



## Covariational Reasoning in Calculus: development from the use of tasks

### Raciocínio covariacional em Cálculo: desenvolvimento a partir de tarefas

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#### Abstract

The design of coordinated analysis as variables of two interdependencies (covariational reasoning - CR) is a central aspect of Calculus learning, the most discussed skills are mobilized amid the mathematical routines triggered by work with tasks. Realization of a qualitative research, interpretive, from records of written production and excerpts from dialogues of a group of students without work with a task in conjunction with a bottle that fills. The task made possible the exploration of the skills including: quantity and reasoning about its measurement process; imagine minute varying continuously; coordinate pressure pairs that can vary together (time, height, volume and radius). In addition, students were able to establish a relationship between height and volume, building a representation in which the inversion in the concavity of the graph showed an understanding of the existence of a change in the rate of change in this situation.

**Keywords:** Mathematics Teaching; Teaching Differential and Integral Calculus; Mathematical tasks; Covariational Reasoning.

#### Resumo

Visto que analisar de forma coordenada as variações de duas grandezas interdependentes (raciocínio covariacional – RC) é um aspecto central da aprendizagem do Cálculo Diferencial e Integral, discutimos habilidades do RC mobilizadas durante discussões matemáticas desencadeadas pelo trabalho com tarefas. Realizamos uma investigação qualitativa, de cunho interpretativo, a partir de recortes da produção escrita e trechos de diálogos de um grupo de estudantes de Engenharia que cursaram essa disciplina, no trabalho com uma tarefa envolvendo uma garrafa que enche. A tarefa possibilitou a exploração de habilidades como: constituir quantidades e raciocinar sobre seu processo de medição; imaginar quantidades variando continuamente; coordenar pares de quantidades que variam juntas (tempo, altura, volume e raio). Além disso, os estudantes foram capazes de estabelecer uma relação entre a altura e o volume, construindo uma representação

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na qual a inversão na concavidade do gráfico mostrou compreensão da existência de uma mudança na taxa de variação nessa situação.

**Palavras-chave:** Ensino de Matemática; Ensino de Cálculo Diferencial e Integral; Tarefas Matemáticas; Raciocínio Covariacional.

## Introduction

During basically all the three years of High School and when entering Higher Education, in the Differential and Integral Calculus (*Cálculo Diferencial e Integral*, CDI) course, students deal with a “formal” concept of functions that rarely “carries” its essence: the covariational aspect of the magnitudes involved. According to Barufi (2002, p. 69), “for most students, the School Mathematics in High School has little or nothing to do with what they learn in Calculus, and the character of analysis with which they are confronted seems to be a great difficulty”.

A function is related to a mathematical concept that describes how two or more quantities vary from one another and can be described by different types of representations, including words, mathematical symbols, graphs or tables. As Mastre (2014, p. 71) points out, the “genesis of functional thinking happens when the student engages in an activity, chooses to pay attention to the quantities that vary, and starts to focus on the relationship between these quantities”.

In this regard, the so-called covariational reasoning (CR) can be characterized in terms of coordinating the images of two variables as they change (Confrey & Smith, 1994, 1995), or because of the values of individual quantities as variables, therefore conceptualizing two or more quantities as varying simultaneously (Thompson & Carlson, 2017). According to the latter researchers, students’ quantitative and covariational reasoning is essential for learning many mathematical topics, such as function and variation rate. They also argue that ideas of variation and covariation are epistemologically necessary as conceptual bases to create CDI concepts. Although there is an increasing amount of research on the quantitative and covariational reasoning of students at the high school level, research on the quantitative reasoning of students at the undergraduate level is scarce (Speer & Kung, 2016; Mkhathshwa, 2020).

According to Thompson (2011), for students to get involved in CR, they must (1) build two quantities, (2) imagine that the measurements of these quantities vary smoothly, and (3) join the measurements of two quantities by building a multiplicative object that simultaneously represents both measurements. For his part, Frank (2017) highlights that working with investigative tasks enables students to develop these CR skills, with a focus on designing graphics as emerging representations of constantly changing phenomena.

In this regard, considering the importance of the topic, we list as a research problem the investigation of the possibilities of developing CR skills promoted by working with

mathematical tasks, in engineering students who attend CDI1<sup>5</sup>. This article aims to present and discuss CR skills that were mobilized during the discussions triggered by the work with mathematical tasks. For this purpose, it is organized in introduction; presentation of the theoretical framework related to CR themes and mathematical tasks; methodological procedures for research and data description and analysis, organized from some of the tasks that composed the CDI1 teaching and learning environment.

## The Covariational Reasoning

We have summarized here, based on the texts of Thompson, Carlson and collaborators, published since the 1990s, ideas that circumscribe the concept of covariational reasoning - CR. We recognize that, in Thompson's works, in the decade in question, there is a prefatory idea of CR presented with the "quantity" and "quantitative reasoning" topics, with the purpose of characterizing the concept of rate, a key aspect when thinking about function under the covariational reasoning. Thompson (1990, 1992, 1994) presents an organized theoretical model about stages to characterize the quantitative reasoning, supported by Piaget's reflective abstraction, in research about additive and multiplicative conceptual fields.

In this model, a quantity is defined as a quality of something possible to be measured. Quantification is the process through which numerical values are assigned to the qualities of an object. A quantitative operation is a mental action through which a quantity such as that whose value varies (or may vary) is recognized and a new quantity is conceived in relation to one or more already known quantities. A quantitative operation is not necessarily numerical, because it has to do with understanding a situation (recognizing that a quantity grows/decreases, or that the "way" how it grows/decreases can change, or even that two quantities grow, or not, in the same direction). A numerical operation, in turn, can be used to quantify/measure these changes.

Thompson (1994) points out that the development of the concept of variation rate initially involves the recognition of change in some quantity, progresses to a slightly coordinated image between two quantities and consolidates into an image of the covariation of two quantities whose measures remain in constant proportion. A more systematized definition of covariation found in Saldanha and Thompson (1998, p. 299, our translation) is as follows: "Our idea of covariation is that of someone who has in mind a sustained image of the values of two quantities [that vary] simultaneously".

In this way,

In the initial development, values of two quantities are [mentally] coordinated – think of one, then the other, then the first, then the second, and so on. Later covariation images imply the understanding of time as a continuous quantity, so that in its image the two values of quantities persist (Saldanha & Thompson, 1998, p.289).

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<sup>5</sup> 1<sup>st</sup> semester subject of Engineering courses at UTFPR, institution where the research was developed, including the study of real functions of a real variable, limits, derivatives and integrals, with a workload of 90 hours.

Carlson, Jacobs, Coe, Larsen, and Hsu (2002) propose a theoretical framework to describe CR skills, including five categories of mental actions observed in students, by applying it in the context of representing and interpreting a graphic model for a dynamic event. More recently, Thompson and Carlson (2017) reviewed the previous covariational structures in two ways: (i) by analyzing students' variational reasoning separately from their covariational reasoning; (ii) by analyzing how students coordinate their images of varying quantity values.

Frank (2017) developed tasks that sought to develop CR skills, conceptualizing graphs as emerging representations of constantly changing phenomena. To do so, the student must first conceptualize two quantities involved in a certain situation (recognizing an attribute that they believe to have a measurable magnitude). Then they must imagine the measurements of these two quantities varying continuously, focusing on the variable magnitude rather than the numerical values that the quantity measure can assume. Next, they must coordinate how the two quantities vary together, imagining the relationship between their measurements at a given moment in their experimental time, building a multiplicative object that joins the measurement of these two quantities. Finally, the student must coordinate their conception of the smooth variation of the measurement of each quantity in order to engage in the smooth continuous covariation. Such processes are essential in working with tasks that involve changes occurring in two interdependent variables, such as those we considered in our research, and which will be presented below.

## **Working with mathematical tasks and the role of mathematical discussions in learning CDI**

Pedagogical proposals for teaching CDI that are in line with the latest recommendations for engineers education (NAE, 2005; Brockman, 2013)

ask for tasks that favor the student to develop connection and reflection skills, beyond mere reproduction and memorization. Such tasks must be based on the fact that mathematical knowledge is dynamic and built from the relationships, justifications, analysis and validations established by those involved and not as something ready and finished (Trevisan & Mendes, 2013, p. 137).

In the direction proposed by Ponte (2014, p.14), tasks are “organizing elements of those who learn, which are mostly proposed by teachers and, once proposed, must be interpreted by students and may give rise to several activities”. The author also says that they are “fundamental tools of mediation in teaching and learning mathematics”, and may or may not have “potential in terms of mathematical concepts and processes that can help mobilize” (Bridge, 2014, p. 16).

Gafanhoto and Canavarro (2014) point out that, in most cases, the choice of tasks to be proposed to students is directly influenced by textbooks and other accessible curriculum mediators, especially the Internet. The authors point out, however, that “these resources are not always best suited to the students of a given class and the teachers' purpose of teaching”,

and they also reinforce that “the selection, adjustment or creation of good classroom assignments is a challenge for many teachers” (Gafanhoto & Canavarro, 2014, p. 115).

Mathematical tasks can be classified under different perspectives: their nature, their characteristics, strategies for their resolution, their cognitive demand, the types of reasoning required for their resolution. Based on the works of Stein and Smith (2009), we point out that tasks contemplating higher levels of cognitive demand (which require the use of procedure in connection with meaning and doing mathematics) are desirable in proposing tasks to students.

Since “students do not learn so much from the practical activities, but from the reflection they make on what they have done during these practical activities” (Bridge, 2005, p. 15), it is important to highlight the role of mathematical discussions in working with mathematical tasks, either in small groups or in plenary moments. The discussions are

moments of work in the classroom with great potential for students’ learning, as they favor their engagement in presenting, justifying, arguing and negotiating meanings for their reasoning when working with mathematically significant tasks (Rodrigues, Menezes & Ponte, 2018, p. 399).

Ponte (2017) argues that, as opposed to a class model that assumes the exposure of concepts and a practice that leads to uncritical memorization and mechanization, both the research in Mathematics Education and the professional practice of teachers point out mathematical discussions as essential for students’ understanding. Starting from a “state of the art” regarding the subject, the author highlights that the appreciation of discussion moments when teaching Mathematics is already a strongly established idea, at least since the 1980s.

With the aim of “operationalizing” a class model “of viable implementation in our educational model”, which values mathematical discussions, we have advocated the organization of learning environments based on episodes of task resolution (Couto, Fonseca, & Trevisan, 2017; Couto & Trevisan, 2017; Trevisan & Mendes, 2017; Trevisan & Mendes, 2018). The expression “learning environment” refers to the context where the individual is offered opportunities to learn. The episodes (moments) are organized based on the proposition of tasks developed and/or adapted from curricular materials, which are not preceded by definitions or similar examples, and whose development in groups in the classroom contributes to the intuitive exploration of mathematical ideas and subsequent systematization (Trevisan & Mendes, 2018).

According to Trevisan and Mendes (2017), such dynamics of work aims at inviting students to explore some concepts intuitively, i.e., there is no previous explanation of concepts prior to the task that will be proposed (the central concern at this time is not to bring the formal definition of the concept in question). In this context, during autonomous work or in small groups, students are able to develop their own conjectures and thus present their conclusions to their classmates. After these discussions and a collective discussion with the whole class, the teacher can intervene in order to “refine” the concepts that underlie the

content to be explored. In the case of this article, our focus is on discussions that take place in small groups, when students are working independently (Ponte, 2014; Stein et al, 2008).

## Methodology procedures

### *Research characterization and contextualization*

This is a qualitative investigation of an interpretative character (Bogdan & Biklen, 1994). It is an excerpt from a work developed within a broader project under the coordination of the second author, whose general objective was to investigate the processes involved in characterizing, implementing and evaluating a teaching and learning environment for the subject of CDI, considering the real conditions to which we are subject.

To this end, several specific objectives are unfolded, including organizing mathematical tasks that make up this environment. Thus, we have worked with the creation of tasks (and, intrinsically associated with this, investigated ways of using them in the classroom, as well as the learning provided by them) that provide students with the opportunity to reinvent the CDI, which allow the creation of fundamental concepts and theorems used intuitively before they are accurately described or proven (Trevisan & Mendes, 2017).

### *Context of investigation*

With the purpose of achieving the proposed objective for this article, five task resolution episodes were organized with a CDI1 class from the Engineering course of a Federal University, under the responsibility of the second author (and advisor of the broader paper). Five episodes of task resolution (conducted by the first author, as part of his Teaching Internship, together with a Vocational Master's Degree Program in Mathematics Teaching) were organized throughout the semester, each consisting of a meeting of three 50-minute classes, as the "formal" concepts of limit, derivative and integral were explored and systematized. The 50 students who attended the course, organized in groups of three to five members (according to their choice) were offered sequences of tasks without any prior explanation of content. The course was planned following the "unusual" curriculum structure proposed by Trevisan and Mendes (2017, p. 265).

When organizing the tasks, we took into account a set of CR skills that could be mobilized in the proposed tasks: (i) Making quantities involved in the situation (recognizing attributes of a situation that can be measured); (ii) Reasoning about the process of measuring these quantities; (iii) Imagining measures of quantities varying continuously; (iv) Coordinating two quantities that vary together: (a) recognizing that quantities are related; (b) recognizing the direction of growth - both grow/decrease, for example; (c) recognizing the existence of variation rates - grow faster/ slower, or grow at an increasing or decreasing rate; (d) identifying possible changes in the growth rate.

Since these students had already had contact with a formal definition of function



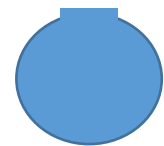
during high school, our purpose in performing these tasks was for them to (re)signify this concept. In all tasks, we tried to mobilize the articulation between multiple representations (natural language, table, graph), in an attempt to coordinate the variation of the quantities involved, recognizing the existence of variation rates and possible changes in these rates. The way the tasks were organized intended the students to deal with situations without the need to take/adopt specific values for the quantities involved, i.e., imagine measures of quantities varying continuously (item iii).

The set of tasks prepared and proposed is found in Gonçalves (2018). In Table 1, we present the task selected for analysis in this article because it involves a greater diversity of CR skills and also because it allows the articulation between different representations (natural language, table, graphic and algebraic expression). It enables: (i) Making quantities (such as volume or height of water in the bottle, time to fill the bottle); (ii) Reasoning about the process of measuring these quantities (choice of appropriate units of measurement, for example); (iii) Imagining measures of these quantities varying continuously and (iv) Coordinating two of these quantities, for example: (a) relating time and height; time and volume; height and volume; (b) observing the increasing behavior in the several relationships listed; (c) recognizing that time and volume relate to a constant rate, while time and height, and height and volume relate to variable rates and (d) in the case of time and height (and height and volume), recognizing that there is a change in the variation rate (representing a graph with change in concavity – existence of inflection point).

Table 1 - Task selected for analysis.

*Water is poured into a bottle/vessel at a constant rate. Use this information and the shape of the vessel (figure on the right) to answer the following questions.*

- a) What do you understand by a constant rate of water spill in this situation?*
- b) Imagine the scene of the vessel being filled and write down what you think can be measured in this situation.*
- c) Draw a chart that relates the height of the water in the bottle over time. Explain the reasoning that led to your sketch.*
- d) Build graphs that relate the different magnitudes involved in this situation.*



Source: Gonçalves (2018, p. 25).

One of the procedures for data collection included using the written production and audio from the groups while working with the tasks. Another procedure consisted of participant observation (Gil, 2010) by the first and second authors, both when working in small groups and in collective discussion, implying living with the researched group for some time. In the case of the first author, he fit as an “outside” individual, since his presence was not routine in classes, and the second author, as a “native”, since he was the teacher responsible for the subject. As a form of data records, to be later analyzed, notes were made by researchers in their individual field diaries, as they followed the work in groups circulating through the class, and also after the class. These observations were intended to identify aspects of CR skills that were present during the discussions.

### *Data organization and analysis mode*

For analysis purposes, we organized excerpts of written production and dialogue pieces during the work of one of the groups, which illustrated the potential of the tasks in terms of fostering discussions involving CR skills. This choice was made based on the records made by the researcher in his field diary, when following the groups' work in class. For the choice of episodes, we took as criteria those that showed a greater involvement of students in "presenting, justifying, arguing and negotiating meanings" (Rodrigues, Menezes and Ponte, 2018, p. 399), while working with the tasks. The researcher intervened with the groups at some moments of the discussions, in order to understand how they were thinking and, when applicable, to bring new elements that would make it possible to deepen the discussion, or to put incorrect hypotheses "in check".

In this article, we bring excerpts of protocols from this group of students, whose members are indicated as A1, A2 and A3. The group was chosen because of their involvement with the proposal, the content of the discussions and the diversity of ideas among the members who got involved in these discussions.

Once the groups had been selected, the data were analyzed. To this end, we adapted the steps of the model proposed by Powell, Francisco and Maher (2004) for data analysis in the context of investigations on mathematical work and the development of thinking of students engaged in mathematical investigations (in the case of our work, the model was adapted for audio analysis). To become familiar with the content of the audios, we fully listened to the audios of the selected teams (along with our written protocols and field diaries). From there, we selected and took note of specific parts of the dialog to listen to them again, creating brief descriptions of the content of each part and identifying significant moments, which we call critical events (Powell, Francisco and Maher, 2004).

With the transcripts on our hands<sup>6</sup>, we focused on the content of these critical events, identifying topics that would help us interpret the data collected (coding). As Powell, Francisco and Maher (2004, p. 29-30) point out, "just like the identification of critical events, the codification is directed by the theoretical perspective of the researchers and the research questions". Thus, we tried to unveil CR skills that were mobilized during the mathematical discussions in each of these critical events, triggered by the work with mathematical tasks, bringing to analysis the full transcription of 5 of these critical events (some speeches that were not considered to be relevant were omitted from one event to another). According to the aforementioned authors, the proposed model is compatible with the implementation of codes defined *a priori*, which occurs in our case.

Aiming at facilitating such identification and subsequent interpretations, we highlighted in the students' speech, excerpts transcribed within each critical event, which, in

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<sup>6</sup> In the transcripts, the symbol [...] indicates a cut of speech considered not relevant, and the symbol [hypothesis] represents a hypothesis from the context of speech.



our understanding, highlight the skills mobilized, which are organized according to Table 2.

Table 2 - Skills mobilized/unveiled when working with the task

Symbols Used	Mobilized/Unveiled Skills
i	Making quantities involved in the situation.
ii	Reasoning about the process of measuring these quantities.
iii	Imagining measures of quantities varying continuously.
iv (a)	Coordinating two quantities that vary together: recognizing that quantities are related
iv (b)	Coordinating two quantities that vary together: recognizing the direction of growth - both grow/decrease.
iv (c)	Coordinating two quantities that vary together: recognizing the existence of variation rates - grow faster/ slower, or grow at an increasing or decreasing rate.
iv (d)	Coordinating two quantities that vary together: identifying possible changes in the growth rate.

Source: authors.

## Results and Discussion

The results presented and discussed here are organized in five critical events identified based on the mathematical discussions that took place in the student group that was analyzed.

### *Event 1 - presenting the situation of the bottle*

The researcher started the task by reading the items for the entire class. When he finished reading, he asked a question: *Do you believe the height inside the bottle will be constant?* The researcher's intention was to allow the students to start the situation with the central issue of the task in mind. One of the members of the group started by answering the question, as transcribed below.

[1.1] A1: Nooo	
[1.2] A2: No?	
[1.3] A1: No, because the radius starts small [...] then it increases and then decreases.	i, iii, iv(b)

In this event, the initial understandings among the members were different and A1 tried to explain his opinion with his argument. We can highlight that the explanation brings with it the identification of the magnitude (radius) and its behavior in the course of the situation, as well as the fact that this magnitude interferes at the height. Thus, the situation presented in the task allowed the group to individually conceptualize the values of each of these magnitudes (radius and height), a necessary step for a later recognition of simultaneous variation (Thompson & Carlson, 2017).

*Event 2 - spill in or out?*

The discussion of the task items started by reading item (a).

[2.1] A1: Constant spill rate [...] What do you understand by a constant spill rate? I'll put it this way: Volume that goes "inside" the container, the [...] entry rate.	i
[2.2] A2: The amount of water that comes in is the same amount that comes out, to fill, isn't it?	
[2.3] A1: No, no, I'm just talking about the entrance of the situation. Constant rate means that the same amount of water will enter per unit of time.	i,iii, iv(a), iv(b), iv(c)
[2.4] A2: "Uhum" [...] I had understood it spilling out from the vessel.	
[2.5] A1: It is a spill from spilling into the vessel. [...] I wrote like this: constant spill rate means the same amount of water will be spilled.	
[2.6] A2: Per unit of time.	
[2.7] A1: Per unit of time [emphasizing]	

The term "spill" raised questions among the members of the group, as it was understood from two perspectives: if the vessel was empty, it would be filled with water; if the vessel was full, it would be emptied. Based on A1's comment in [2.1], when he says "volume", we can see that the student established that the magnitude to be observed is related to the volume, while in the statement of the task we presented the expression "water is poured into a vessel at a constant rate", that is, we did not explicitly mention that we would be interested in the volume of water inside the vessel. In this regard, the discussion that took place in the group proved to be an essential aspect for the negotiation of meanings and to understand the mathematical situation under study (Ponte, 2017; Rodrigues, Menezes, & Ponte, 2018).

Additionally, the expression "constant rate" reveals that, for the student, the same amount of water will be poured over the entire period observed (the student called this fact "per unit of time"). This is a particular type of growth that can be explored intuitively, without any prior explanation of concepts prior to the proposed task (Trevisan & Mendes, 2017).

*Event 3 - identifying more magnitudes involved*

Group thinking about item (b) of the task:

[3.1] A1: Imagine the scene of the vessel being filled...	
[3.2] A2: Volume, radius, diameter.	i
[3.3] A1: Volume? "quantiii" (at this moment, A1 seems to doubt this fact)	
[3.4] A2: Yeah, volume [...] diameter of the vessel can also be seen.	
[3.5] A1: When it is being "filled" is when one can measure?	

[3.6] A2: The volume, because it depends on the amount of water to measure the volume.	iv(a)
[3.7] A3: You can calculate the mass, imagine it tied to a rope and spilling water.	i, iv(a)
[3.8] A2: What else?	
[3.9] A1: Volume, diameter [...]	
[3.10] A2: Mass	
[3.11] A1: [...] what else? [...] filling speed.	i
[3.12] A2: I don't think there is any more.	
[3.13] A1: That's good, that's enough already (laughs).	

Observing event 3, we can see that even before A1 [3.1] finishes reading, A2 [3.2] already answers the statement. A third member (A3), who had made no comment before, comments on the situation in [3.7]. This suggests that the context of the situation, together with the previous item, has made it possible for students to mobilize the recognition of individual quantities involved (volume, radius, diameter, mass, filling speed) as variables, as well as ideas about possible relationships between these quantities (Thompson & Carlson, 2017). Thus, the discussion favored the engagement of students in presenting and explaining ideas (Rodrigues, Menezes & Ponte, 2018), enabling them to “imagine” what was happening in the situation.

*Event 4 - what will be related to what? Will it be a straight line?*

The group started by reading the statement in item (c).

[4.1] A2: In the beginning it is a little bit bigger.	iv(c), iv(d)
[4.2] A1: Fast	
[4.3] A2: Bigger... then it gets smaller.	iv(c), iv(d)
[4.4] A3: Then? It depends a lot what you're looking at on the graph, if it's by...by radius and volume, you have to see how you're assembling the graph...if it's by radius...if it's by volume...	iv(a)
[4.5] A2: Yeah, so, because it grows fast at the beginning, then it decreases and then it comes back faster, because of here, like...	iv(c), iv(d)
[4.6] A3: So, but then, if you measure the height of the thing by the diameter per unit of time, that's a straight line.	
[4.7] A2: No, no, because like that, look! the radius here is smaller than the one here (at this moment they are looking at the shape of the bottle presented in the task) so the amount of water that will be put here will already appear here, so by the time it starts to fill this amount of... this diameter, it will take a while to grow and here it will grow faster (they keep looking at the diameters of the bottle as it is being filled with water)	ii, iv(b), iv(c), iv(d)
[4.8] A2: then I think it would be this graph (pointing to the graph sketch) it is this graph there (graph shown below as Figure 1).	
[4.9] A2: Because, like, when you start putting water here, it will grow fast (A2 is watching, at this moment, the water at the beginning of filling the bottle)	ii, iv(c), iv(d)

because it will slow down in speed, look at the size of this (looking more at the center of the bottle), so the time it starts narrowing here, it will grow faster.	
[4.10] A1: Height of water, this graph here (the graph shown in the figure) would be of water height. The volume will be constant, it is always filling in the same amount regardless of.	iv(b), iv(c)

The graph shown by the group, as mentioned in [4.8], is shown in Figure 1.

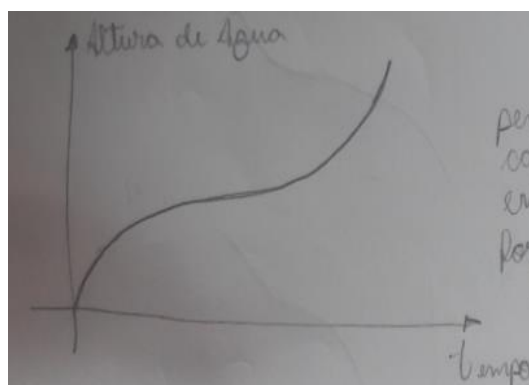


Figure 1 – Sketch proposed by the group for the relation between time and height

Source: Students' written production.

The discussion allowed the group to think about quantities that vary not from tables of numbers that increase or decrease their values, but rather from objects that change continuously, an essential aspect of CR. From the way he made the strokes on paper, we infer that the student understands that, in this situation, the quantities involved vary continuously, since, instead of marking points and joining them, he builds a continuous curve, showing that understands the continuous and smooth variation of the magnitudes involved (Thompson & Carlson, 2017; Frank, 2017).

If we observe the A2 lines, the student gradually refines his statements. In section [4.2] and later in [4.7], he reformulates his arguments. It is worth noting that in [4.9], the student argues, making sections at different heights of the bottle, and tries to convince the other members, and his suggestion of representation converged with his arguments. Here, we stand out the “argument and negotiation of meanings for his reasoning” (Rodrigues, Menezes & Ponte, 2018, p. 399) contributing to conceptualize two or more quantities that vary simultaneously (Thompson & Carlson, 2017).

#### *Event 5 — relating other magnitudes*

The group started the discussions from item (d)

[5.1] A2: Build graphs that relate the different magnitudes involved in this situation.	
[5.2] A1: Radius and [...]	i
[5.3] A2: Radius and graphhhh, radius is constant.	
[5.4] A1: Radius is not constant.	ii

[5.5] A2: Radius is constant, it doesn't do that, look. (it was not possible to identify where he points to).	
[5.6] A1: Indeed.	
[5.7] A2: Diameter that is not constant	
[5.8] A1: Diameter is only when there are two radii.	
[5.9] A2: So both are constant. [...] No need to calculate, just think.	iv(a)
[5.10] A1: But yeah, the radius, when sectioning this vessel into horizontal parts, endless horizontal parts, these radii from the sectioned parts will change, I just don't know how to write that.	
[5.11] A2: Go like this, look (points to the graph in figure 4).	

Observing the A2's lines in [5.3], [5.5] and [5.7], we notice that, at all times, the idea was that the radius was constant and the diameter was not, however, with A1's comment in [5.8], A2 changes his mind, just observe [5.9]. The graphic representation of the above situation was symbolized by a parabola with downward-facing concavity. This idea is the result of A1's discussions, mainly in [5.10], and shows a reflection made by this student in order to recognize the existence of an invariant relationship of two quantities that vary simultaneously, so that, "in the person's conception, each value of a quantity determines exactly one value of the other" (Thompson & Carlson, 2017, p.444). Below is the graph shown by the group, Figure 2.

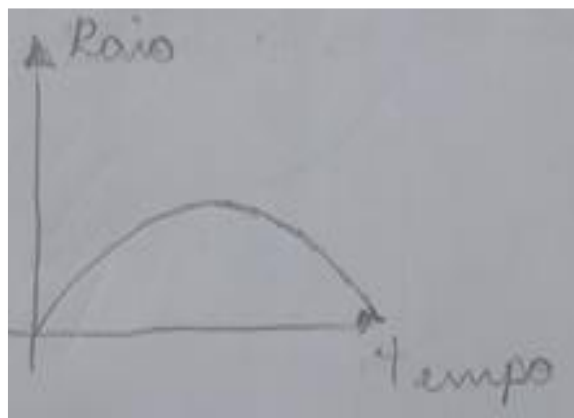


Figure 2 – Sketch proposed by the group for the relation between time and height.

Source: Students' written production.

The group continued to discuss item (d) of the task, now thinking about the graphic representation that relates the volume of water as a function of time.

[5.11] A2: Volume is constant too, right?	ii, iii
[5.12] A1: The vessel volume is constant.	
[5.13] A2: Yeah, so you can do it in the time and volume graph.	iii, iv(a)
[5.14] A1: Wait, then, it's the volume of water.	
[5.15] A2: Then, to do it in the graphic [...] it is a straight line, the volume is the same. Height that is like that [mentioning the graphic of item c]. The volume is constant.	

The group seems to have no doubts in stating that the graph of the situation will be a straight line, since the two magnitudes observed are constant [5.11], and shows the graph of Figure 3 as a result of the brief conversation.

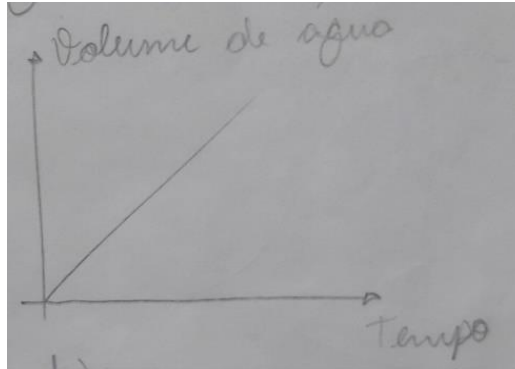


Figure 3 – Sketch proposed by the group for the relation between time and volume.  
Source: Students' written production.

After thinking about what magnitudes could be measured in the bottle situation, they were faced with two different magnitudes from those that had been foreseen in the planning. The group then started an interesting and intriguing discussion.

[5.16] A1: If the radius increases... the growth	i
[5.17] A2: I think this relation that we made with this radius will complicate life.	ii
[5.18] A1: No, it will be the same here. Look, if the radius increases... speed of growth... It's not just height, it's speed of growth.	iv(a)
[5.19] A2: If the radius increases, the growth decreases.	
[5.20] A3: The volume as a function of time?	
[5.21] A2: No, volume is constant...the radius...like, if you take every little dot.	
[5.22] A2: Decreases... When the radius is larger, it takes longer to fill.	iv(a), iv(b)

If we look at [5.17], A2 realizes that correlating the magnitudes in question will be a handful, perhaps because the radius grows and decreases after a certain moment (when it reaches the largest circle of the cross section). But A1 is convinced of the idea of the situation, according to [5.18]. We present Figure 4 to illustrate the graph resulting from the comment [5.22].



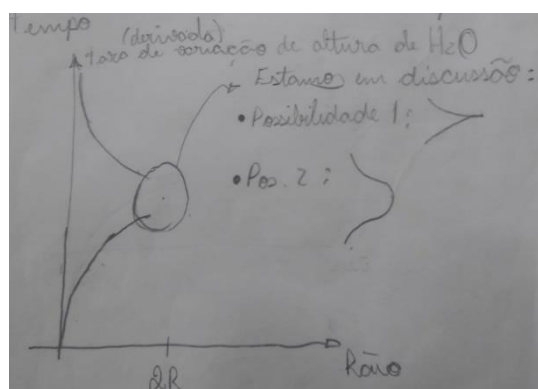


Figure 4 – Sketch proposed by the group for the relation between radius and time.  
Source: Students' written production.

The group discussions continue, now with the presence of the researcher.

[5.23] A1: So, the radius is going to be a magnitude that will constantly increase, but the speed.	iii
[5.24] Researcher: Does it (radius) grow all the time?	
[5.25] A1: Yeah, so, up to half here. The radius only goes up to one point, it has a limit,	iv(d)
[5.26] A2: It grows and then decreases again.	iv(b)
[5.27] A1: The radius has a limit. But the speed...	
[5.28] A2: The speed, it is fast, like, in the beginning, in the first section, it increases quickly, it is higher and then, when it gets closer to the diameter, it decreases, it gets smaller, then it increases again.	iv(b)

Looking at the dialogues, we can see that for A1, the closer the sections are to the diameter, the lower the growth rates of the water height inside the container [5.38]. In this situation, a quantitative operation takes place, which shows the understanding of the situation (recognition that a quantity grows/decreases, as well as the “way” how it grows/decreases).

We can see that the student has as “obstacle” the question of the radius increasing and decreasing over time ([5.26]). By relating it to height, one reaches the representation of a situation that is mathematically a function, and the “obstacle” is possibly due to the fact that, often, such relations are ignored (or artificially constructed) in Mathematics classes in Elementary Education and also in the CDI classes, although essential to the development of CR.

## Conclusions and final considerations

In this section, we showed a summary of the results observed, based on the analysis of the events where the data were organized. To this end, we sought to answer the research objective, i.e., to discuss CR skills that were mobilized during the mathematical discussions triggered by the work with mathematical tasks.

Although there are items in the task that explicitly involve the time variable, its main objective was to explore the relation between the height and volume of the bottle. Thus, instead of asking to build a graph that would relate these two variables (Carlson & Thompson, 2017), the task was organized through items that mobilize the elements necessary for this construction. Initially, the following requests were made: an explanation of the concept of constant spill rate; the identification of magnitudes that could be measured in the situation; the construction of a graph that related the height of the water to time and the construction of graphs that related the magnitudes involved, identified by the group of students.

For the group of students under review, we recognize that the task enabled the exploration of the skills expected. The group recognized that the radius can be thought of as an amount that varies with time, as well as volume. The students also established a relationship between height and volume, constructing a representation in which the inversion in the graph concavity showed an understanding of the existence of a change in the variation rate in that situation.

For example, when one of the students mentions that “it (the height) grows fast in the beginning, then it decreases and then it comes back faster”, it seems to us that this speech becomes a “trigger” for a discussion among some members of the group. With this, we see that the students recognize that the magnitudes of time and height would covariate and that there is a change in the growth of height over time. By observing the graphical representation used, as well as the mathematical discussions made, we understand that the students were able to graphically represent the situation without thinking discreetly, a fact necessary for developing CR (Carlson, 2002; Thompson, 2016). The idea of covariation between the “height and radius” quantities of the cross sections (aspect not foreseen when preparing the task) was present in the group’s discussions. While the height is an ever increasing amount, the radius sometimes increases, sometimes it decreases over time.

Regarding the skills unveiled when working on these tasks, the data analysis showed that, to a lesser or greater extent, all those that were previously listed were present in the discussions of the selected group. Of these, the *constitution of quantities* involved in the situation and *reasoning about the process of measuring* these quantities were ideas that were present in the group’s discussions, when dealing with the task. In this discussion, the quantity (volume) and height of the water in the bottle were recognized as attributes of the situation, capable of measurement, as explicit variables that would covariate over time. It is also worth mentioning the recognition of the radius variation of the cross sections of the bottle over time, an amount that had not been thought of while preparing the task. Such identification was essential for the investigation of the graphic representation of the relationship between these variables, as well as the investigation of these graphs’ concavity. Furthermore, they mobilized the understanding of the variation rates involved in the proposed context. The construction of the graph that related time to the radius of the water surface in the spherical

part of the vessel led the students to recognize a situation that, mathematically, is not a function.

Recognizing that the group under analysis was able to *imagine measures of quantities varying continuously* was an aspect we had some difficulty dealing with, especially due to the way the data was collected (use of audio). Such inference occurred because, at several times, in their many representations, the group sketched continuous curves (Figures 3 to 6) representing the relationship between the variables involved, showing the recognition of a variation.

The task allowed students to move between several forms of mathematical representation (Thompson & Carlson, 2017; Bridge, 2005), thus enhancing the exploration of ideas related to the *coordination between the quantities involved*, in particular, the recognition that there were quantities involved in the situation that could be related. We recognize that this aspect highlights the potential of the task, proposals to redefine the concept of function, since they broaden the approach usually present in both high school and CDI text books that deal with this concept, since, in general, they give too much priority to algebraic language.

The recognition of the direction of growth, the existence of variation rates and any changes in this rate was present in the discussions. Thus, reflections based on expressions present in the statement, for example: “continues to grow, but at a decreasing rate”, were important in mobilizing these ideas.

However, at several times, although the students seemed to verbalize their understanding of the situation, they failed or “got stuck” in the representations. The comment “I think this relationship that we made with this radius will complicate life” arises when they try to draw a possible graph that does not seem to illustrate a situation that can be represented. At several times, while doing the task, the group came across situations that, to them, seemed not to be capable of graphic representation.

In an informal way, it seems that, although in their speeches they seemed to recognize the covariation among the magnitudes involved, they were not able to create a graphic representation. This occurred at the time when the group sought to represent the relationship between the radius of the cross sections and time, a relationship that is not classified as a function (there are always two symmetrical “moments” in relation to the middle of the bottle in which the radii coincide). Thus, the concavity of the graphs “appeared” more as a consequence of linking points marked in the Cartesian plane, or by inferring the growth relationship between the variables involved, than by analyzing how these variables grew.

We also highlight the role played by the discussion among the students during the task (Bridge, 2017), creating conjectures regarding the representations that were reviewed and redrafted during the work and, also, the potential of the proposed task, which, due to its exploratory character, proved to be of high cognitive demand (Stein & Smith, 2009). We also realized that the tasks of high cognitive demand enabled the development of CR skills and the

design of graphics as emerging representations of constantly changing phenomena (Frank, 2017).

Finally, if we look at the several graphical representations mobilized, we can infer that, although the analyses showed that the students understood elements related to CR, they did not know how to graphically represent the relationship between the enmeshed magnitudes. This is an aspect that has emerged from our data and remains open, demanding greater theoretical depth, both in the pursuit to understand it, and in order to redraft the tasks statements or sequencing.

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## References

- Barufi, M. C. B. (2012). *A construção/negociação de significados no curso universitário inicial de Cálculo Diferencial e Integral*. Tese de Doutorado em Educação. São Paulo: Universidade de São Paulo.
- Bogdan, R. C., & Biklen, S. K. (1994). *Investigação qualitativa em educação*. Porto: Porto Editora.
- Brockman, J. B. (2010). *Introdução à engenharia: modelagem e solução de problemas*. Rio de Janeiro, RJ: LTC.
- Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: a framework and a study. *Journal for Research in Mathematics Education*, 33 (5), 352-378.
- Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics*, 26, 135–164.
- Confrey, J., & Smith, E. (1995). Splitting, covariation and their role in the development of exponential function. *Journal for Research in Mathematics Education*, 26, 66–86.
- Couto, A. F., & Trevisan, A. L. (2017). Cálculo interativo: um ambiente virtual de suporte às aulas de cálculo diferencial e integral. *Hipátia*, 2(1), 16-25.
- Couto, A. F., Fonseca, M. O. S., & Trevisan, A. L. (2017). Aulas de cálculo diferencial e integral, organizadas a partir de episódios de resolução de tarefas: um convite à insubordinação criativa. *REnCiMa*, 8(4), 50-61.
- Frank, K. M. (2017). *Examining the Development of Students' Covariational Reasoning in the Context of Graphing*. Dissertation of Doctor of Philosophy. Tucson: Arizona State University.
- Gil, A. C. (2010). *Métodos e Técnicas de Pesquisa Social*. São Paulo: Atlas.
- Gonçalves, W. J. (2018). *Raciocínio covariacional em aulas de Cálculo Diferencial e Integral: possibilidades de desenvolvimento a partir do uso de tarefas*. Dissertação de

Mestrado Profissional em Ensino de Matemática. Londrina: Universidade Tecnológica Federal do Paraná.

- Mestre, C. M. M. V. (2014). *O desenvolvimento do pensamento algébrico de alunos do 4.º ano de escolaridade: uma experiência de ensino*. Tese de Doutorado em Educação. Lisboa: Universidade de Lisboa.
- National Academy of Engineering (NAE). (2005). *Educating the Engineer of 2020: Adapting Engineering Education to a new Century*. Washington, DC: National Academies Press.
- Mkhatshwa, T. P. (2019). Calculus students' quantitative reasoning in the context of solving related rates of change problems. *Mathematical Thinking and Learning*, 22 (2), 139-161.
- Ponte, J. P. (2005). Gestão curricular em Matemática. In: GTI (Ed.), *O professor e o desenvolvimento curricular*. Lisboa: APM.
- Ponte, J. P. (2014). Tarefas no ensino e na aprendizagem da Matemática. J. P. PONTE (Org.), *Práticas Profissionais dos Professores de Matemática* (pp. 13-27). Lisboa: Instituto de Educação da Universidade de Lisboa.
- Ponte, J. P. (2017). Discussões coletivas no ensino aprendizagem em Matemática. In: GTI (Ed.), *A prática dos professores: planificação e discussão coletiva na sala de aula* (pp. 33-56). Lisboa: APM.
- Powell, A. B., Francisco, J. M., & Maher, C. A. (2004). Uma abordagem à análise de dados de vídeo para investigar o desenvolvimento de ideias e raciocínios matemáticos de estudantes. *Bolema*, 17(21), 81-140.
- Rodrigues, C.; Menezes, L., & Ponte, J. P. (2018). Práticas de Discussão em Sala de Aula de Matemática: os casos de dois professores. *Bolema*, 12 (61), 398-418.
- Saldanha, L., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In: S. B. Berensah & W. N. Coulombe. (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education*. Raleigh, NC: North Carolina State University.
- Speer, N., & Kung, D. (2016). The complement of RUME: What's missing from our research? In T. Fukawa-Connelly, N. Infante, M. Wawro & S. Brown (Eds.), *Proceedings of the 19th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp. 1288–1295). Pittsburgh: Pennsylvania.
- Stein, M.H., & Smith, M.S. (2009). Tarefas matemáticas como quadro para reflexão. *Educação e Matemática*, 105, 22-28.
- Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, 10(4), 313-340
- Thompson, P. W. (1990). *A theoretical model of quantity-based reasoning in arithmetic and algebraic*. Center for Research in Mathematics & Science Education: San Diego State University.
- Thompson, P. W., & Thompson, A. G. (1992). *Images of rate*. Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco, CA.

- Thompson, P. W. (1994). Students, functions, and the undergraduate mathematics curriculum. In: E. Dubinski, A. H. Schoenfeld & J.J. Kaput (Eds.), *Research in collegiate mathematics education I* (pp. 21-44). Providence: American Mathematical Society.
- Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In: L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in Mathematics Education*. (pp. 33-57). Laramie, WY: University of Wyoming Press.
- Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp.421-456). Reston, VA: National Council of Teachers of Mathematics.
- Trevisan, A. L., & Mendes, M. T. (2013). Possibilidades para matematizar em aulas de Cálculo. *Revista Brasileira de Ensino de Ciência e Tecnologia*, 6, 129-138.
- Trevisan, A. L., & Mendes, M. T. (2017). Integral antes de derivada? Derivada antes de integral? Limite, no final? Uma proposta para organizar um curso de Cálculo. *Educação Matemática Pesquisa*, 19(3), 353-373.
- Trevisan, A. L., & Mendes, M. T. (2018). Ambientes de ensino e aprendizagem de Cálculo Diferencial e Integral organizados a partir de episódios de resolução de tarefas: uma proposta. *Revista Brasileira de Ensino e Tecnologia*, 11(1), 209-227.