ON A NEW APPROACH TO PEIRCE´S THREE-VALUE PROPOSITIONAL LOGIC[1]

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Abstract: In 1909, Peirce recorded in a few pages of his logic notebook some experiments with matrices for three-valued propositional logic. These notes are today recognized as one of the first attempts to create non-classical formal systems. However, besides the articles published by Turquette in the 1970s and 1980s, very little progress has been made toward a comprehensive understanding of the formal aspects of Peirce's triadic logic (as he called it). This paper aims to propose a new approach to Peirce's matrices for three-valued propositional logic. We suggested that his logical matrices

[1] This paper presents partial results of Postdoctoral research in Philosophy at the University of São Paulo (FFLCH-USP), named "Studies in Charles S. Peirce's three-valued logic" (2021-2022).

give rise to three different systems, one of them – which we called $\mathbf{P_3}$ – is an original and non-explosive logic. Besides that, we will show that the $\mathbf{P_3}$ system can easily be transformed into paraconsistent and paracomplete calculi, adding to it, respectively, unary operators of consistency and intuitionistic negation. We conclude with a discussion about philosophical motivations.

1. Introduction

Max Fisch and Atwell Turquette (1966) published, over fifty years ago, three pages of Charles S. Peirce's unpublished manuscripts containing what would be a fully developed many-valued system. The pages from Peirce's logic notebook, written at beginning of 1909, include descriptions of matrices with three truth-values, which Peirce called *verum* (V), *falsum* (F) and *the limit* (L). He called it "triadic logic".

This discovery placed Peirce not only as a pioneer in the use of matrix method applied to bivalent propositional logic (Anellis, 2004, 2012), but also as the creator of non-classical systems, a decade before the seminal works on many-valued logic released by J. Łukasiewicz (1920) and E. Post (1921)[2].

In the following years, Turquette published several papers focus on Peirce's notes (Turquette, 1967, 1969, 1972, 1976, 1978, 1981). This outstanding analysis was based on the same Hilbert-style axiomatic that he developed with Rosser years back and became standard in many-valued logics literature (Rosser and Turquette, 1952).

Nevertheless, besides Turquette's articles, we can barely find any considerable progress in the analysis of Peirce's

[2] There is, however, some strong indications that Peirce was working on a tree-valued logic at least since 1903 (Eisele, 1976, p. xvii).

three-valued formal system.[3] This is very unusual, considering that Peirce is recognized for his original works on non-classical and many-valued logics (Rescher, 1968, p. 55; Bolc and Borowik, 2013, p. 23). There is, however, a well-established debate over what philosophical concerns Peirce had in mind when he developed his triadic logic. Fisch and Turquette (1966), Lane (1999), and Odland (2020) have different viewpoints on that issue.

Our focus here will be mainly on the formal aspects of the subject. This article aims to provide a new approach to Peirce's three-valued propositional logic. The basic idea is that Peirce's matrices comprise not only one, but three distinct systems. Two of them were later discovered by others logicians, such as Łukasiewicz, Kleene, and Bochvar. The third system, hereafter called $\mathbf{P_3}$, is a unique three-valued calculus. Its main characteristic is that the complex formula only takes an indeterminate value (neither true nor false) in the case both atomic formulas takes indeterminate value as well.

This article is organized into four further sections (five counting the introduction section). The second is dedicated to explaining our interpretation of Peirce's manuscript pages, and the third, to explore some semantical and structural aspects of $\mathbf{P_3}$. In section four, we show that the original system $\mathbf{P_3}$ can be converted into a paraconsistent and paracomplete logics by adding new unary operators. This will allow us to discuss some of Peirce's philosophical motivations in section five.

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[3] At the time we finished this paper, A. Belikov (2021) published an article in which he presented a similar perspective, working on a fragment of Peirce's triadic logic and connecting it to non-classical logics.

2. Peirce's three-valued propositional logic

Two pages of Peirce's notebook, seq. 638 and seq. 640, respectively introduces sets of four unary operators and six binary operators[4]. These logical truth-functional operators are presented in the form of modern truth-tables, showing that Peirce was entirely aware of the technique at that time.

The main characteristic of Peirce's triadic logic is that we can find four types of negations, three types of conjunctions and three types of disjunctions, totalizing 10 different kinds of operators. There is no record of experiments with material implication, despite the importance that he gave to it in his propositional logic[5].

The challenge of Peirce's notes is understand his motives to introduce a third truth-value in propositional logic and why he selected such logical connectives. As Turquette suggested in three papers (1967, 1969, and 1972), Peirce was concerned about algebraic relations of duality and trimorphism between the connectives to compose a functionally complete and symmetric logical system.

However, the three-valued axiomatic systems proposed by Turquette it is just a possible interpretation for the basic idea that Peirce's triadic matrices are generalizations of conjunctions and disjunctions of the classical propositional logic. Besides that, Turquette's approach has some drawbacks.

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[4] "The Logic Notebook" is available online at Houghton Library. See Peirce (2021).

[5] In "On the algebra of logic" (1885), Peirce developed a complete propositional logic system that has the operators of negation and implication as primitive ones (Peirce, 1885). For a detailed exam on this point, see Rodrigues (2017).

1. First, he needs to add to the system two partial negations (partial in the sense that such negations do not transform all the truth-values). The reason is to make the system functionally complete.

2. Second, his approach is flawed about the use of two operators, Φ and Ψ, that he considers "mysterious" because the functions of the operators are unclear to him (Fish and Turquette, 1966).[6]

3. From a philosophical point of view, as shown by Rodrigues (2017), Peirce was more concerned with how to deal with logic without reducing it to a mere calculus, as Boole's algebraic tradition did it. Peirce's main point instead is how to symbolize logical inference without erasing logic's proper identity as the study of reasoning.

4. Finally, the calculus proposed by Turquette is too complex and hard to follow. So, our problem is how we find a more intuitive calculus[7] for Peirce's triadic logic, avoiding such difficulties and at the same time filling in the gaps of the system.

Parks' article (1971) gives us a hint that Peirce's triadic logic could be viewed as three different finitely three-valued

[6] According to him, these binary operators appear to be "a slight variation" of theta (Θ) and zeta (Z), and, for this reason, pointless to the system (Fisch and Turquette, 1966, p. 76). Parks (1971) demonstrated that these operators correspond to disjunctions and conjunctions in the system of Sobocínski and Cooper's logic of ordinary discourse.

[7] "Intuitive" in Gentzen's sense.

propositional systems based on negation, conjunction, and disjunction operators.

The key idea is that the duals operators, displayed in pars {{Φ, Ψ}, {Θ, Z}, {ϒ, Ω}}, should be understood as three different finitely many-valued propositional calculi. These systems will differ in grades of indetermination regarding the truth-value L.

Therefore, each group of conjunction and disjunction gives rise to a different calculus, denoted by the sets of ordered pairs. We choose only Peirce's bar negation (−) for the systems presented below, but depending on the motivations any other negation could be used.

In the matrices for negations, Peirce anticipates some form of negations presented in many-valued calculus. The bar negation is now the standard Łukasiewicz's three-valued negation, and in the last column, we have the Słupecki's tertium operator-T (1936), in which all the values take ½ no matter what.

The other two negations, called "total negations" by Turquette, are often compared to cyclic rotate Post's negation for his n-valued logic (Post, 1921; cf. Fisch and Turquette, 1966; Lane, 2001; Odland, 2020). For standard notations, we will use the set of truth-values number {1, ½, 0}, instead of the letters {V, L, F}:

α	/α	\α	−α	∸α
1	½	0	0	½
½	0	1	½	½
0	1	½	1	½

Table 1: Peirce's negations.

The purpose of these unary operators in the triadic logic it is unclear. In this paper, we will use only the standard one (−α) and the first one of the total negations (/α).

Binary connectives, as already said, are presented by Peirce in pars of sets {Φ, Ψ}, {Θ, Z}, and {ϒ, Ω}[8]. Our main idea is that these operators, which correspond to disjunctions and conjunctions, can be combined with Peirce's bar negation to create three distinctive systems.

The first two calculi are well-known in the literature on many-valued logics. The matrices for {Θ, Z} are developed later by Łukasiewicz (**Ł$_3$**), Kleene (strong **Ks$_3$**), and the matrices for {ϒ, Ω} give us Kleene's weak **Kw$_3$** and Bochvar's internal **Bi$_3$**. The other system is original. Its main characteristic is that formula takes the truth-value ½ (limit) only when the components of the formula take ½ as well. We call it **P$_3$**. This logic is generated taking the set {Φ, Ψ} as logical connectives. In the following, we will use standard notation, instead of symbols (Greek letters) introduced by Peirce to show all these matrices:

	¬
1	0
½	½
0	1

∨	1	½	0
1	1	1	1
½	1	½	½
F	1	½	0

∧	1	½	0
1	1	½	0
½	½	½	0
0	0	0	0

Table 1: System 1 (**Ł$_3$ / Ks$_3$**).

	¬
1	0
½	½
0	1

∨	1	½	0
1	1	½	1
½	½	½	½
0	1	½	0

∧	1	½	0
1	1	½	0
½	½	½	½
0	0	½	0

[8] Looking at the plate (seq. 640) is unclear the exactly order we should read the binary operators. At first glance seems that the correct order is {Φ, Θ}, {Ψ, Z}, {Ω, ϒ}. However, if we think about conjunctions and disjunctions, {Φ, Ψ}, {Θ, Z}, and {ϒ, Ω} would be the right order.

Table 2: System 2 (**Kw$_3$ / Bi$_3$**).

	¬
1	0
½	½
0	1

∨	1	½	0
1	1	1	1
½	1	½	0
0	1	0	0

∧	1	½	0
1	1	1	0
½	1	½	0
0	0	0	0

Table 3: System **P$_3$**.

Now it turns out that these three systems generated by Peirce's matrices have the following general semantic proprieties:

1. (*Normality*) Whenever the components of a complex formula take classical truth-values 0 or 1, the resulting complex formula takes classical truth-values too;

2. If both components of a complex formula have value ½, so does the formula as a whole;

3. The third value ½ "spread" into the systems, which ranges from less indetermination (**P$_3$**) to the "infectious" **Kw$_3$ / Bi$_3$** system (we shading the truth-value ½ in the matrices above to make clear this point).

Because both systems (1) and (2) have been widely studied by logicians (Malinowski, 1993), we focus next on some theoretical features of **P$_3$**.

3. The Peirce's three-valued system $\mathbf{P_3}$

In this section, we presented the syntax and semantic of $\mathbf{P_3}$ in a standard way (see Burris & Sankappanavar, 1981; Wójcicki, 1988).

Let $\mathcal{L}$ be a usual language of propositional logic with the set *Var* = {p, q, r…}, the elements of which are called propositional variables or atomic formulas, and the set of n-ary connectives $c = \{\neg, \vee, \wedge, \rightarrow\}$. A finite sequence of elements of *Var* defines a set of formulas, denoted as *For*. We use lowercase Greek letters (α, β, γ, … etc.) as variables for formulas, and uppercase Greek letters (Γ, Δ, Σ…) for sets of formulas.

Definition 2.1: A propositional logic is a pair $L = <\mathcal{L}, \vdash>$, where $\mathcal{L}$ is a propositional language and $\vdash$ is a logical consequence relation between sets of formulas and formulas of $\mathcal{L}$ that satisfies the following proprieties, for all $\Gamma \cup \Delta \cup \{\varphi\} \subseteq$ *For*.

1. If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$;
2. If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$;
3. If $\Delta \vdash \varphi$ and $\Gamma \vdash \psi$, for every $\psi \in \Delta$, then $\Gamma \vdash \varphi$;
4. If $\Gamma \vdash \varphi$, then $\sigma(\Gamma) \vdash \sigma(\varphi)$, for every substitution σ.

Definition 2.2: A matrix is a triple $\mathcal{M} = <\mathcal{V}, \mathcal{D}, A>$, where:

1. $\mathcal{V}$ is a non-empty set of truth-values;
2. $\mathcal{D}$ is a non-empty subset of $\mathcal{V}$ whose elements are called designated values;
3. A is an abstract algebra of type L such that, for every connective of $\mathcal{L}$ with n-arity, there is a truth-function $c: \mathcal{V}^n \longrightarrow \mathcal{V}$ associated with it.

Definition 2.3: A $\mathcal{M}$-valuation for L is a function *v*: *Var* ⟶ $\mathcal{V}$ that maps propositional variables into elements of the set of truth-values.

Definition 2.4: A $\mathcal{M}$-valuation *v* is a $\mathcal{M}$-model of a formula φ, denoted by $v \vDash_{\mathcal{M}} \varphi$, if $v(\varphi) \in \mathcal{D}$. A formula is φ called a tautology if every valuation is a model of φ.

Definition 2.5: Let $\mathcal{M}$ be a matrix for $\mathcal{L}$; Γ is a set of formulas and φ a formula of *For*. We say that φ is a consequence of Γ, or $\Gamma \vDash_{\mathcal{M}} \varphi$, iff every model of Γ is a model of φ.

The three-valued propositional logic $\mathbf{P_3}$ of Peirce is characterized by the following matrix:

$\mathcal{M}\mathbf{P_3} = \langle\{1, ½, 0\}, \{1, ½\}, \neg, \vee, \wedge\rangle$

in which the connectives are defined as:

1. $v(\neg x) = 1 - v(x)$;
2. $v(x \vee y) = \max(v(x), v(y))$;
3. $v(x \wedge y) = \min(v(x), v(y))$.

Now we have to clarify two points regarding our choices for implication and designated values that take us beyond Peirce writings:

1. As was said previously, there is no record of conditional connectives in Peirce's notes on three-valued logic. This sounds unusual considering that implication plays a central role in his two-valued propositional logic. Turquette suggests three types of conditionals: Łukasiewicz, Kleene, and Gödel-Heyting's implications, ultimately giving preference to the last one (Turquette, 1976). Nevertheless, we think it is more

intuitive to choose the following one, obtained from the primitive operators. In this way, $\mathbf{P_3}$'s implication is defined as (¬α ∨ β) or ¬ (α ∧ ¬β)[9]. This gives us the following table:

→	1	½	0
1	1	0	0
½	1	½	0
0	1	1	1

Table 4: Implication for $\mathbf{P_3}$.

2. There are, basically, two types of three-valued logics (Avron, 2003): those in which value ½ belongs to the set of designated values, and those in which ½ does not belong to it. However, it is unclear what would be Peirce's choice for the set of designated values. If we select $\mathcal{D}$= {1}, the $\mathbf{P_3}$ would have no theorems at all.[10] It is easy to find out why: whenever the components of a compound formula are assigned with truth-value ½, the formulas takes the same truth-value. In every formula, at least one valuation of both atomic formulas has the truth-value ½. Therefore, no formula would be a tautology (or a contradiction) in $\mathbf{P_3}$. This would be against Peirce's claims that his triadic logic is a generalization of classical logic. Peirce had wanted to

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[9] In the paper "On the algebra of Logic" (1880, p. 24), Peirce gave the first formulation of the residual property (also known as deduction theorem), which asserts that from x, y ⊢ z the obtain x ⊢ y → z, i.e., the implication y → z is a *residual* of the conjunction x ∧ y. I would like to thank the anonymous referee for the mention of this important point.

[10] In fact, there would not be any tautologies or contradictions, just the same as Łukasiewicz and Kleene's logics, for example.

develop a logic that is closer to two valued one, as he states that his "[…] triadic logic does not conflict with Dyadic Logic; only it recognizes what the latter does not […]" (Peirce, 2021, seq. 645). Therefore, he would like to maintain as many classical theorems as possible in his project of a non-Aristotelian logic. For this reason, we selected $\mathcal{D}$= {1, ½}.

An important semantical consequence of these considerations is that $\mathbf{P_3}$ is functionally complete, in the sense that every truth-functional connective of the system can be defined in terms of his own set of connectives, primitive or derivable.

Definition 2.6: If can be shown that every propositional connective of a logic L is definable through a set of pairs {∧, ¬} or {∨, ¬}, the system L with {∧, ∨, ¬} is functionally complete.

Theorem: $\mathbf{P_3}$ is functionally complete.

Proof: The proof is trivial through the definitions above and using the simple method of truth-tables.

Finally, from the notions of tautology (Def. 2.4) and consequence (Def. 2.5) we can prove that some important theorems of classical logic hold in $\mathbf{P_3}$, such as the Principle of Identity (PI), Principle of non-Contradiction (PC) and Principle of Excluded Middle (PEM).[11] At the same time, we can prove inferences like Modus Ponens. In fact, we can demonstrate that $\mathbf{P_3}$ is a conservative extension of Classical Propositional Logic (**CPL**), such that $\mathbf{P_3} \subseteq \mathbf{CPL}$.

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[11] In Salatiel (2022) we showed proof in analytical tableaux that the $\mathbf{P_3}$ system is sound and complete.

Definition 2.6: A three-valued propositional system L_3 is a conservative extension of the two-valued propositional logic L if the connectives of L_3 are *normal* in the sense that behaves exactly as the connectives of L.
Theorem 2: $\mathbf{P_3}$ is a conservative extension of **CPL**.
Proof: All the connectives of $\mathbf{P_3}$ behave exactly as the connectives of **CPL**, in the sense that, if we ignore the third truth-value ½ in the first system, we obtain the connectives of classical truth-tables.

4. Peirce on a paraconsistent/paracomplete road

The system $\mathbf{P_3}$ shared a characteristic with others two systems developed by Peirce: it has no tautology or contradictions, because the indeterminate truth-value ½ prevents formulas to be truth (our false) in every valuation. One possible way to avoid this unwanted result is to take the set {1, ½} as designated values, as seen in the previous section.

This is a way to reinforce $\mathbf{P_3}$, but surprisingly gives us a suggestion that Peirce was close to discovering a paraconsistent system.

It is first noted that the matrices of $\mathbf{P_3}$ propositional calculus correspond exactly to the matrices proposed by B. Sobocinski (1952). In Sobocinski's system the primitive operators are $\{\sim, \rightarrow\}$ and the designated values are {1, ½}. Although Sobocinski's three-valued logic was not established to be a paraconsistent calculus, it results in a relevant logic **RM3** (Anderson and Belnap, 1975).

Paraconsistent logics are formal systems in which the presence of some contradictory statements does not lead to trivialization. Most of these logics do not invalidate the PC, because they were intended to be an expansion of classical logic. So, to avoid the inconsistency implies trivial deductive theories, they reject the Principle of Explosion (PE)[12].

Peirce's logic is non-explosive in the sense that PE does not hold. It is easy to see that the schema formula for PE $(\alpha \rightarrow (\sim\alpha \rightarrow \beta))$ is not valid from the assignment of both $v(\alpha)$ and $v(\sim\alpha) = \frac{1}{2}$ and $v(\beta) = 0$. But PC holds in $\mathbf{P_3}$ for $\mathcal{D} = \{1, \frac{1}{2}\}$, so the PEM and the PI as well.

To achieve the paraconsistent transformation is enough to convert Peirce's (⁄) negation into an operator of consistency "∘". Then, adding the operator of inconsistency to our system (⊥) we define the operators of consistency as follows:

$\circ := (\alpha \rightarrow \bot) \vee (\neg\alpha \rightarrow \bot)$

This operation gives us the following matrix for unary operator:

	∘
1	1
½	0
0	1

Table 5: Matrix for ∘.

Now it is easy to prove postulates of paraconsistent calculus outlined below are all theorems of $\mathbf{P_3}$:

1. $(\neg\alpha \rightarrow \circ(\neg\alpha))$
2. $((\circ\alpha \rightarrow (\alpha \rightarrow (\neg\alpha \rightarrow \beta))$
3. $(\circ\alpha \wedge \circ\beta) \rightarrow \circ(\alpha \wedge \beta)$
4. $(\circ\alpha \wedge \circ\beta) \rightarrow \circ(\alpha \vee \beta)$
5. $(\circ\alpha \wedge \circ\beta) \rightarrow \circ(\alpha \rightarrow \beta)$

[12] The terms "principle" and "law" are used interchangeably in this article, with a preference for the former, as employed in Peirce's writings.

The same holds for the Gentle Principle of Explosion, which is proved in the following diagram:

α	β	$\neg\alpha$	$\circ\alpha$	$\neg\alpha \to \beta$	$\alpha \to (\neg\alpha \to \beta)$	$\circ\alpha \to (\alpha \to (\neg\alpha \to \beta))$
1	1	0	1	1	1	1
½	1	½	0	1	1	1
0	1	1	1	1	1	1
1	½	0	1	1	1	1
½	½	½	0	½	½	1
0	½	1	1	1	1	1
1	0	0	1	1	1	1
½	0	½	0	0	0	1
0	0	1	1	0	1	1

$\mathbf{P_3}$ belongs to the same class of non-explosive many-valued logics, such as Asenjo's logic of antinomy (1966), D'Ottaviano and Da Costa's J_3 calculus (1970), and Priest's logic of paradox (1979). However, Peirce's triadic system matches with another many-valued paraconsistent logic that belongs to the family of **LFI** (Logic of Formal Inconsistency).

LFIs are paraconsistent logic that has a primitive (or derivable) operator for consistency which allows the system to hold some kind of explosion, called a *gently explosion*. In this way, it could separate contradictory statements into those which involve an *explosion* and those that are not explosive (Carnielli and Coniglio, 2016).

In fact, because the following inferences hold, $\mathbf{P_3}$ can be viewed as a "strong" **LFI**, such that Da Costa's C-calculi:

1. $\circ\alpha, \alpha \nvDash \beta$
2. $\circ\alpha, \neg\alpha \nvDash \beta$
3. $\alpha, \neg\alpha \nvDash \beta$
4. $\circ\alpha, \alpha, \neg\alpha \nvDash \beta$

Yet it is surprising to find that paraconsistent $\mathbf{P_3}$ was first presented by Carnielli, Marcos, and De Amo (2000) in a form of a logical matrix for a logic called **LFI2**, with the set of operators {∨, ~, •} as primitives. The symbol • denotes a strong negation for inconsistency. In that paper, the authors state that the matrices "[…] define a brand new three-valued logic, different from $\mathbf{J_3}$ and from any other logic we have heard of" (2000, p. 145). After that, Carnielli and Marcos (2002) presented the same logic, under the logic **Ciore**, with the primitive operator of consistency "∘".

Finally, we can turn $\mathbf{P_3}$ under $\mathcal{D}= \{1\}$ into a paracomplete logic if we turn out Peirce's bar negations into the strong (intuitionistic) negation "~", defined as: $\sim \coloneqq (\alpha \rightarrow \bot)$. This operation gives us the following matrix for the unary operator:

	~
1	0
½	0
0	1

Table 5: Matrices for ∘.

Now it is easy to check in a simple truth table that the Principle of Excluded Middle (PEM) does not hold in our logic, same as the Double Negation. But others theorems, such as the Principle of non-Contradiction (PC), remain valid, as we expected for intuitionistic calculi.

5. Philosophical remarks

Aside from formal aspects, one could ask whether Peirce was intended to create many-valued paraconsistent and paracomplete systems. There is a current debate about the philosophical motivations of Peirce's triadic logic (Fish and Turquette, 1966; Lane, 1999; and Odland, 2021), which is hard to deal with because of the lack of unity in the philosopher's works.

Nevertheless, there is clear textual evidence that Peirce was thinking about his theory of continuity at the time he introduced his three-valued logic[13]. Therefore, in our view, Peirce's rejection of the principle of bivalence is motivated by his studies on continuity and modalities[14].

The notion of continuity is central in Peirce's philosophy, which at some point is named as *synechism*, from the Greek *synechés* or continuous (CP 1.172 [c. 1897][15]). He applied both sophisticated mathematical tools and philosophical analysis to understand the concept of *continuum* over almost his entire

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[13] The main evidence comes from the third annotation on triadic logic in Peirce's notebook (seq. 645). After defining the triadic logic, Peirce gave his notorious example of an inkblot: "Thus, a blot is made on a sheet. Then every point of the sheet is unblackened or blackened. But there are points on the boundary line; and those points are insusceptible of being blackened or of being unblackened [...]". The same example, with some variations, appears many times in the context of Peirce's theory of continuity (Lane, 1999).

[14] The recent work by Odland (2020) has compelling arguments in such a direction.

[15] PEIRCE, C. S. *Collected Papers*. 8 vols. HARTSHORNE, Charles; HEISS, Paul and BURKS, Arthur (eds.). Cambridge: Harvard University Press, 1931-1958. Hereafter cited as CP followed by volume and paragraph.

career, and seems never satisfied with the results (see Havenel, 2008).

Even so, we can detach two main original ideas of the philosopher that is relevant for our discussion. First, by 1897 Peirce was already ruled out the notion of continuous as being made of discrete entities, like individual points, and came to a conception of a continuous line as an aggregate or collection (set) of *potential* points, which are only capable of determination (PM, 185 [1898][16])[17]. In other words, the continuous is *non-punctate*, in the sense that it cannot be composed of discrete points[18].

For Peirce, any existential or individual point mean a breach on the continuity, which he associated with generality (and vagueness). Consequently, it is important for him that "all possible points are not distinct from one another;

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16 PEIRCE, C. S. *Philosophy of Mathematics*: selected writings. MOORE, Matthew E. (ed.). Indiana University Press: Bloomington and Indianapolis, 2010. Hereafter cited as PM followed by page.

17 Although that conception was prefigured in Peirce's early works, such as "On a new class of observations, suggested by the principles of logic" (EP 1: 106-108 [1877]). PEIRCE, C. S. *The Essential Peirce*: selected philosophical writings, v. 1 (1867-1893). HOUSER, N. and KLOESEL, C. (eds.) Indiana University Press: Bloomington and Indianapolis, 1992. Hereafter cited as EP followed by volume and page (I would like to thank the anonymous reviewer for this useful observation).

18 In this sense, it is contrary to Dedekind and Cantor's views that there is a one-to-one correspondence between a continuous line and discrete series of numbers. Peirce, oppose to them, emphasizes that "numbers cannot possibly express continuity" (PM, 196 [1897]).

although any possible multitude of points, once determined, become so distinct by the act of determination" (PM, 204 [1900]). This notion is strongly related to his (Aristotelian) metaphysic of real possibilities, that Peirce achieved at the same period, and his theory of modalities (possible and necessary)[19].

Now, this takes us to the second point. Peirce's account on continuity concerns the questioning of the laws of classical logic, namely, the PC and PEM. According to Peirce there is two modes of indetermination: *generality* and *vagueness*. Peirce states the difference between these two types of indetermination on the basis of the principles of excluded middle and contradiction: "[...] anything is *general* in so far as the principle of excluded middle does not apply to it and is *vague* in so far as the principle of contradiction does not apply to it" (EP 2: 351 [1905])[20]. In his opening statements about triadic logic (seq 645), dated February 23, 1909, he writes:

> Triadic logic is that logic which, though not rejecting entirely the principle of excluded middle, nevertheless recognizes that every proposition, S is P, is either true or false, or else has a lower mode of being such that it can neither be determinately P, not determinately not P, but is at the limit between P and not P.

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[19] See Lane, 2007.

[20] Lane (1999) has an interesting analysis on the distinction between proposition which a logical principle *does not apply* and those which a logical principle is *false* regarding it, in Peirce's writings. According to him, what Peirce intends to express by a third truth-value is a proposition concerning to both PC and PEM *apply*, but PEM is *false*. We will not follow this line of interpretation in this paper.

Why Peirce says that triadic logic does not reject entirely the PEM? Peirce usually defines PEM in terms of predicate logic: given a predicate P, for any subject-term S, "S is P" or "S is not P" is true. But this means that PEM does hold only for individuals and, therefore, fails to apply to anything general, because the general is a mode of indetermination (CP 1.434 [c. 1896]; CP 6.168 [1903]; NEM II: 514 [c. 1904][21]; CP 5.448 [1905]). Now the continuous, as he sustains, is a general, and so any individual introduces an element of discreteness in the line. "The principle of excluded middle only applies to an individual […]. But places being mere possible without actual existence are not individuals." (PM, 138 [1903]). This is a "lower mode of being" (general) which PEM is not valid[22].

However, the continuous has another "lower mode of being" indeterminate: the possible. And we will find there the same restrictions on applying of PC.

> Whatever actually is, however, is discrete. Hence, instants, -- the ordinary instants of a lapse[23], -- have no actual being, unless they are marked and actualized by some fact. But to say

[21] PEIRCE, C. S. *The New elements of mathematics*. The Hague: Mouton Publishers, 1976. Hereafter cited as NEM followed by volume and page.

[22] Another way to put this: suppose a line R in which, for any real number x, either x= 0 or x ≠ 0. Then consider a function f that takes the value 1. If PEM holds, then f(x)= 1 if x= 0 and f(x)=0 whenever x ≠ 0. As a result, PEM allows for a discreteness function in the continuous R (Bell, 2008).

[23] Here Peirce refers to a temporal continuity.

> that they are not discrete is to say that they are not unconditionally subject to the principle of contradiction. […]. Now, what mode of being is that which escapes the principle of contradiction? It is *possibility*. It is possible for me to sin and possible not to sin. (NEM IV: 258 [1904]).

Therefore, some laws of classical logic, such as PEM and PC, are incompatible with the mathematic of continuous, as Peirce states in many passages where he mentioned both principles. But such a failure may not occur simultaneously.

Peirce has something to say about PC in the same entrance of triadic logic (seq. 645), in the paragraph following the one mentioned early: "Of course it remains true, as far as the principle of contradiction is concerned that the state of things represented by the proposition cannot be V and F, *verum atque falsum* and must be V + F if by F is meant L + F."

Here "+" stands for the usual Boolean disjunction. Therefore, he says that for $\boldsymbol{\mathcal{D}} = \{1\}$ ("F is meant L + F"), PC remains true, and the limit means truth-value *gap* ("neither be determinately P, not determinately not P"). That is the intuitionistic calculi we propose above (section 4). But Peirce does not say what happens when we face a continuous of modal possibility and, consequently, a truth-value *glut* – both P and not P is true. In this case, we suggest that Peirce would go for a paraconsistent calculus.

These considerations would be the motive to Peirce explore non-classical logics and, in particular, the rejection of bivalence. Accordingly, the "limit" as the third value means to express some aspects of the continuous conceived as a kind of generality, which, like the possible, Peirce contrasts with the individual or actual. Therefore, it would be reasonable to assume paraconsistent and paracomplete three-valued logics as the most adequate solution[24].

Conclusion

Peirce's triadic logic is one of the first attempts to build a non-classical system in modern logic at the turn of the 19th Century, alongside logicians as Hugh MacColl and Nicolai Vasiliev. But ever since Peirce's manuscripts on three-valued logic were discovered in the middle of the 1960s by Fish and Turquette, there are few comprehensive analyses of them, aside from Turquette's papers.

In the present article, we suggested a new approach to Peirce's three-valued matrices. Contrary to focus on dualism between the connectives, as proposed by Turquette, we work on the set of connectives as three different systems. Two of these formal systems are well-known today as Łukasiewicz, Kleene, and Bochvar's logics, discovered some years after Peirce has written his notes. The other one, which we called system $\mathbf{P_3}$, generated an original matrix that has the proprieties to be conservative and functionally complete.

As shown in this paper, our view has an advantage to exam some structural and semantic proprieties of matrices in an easier way than Turquette's axiomatic calculi. It also

[24] It could be asked if Peirce is not looking for a modal logic to deal with these modalities. There is a reason for his choice to call "limit" the third value, a mathematical notion commonly associated with continuity, and not simply "possible" or "indeterminate", as Łukasiewicz did it. Besides that, Peirce already has a modal logic in his Gamma Graphs (cf. Zalamea, 2003).

enables us to use straightforward types of proof calculi, such as analytical tableaux (Salatiel, 2022).

Furthermore, working on a fragment of the triadic logic ($\mathbf{P_3}$) allowed us to explore interesting non-classical systems, like paraconsistent and paracomplete logics. As we suggested, such systems are coherent with some of Peirce's statements about his philosophical motivations to reject logical bivalence. Hence, we think that the study of such formal aspects can also help us to understand his motivations at the time.

Yet some difficulties persist in our research. First, we concentrate our analysis on an original fragment of triadic logic, which we have called $\mathbf{P_3}$. However, there is no indication whatsoever that Peirce gives any special attention to it, even if he was thinking about three distinctive systems of logic.

Second, although we can find some evidence that supports our approach, future works will have to refine hypotheses to solve the puzzle about Peirce's philosophical motivation. Besides that, further investigations on Peirce's negations could generate other many-valued logics as remarkable as those presented here.

Finally, we must say that our new approach to Peirce's triadic logic is far from definitive. We would like to conclude with Turquette's own words, written in his last paper on this subject: "In any event, these axiomatic systems have a certain formal elegance which strongly hints at the possibility of still other interesting interpretations" (Turquette, 1981, p. 381).

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