtually the received interpretation of Wittgenstein's views on this question[6].

In conclusion, I would like to mention, very briefly, a more general issue about Wittgenstein's philosophy raised by his views on mathematics, and which Frascolla's book throws light on. This is the question of whether Wittgenstein's well-known, or perhaps one should say notorious, views on the nature of philosophy are wholly consistent with his actual practice when investigating specific first-order philosophical problems. Two aspects are particularly problematic. Firstly, the question of Wittgenstein's attitude towards the possibility of philosophical criticism of an established practice. He famously

[6] Cf. Baker & Hacker (1985), pp.169ff, for powerful and influential arguments in favour of the anti-communitarian interpretation.

declared that philosophy leaves everything as it is, and made a point of adding that it also leaves mathematics as it is ((1953), §124). However, a number of the views which Wittgenstein puts forward about mathematics certainly appear to require the rejection of substantial parts of classical mathematics, in particular, his views on set theory and real numbers. Frascolla's discussion of Wittgenstein's views on real numbers makes it clear that he really is committed to extensive revisions of classical mathematics, and that these views are the result of fundamental aspects of his philosophy of mathematics. (Cf. pp. 85-86). Similarly, in the case of set theory, Frascolla shows that Wittgenstein unambiguously rejects the concept of the completed infinite (pp. 93, 161-162), a view with equally obvious revisionary consequences, and that this rejection also springs from central aspects of Wittgenstein's philosophy of mathematics. Frascolla thus presents a challenge to those who claim that there is no tension between Wittgenstein's metaphilosophical principles and his actual practice to show precisely how these aspects of his philosophy of mathematics, both absolutely central and in no way marginal views, can be reconciled with Wittgenstein's claim to leave everything, mathematics included, as it is.

The second aspect of Wittgenstein's general conception of philosophy which seems difficult to reconcile with much in his philosophy of mathematics is his claim that the philosopher should not and cannot put forward substantial, much less controversial, theses (Cf. (1953), §128). Frascolla poses this problem in a very sharp way by demonstrating that it is in fact very easy to formulate many quite definite, substantial, not to mention controversial, theses which Wittgenstein gives every appearance of subscribing to about the nature of mathematics. To cite only a single, especially clear, example it is hard to see how Wittgenstein's distinctive constructivist view of the nature of real numbers can, by any stretch of the imagination, be described as "something everyone would agree to", or as not being a definite and substantial philosophical thesis. To anticipate the possible objection that this is an unfair example because Wittgenstein only held this view in his middle period when he had not yet arrived at his later view of the nature of philosophy, and that when he had he no longer maintained this view, it suffices to point out, first, that he was indeed putting forward this, and other, equally problematic aspects of his "later" view of philosophy at precisely the time he was working most intensely on the implications of this view of real numbers (Cf. Hilmy (1987), p. 35), and, second, that even if later it was no longer the focus of such concentrated attention, Wittgenstein never abandoned this view of real numbers (Cf. (1978), pp. 290-291).

His ideas on mathematics have long been the most neglected part of Wittgenstein's philosophy. Frascolla's excellent book should go a long way towards changing this situation. It is clearly and concisely written, and does not get bogged down in discussing the interpretations of other commentators. In the interests of readability it might have been a good idea to have broken up the text into shorter paragraphs. As it is, there are frequently entire pages without a single break, and, in some cases three of four consecutive pages (e.g. pp. 75-77, 128-130, 151-154). The cover is not exactly a masterpiece of graphic art, and if it really is obligatory to put a photo of Wittgenstein on the cover of every book about his philosophy one that has been used less often might have been chosen. I noticed very few misprints. (On p. 175 the reference in note 18 to note 16 should be to note 14). The index could have been more complete. Frascolla's pioneering book is essential reading for any-one with a serious interest in Wittgenstein, and ought to be issued in a cheaper paperback edition to make it available to the widest possible audience.[7]

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