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Untying knots: developing visualization from twisted string experiments

Desatando nós: desenvolvendo a visualização a partir de experimentos com cordas torcidas

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Abstract

This paper presents the results of a qualitative research, whose design fits the strategic principle experiments and quasi-experiments, carried out in 2019, with the aim of investigating how participants in a geometry research group would construe in a intuitive, imaginative and creative way knots in ropes randomly dropped onto a flat surface. Through a rhetorical analysis of the group participants records, taking into consideration the proposed activities, it was possible to conclude that visual skills can be explored in both initial and continuous action with the group participants, starting from the imagination and intuition when raising a hypothesis. Using the cords available the participants could prove or reject their hypotheses. We conclude that even more advanced content, such as homotopy alongside with knot theory, can be explored at different levels of education, which can be a facilitating element not only in geometry studies but also in different fields of mathematical knowledge.

Keywords: Knots theory; Ropes; Geometry.

Resumo

Este artigo apresenta resultados de uma pesquisa de cunho qualitativo, cujo delineamento se enquadra no princípio estratégico por experimentos e quase experimentos, realizada no ano de 2019, com o objetivo de investigar como participantes de um grupo de estudos e pesquisa em Geometria interpreta intuitiva, imaginativa e criativamente nós obtidos por cordões ao serem soltos aleatoriamente sobre uma superfície plana. Por meio de uma análise retórica dos registros dos participantes do grupo, ante às atividades propostas, foi possível concluir que habilidades visuais podem ser exploradas tanto na formação inicial quanto em ação continuada com os participantes do grupo, partindo da imaginação e da intuição no levantamento de hipóteses. A partir de barbantes disponibilizados os participantes puderam comprovar ou rejeitar suas hipóteses. Concluímos que, mesmo conteúdos mais avançados, como homotopias junto à teoria de nós, podem ser exploradas em diversos níveis de ensino, o que pode ser um elemento facilitador não somente na formação geométrica dos indivíduos, como em outras áreas do conhecimento matemático.

Palavras-chave: Teoria de Nós; Cordões; Geometria.

Introduction and Rationale

The Study and Research Group in Geometry – GEPGEO, led by the first author, has been searching, together with the Graduate Program of Sciences and Mathematics Teaching

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of the Franciscana University for theoretical and methodological alternatives for teaching and learning Geometry at different educational stages. We understand that it is not enough to advance Geometry disciplines, either in Mathematics degree, bachelor's degree or even in continuous action, that do not keep up connections with school practice.

When teaching mathematics at any stage, it is possible to work with students differently than traditional. One way to do so is by connecting with the basic intuitions students already have. These connections can occur in two ways. First way, students should be encouraged to make assumptions and produce mathematical knowledge on their own, without the aid of standard strategies or via teacher. A second way is to delineate teaching in order to nurture connections to students pre-existing knowledge through manipulative materials and or experiments.

According to Fischbein (1987), experience plays a fundamental role in shaping intuitions because, under certain circumstances, it shapes stable expectations. It is observed that by doing classroom experiments, students are allowed to contextualize or concretize abstract concepts and to connect with essential elements in mathematics in a more intuitively way.

> Regarding geometry teaching, the same author states that intuition, together with visualization, is one of the main components of our cognitive efforts." It seems that formally based qualities of certainty, coherence, consistency, necessity, etc., do not possess the same kind of stimulating, convincing, and productive capacity as intrinsic credibility, intrinsic structurality and the richness of real phenomena. This is what Hilbert, one of the great founders of axiomatic himself, clearly stated: Who does not always use, together with the double inequality a> b> c, the image of three points following each other in a straight line like the geometric image of the 'in between' concept?" (Fischbein, 1987, p.17).

This way, one may observe that, in mathematics, the popular saying: "a picture is worth a thousand words" is specially true where a picture or some other kind of experiment can be a handy tool in describing a concept.

Being aware of the significance of visualization and experimentation, we justify the present article, that we started from Conway *et al*. (1991), which starts exploring bicycles, pedals and belts in a very intuitive and creative way. From there, the group discussed, analyzed and performed activities, justifying the present article, which aims to investigate how participants in a geometry research group construe intuitively, imaginatively and creatively knots that comes into view when ropes are randomly dropped onto a flat surface.

Theoretical foundation

Initially, we will discuss some considerations about creativity and imagination in geometry and then discuss knots. Granger (1998) already addressed imaginative creation as experience expressing that it "does not consist in a state of passive visualization, but of active experience" (p.11). This, taken to geometry teaching, seems to to be fundamental to achieve the desired changes in this field, specially due to its rejection in school environments and

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teacher education as indicated by Leivas (2009). In turn, the author continues his speech as follows:

> [...] even in mathematics, the imagery quantity was introduced by Descartes in his geometry to designate the roots of the irreducible cubic equation, which cannot be calculated by Cardano's formula because they would require the extraction of square roots from complex numbers. Such entities do not correspond, therefore, to objects of ordinary arithmetic. But the operation then possible proves out in a new object system, the 'complex' numbers. [...] (Granger, 1998, p.11).

Yet invoking Descartes, we take back creative imagination concept by enlarging the dimensions of the Euclidean geometric entities: point, straight line and space, perceptible in the real world, to dimensions larger than 3, existing only in the theory of ideas, through ordered n-tuplets.

Prior to this, Hilbert and Cohn-Vossen (1999) associated geometry with imagination as follows:

> [...] it is our goal to present geometry, as it stands today, in its visual and intuitive aspects. With the help of visual imagination, we can illuminate the variety of facts and problems of geometry and, furthermore, it is possible, in many cases to portray the geometric outline of research and demonstration methods, without necessarily going into details related to the strict definition of concepts and with real calculations (Hilbert & Cohn-Vossen, 1999, p. iii).

Regarding the theory of ideas, Leivas (2009, p.111) portrays imagination "as a form of mental conception of a mathematical concept, which may come to be represented by a symbol, a visual, algebraic, verbal layout or a combination thereof, with the purpose of communicating the concept to oneself or to another person" (). The author goes further by defining visualization "as a process of forming mental images, with the purpose of constructing and communicating a certain mathematical concept, with a view to assisting in analytical or geometrical problems resolution" (Idem, p.111). Apparently the author goes against Hilbert and Cohn-Vossen, as well as Galton in approaching the visual creative imagination.

As for the influence of creativity in teaching-learning mathematics, Segura (2012, p.70) wonders about resorting to creativity in this field. She states that "An education can be said to be creative when the teacher who carries it out encourages and empower the students so they can investigate and rediscover their own knowledge, induce co-working, set up their own knowledge".

According to Moreno & Azcárate (2003) traditional teaching methods that involve definition, demonstration, and activities with closed problems with predetermined answers, produces ill-prepared students in mathematics. Teachers, by teaching this subject without creativity, do not allow students to appreciate its beauty.

Regarding excessive stringency of formulas, Carvajal (1981) states that it is necessary, among other aspects, to exercise imagination, stimulate mental activity, not giving

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priority to hard-lines techniques in learning, and that this has been the case, for a long time, in teaching of drawing, and what we agree to be still present in teaching of mathematics, in general, and with a greater intensity in geometry, in which, almost exclusively, the Euclidean is known both in primary education and in math teachers training.

In addressing visualization, Cunningham (1991) refers to this ability, in a very broad sense, even in ancient Greece, when geometers sketched their diagrams in the sand. For this author, mathematics began to move beyond intuitive areas. However, visuals were considered weaknesses. For example, deficiencies in Euclid's axiomatic were hard to be eliminated since standardized flat figures made Euclidean axioms self-evident. For him, "[...] restoring the visual and intuitive side of mathematics opens new possibilities for mathematical work, especially nowadays when computing has enough power to support it with accurate representations of problems and their solutions." (Cunningham, 1991, p.70). Going a little further, Cunningham (ibid.) states that "[...] adding visualization to mathematics education promotes intuition and understanding and allows a wider range of coverage of mathematics subjects. But it provides more than this."

But what are knots? Obviously, this questioning, out of context, can lead to several different answers. It is our interest, especially in this article, to approach such concept in mathematical context and, more directly, in teaching of Geometry, in a visual, imaginative and creative way, without a mathematical deepening that, in our view, does not apply to the context that we're working on.

It is related to topology, perhaps one of the most complex topic in Mathematics, which is generally not addressed or studied in education programs, but in bachelor's degree programs, concerning to research on pure and applied mathematics field. It plays an important role in fundamental group and covering spaces, therefore, the possibility of verifying when two or more spaces are homeomorphs, that is, finding a continuous bijective inverse function that transforms into each other. In undergraduate courses, this can be explored geometrically, in a simple way, in elementary disciplines of Differential and Integral Calculus, since the study of functions and continuity is part of this area. We can illustrate this as follows.

A closed range [a, b] of real numbers cannot be homeomorphic to an open range (a, b), for instance, because there is no bijection between both. One way is to verify the first one is a compact set while the second one is not, and this is a subject of real analysis branch, usually included in the Mathematics education programs curriculum. A second example can be explored in algebra courses, the absence of homeomorphisms between the straight line (R as a numeric set) and the plane (R^2 as a set of ordered pairs). Removing a point from the plane (a pair in R^2) this space remains connected while removing a point from the line (an R element) it becomes disconnected, so these two sets are not homeomorphs.

Let [0,1] be the range of real numbers, X is any set and f: [0,1]$\rightarrow$X is a continuous function so that $f(0) = x_0$ and $f(1) = x_1$. According to Munkres (1975), the function f is called a path in X from x_0 to x_1. A smoother way to approach this subject is through flat diagrams,

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i.e., starting with what are called pathways. If f and f' are two paths in X, then there is a stronger relationship between them than simple homotopy.

If f and f' have the same start and end points, where I = [0,1] and if there is a continuous function F: I X I → X so that

F(s,0) = f(s) and F(s,1) = f'(s)

F(0,t) = x_o and F(1,t) = x_1, $\forall s \in I$ e $\forall t \in I$, the function F is called path homotopy (Figure 1).

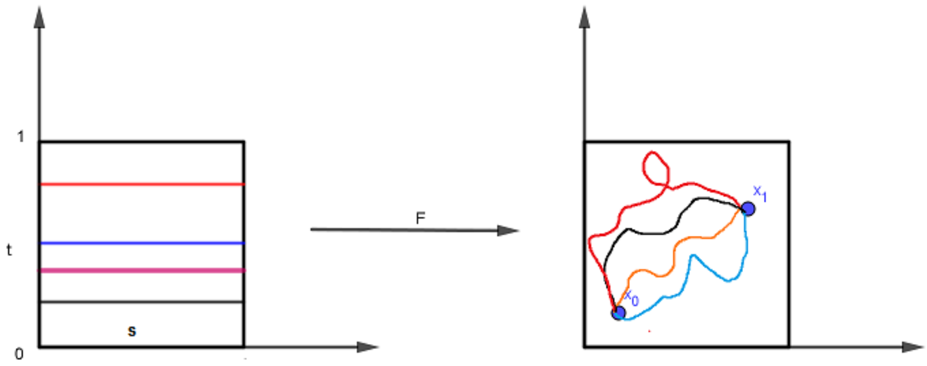

Figure 1 – Homotopy of ways.
Source: adapted from Munkres (1975, p.319) by the authors.

Let the space X be taken as R^2- (0,0), geometrically, the perforated plane and the functions f(s)= (cos(s), 2sin(s)), $s \in [0, \pi]$, whose graphic representation is a semi-ellipse in the upper semiplane; g(s)= (cos(s), sin(s)), $s \in [0, \pi]$, whose representation in the upper semiplane is a semicircle; and h (s) = (cos(s), -sin(s)), $s \in [0, \pi]$ which is the semicircle in the lower semiplane.

In Figure 2, we illustrate the four representations of the paths. In it we observe that f and g are homotopic paths in R^2- (0,0) as well as with the vertical axis. However, the vertical axis does not represent a homotopic path with semicircle in the lower semiplane given by h (s) since there is no image at R^2- (0,0) for all points of this axis.

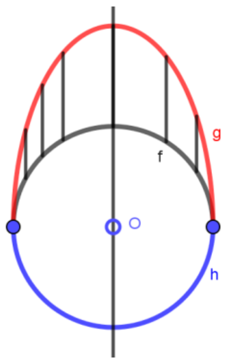

Figure 2 – Homotopy and non-homotopy in the perforated plane.
Source: adapted from Munkres (1975, p.321) by the authors.

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According to Conway *et al*. (1991) we can take an extension cord, and connect one end to the other, which will produce a loop (a knot in mathematical sense). This makes one wonder whether or not it can be untied without disconnecting or breaking the cord. Another intuitive loop (knot) test can be peformed using a thin chain necklace. We can throw it loosely in a crowded drawer, then pick it up without any care and place it on a flat table. We will see that it is all curled up or entangled when we try to make it in a flat closed curve representation. The third experiment is to take a string about 1m long and do the same as with the necklace and the cord.

In these hands-on experiments we will be able to find that the loops (knots) obtained may or may not be undone and, if they may, without a cutting or crossing them with themselves, they are considered 'false knots'. By this, we come to equivalent loops (knots), that is, if it is possible to rearrange each other without cutting and without allowing a passing through itself to which we mathematically associate to the homotopy of paths. (Figure 1).

In the case of loosening the cord on the table, overlapping it and subsequently trying to rearrange it, we will verify the fewest possible crosses that will occur with the cord (Figure 3).

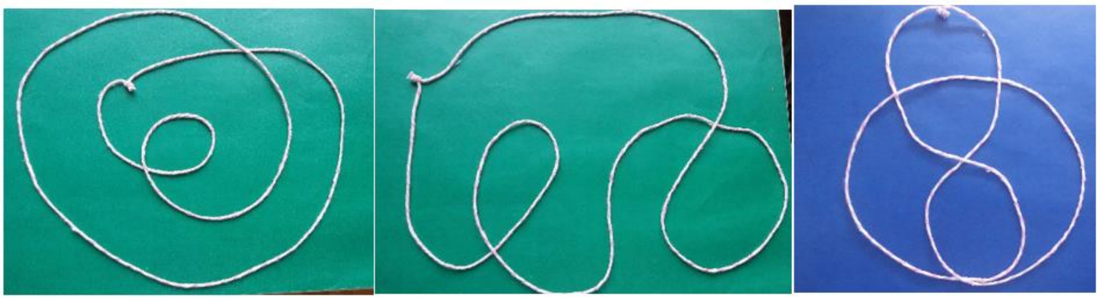

Figure 3 – Two points self-intersected rope.
Source: group collection.

From the experimental, intuitive and visual aspects illustrating loops (knots and paths), in what follows, we delineate the concept. Munkres (1975, p.326) defines it as follows: “Let X be a space and x_0 a point in X. A path in x_0 that starts and ends in x_0 is called a loop based in x_0”.

Methodological procedures

This article aims to investigate how participants in a geometry research group intuitively, imaginatively and creatively interpret 'knots' in ropes randomly dropped onto a flat surface. This is a qualitative research whose design fits the strategic principle ‘experiments and quasi-experiments’ according to Bauer and Gaskell (2015). Regarding the instruments for data collection, we took into account researches’ observation and documents surveyed, whilst the analytical treatment of data was carried out trhough rhetorical analysis, from the perspective of Dittrich (2016). As for knowledge interests, the experiment was based on a consensus building approach and research subjects emancipation. These four

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factors are summarized in Table 1.

Table 1 – The four dimensions of research process

Design principles	Data generation	Data analysis	Knowledge Interests
Experiment	Audiovisual Records (photos)	Content analysis	Emancipation and empowerment

Source: adapted from Bauer and Gaskell (2015, p.19)

As it is an experimental register, we understand that collecting experment images and analyzing them according to the theory, provides possibilities of interpretations *in locus*, which is presented as possibilities for a work group. To Penn (2015, p.319), "[...]Semiology provides the analyst with a set of conceptual instruments for a systematic approach to sign systems in order to discover how they produce meaning. Much of its accuracy comes from a series of theoretical distinctions that are captured through a specific vocabulary".

Thus, exploring the photographic records of participants and experiment, in natural language may help the researcher to interpret them based on the conceptual theoretical foundations of the subject under consideration.

Regarding to content analysis, Bauer (2015) claims that it is just a method for analyzing text within empirical social sciences and, as Barker (1964) states, empiricism needs experience, consisting of experimental rather than rational thinking leading to inductive method. This kind of knowledge needs experiment justification, and this is would be the main focus of the study and research group in question.

Ten individuals took part in the activity. At first, researchers randomly distributed a data entry sheet containing two activities, which will be described and analyzed in the next item. The sheets were identified with one of the letters: A, B, C, D, E, F, G, H, I and Y, not sequenced, in order to preserve participants' identities.

Using arguments, narratives, records and comments, each activity had an initial part with alternatives to choose from and a second part with its justification, as well as photographic records all sent to the researcher. Thus, we will have the research text *corpus*, which, for Bauer (2015, p. 192), "[...] is the representation and expression of a writing community. Based on this, the result of a content analysis is the dependent variable, the thing to be explained." The data collected will be analyzed next.

Data analysis

Initially, using a multimedia projector, researchers displayed the image (Figure 4) and instructed students to observe the trivial knot: a closed loop with zero knots forming a circle, and the trefoil knot: a closed loop with three knots forming a trefoil.

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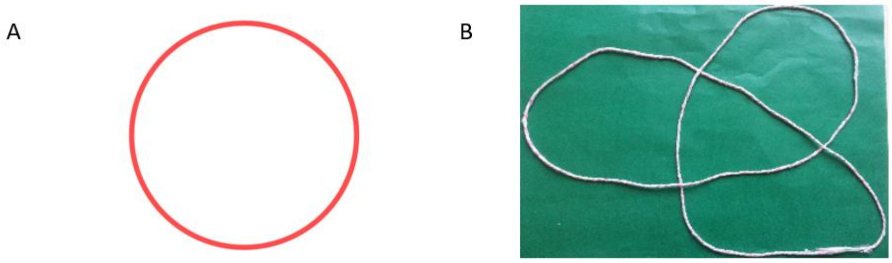

Figure 4 – Projected image.
Source: group collection.

Students were asked then to indicate in the record sheet the answer to the following question.

Can you imagine the transformation from one knot to another without breaking the rope:
( ) yes ( ) no (..) maybe ( ) I can't even imagine ( ) I'd like to try

This activity aimed to verify whether the participants would choose criteria to determine whether or not the knots observed were equivalent.

The data collected showed that none of the participants answered *yes*, *maybe* or *I can't even imagine* to the first question (Chart 1). Only one participant answered *no*; six answered *I'd like to try* and the other three, indicated both *no* and *I'd like to try*.

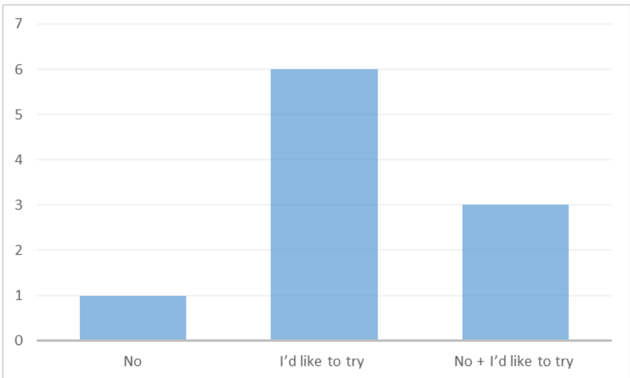

Chart 1 – Answers distribution to activity 1.
Source: research data.

Then, in the same activity, they were asked: If you also answered *I'd like to try* take the rope provided by the teacher and confront your initial hypothesis. We observed that the rope provided by the teacher was apparently a trivial rope (Figure 4A), there was already a knot previously made on it. Thus, participants should try to turn a trefoil knot (Figure 4B) into a trivial knot without untying the rope. The participant G who was the only one to

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answer just *no*, expressed:

G: *When teacher handed me the rope, it had already a knot, i.e. it was not circular (trivial knot). As it was possible to create the trefoil knot, we believe that it will not be possible the transformation from one into another without breaking it.*

Among those who answered *no* and *I'd like to try*, we find the following records:

A: *The transformation could not be performed*.

D: *I figured it wouldn't be possible, because we have knots on different sides and when we try to untie them, we get back to the "initial" situation. In other words, when you undo the knot on top, it comes underneath and vice versa*.

E: *[...]There is no such possibility, since one of the knot lines passes above and the other one passes below. By trial and error, we get to the point of having a knot with another one inside. The following figure confirmed my hypothesis (figure 5).*

Figure 5 – Answer from Student E.
Source: research data.

In these records, we can observe that, even without knowing knot theory, the participants realized something very important and well-grounded in scientific literature, the well-known Reidemeister move that are local transformations (moves) of knots that transform one into another equivalent. There are 3 types of moves as illustrated in Figure 6.

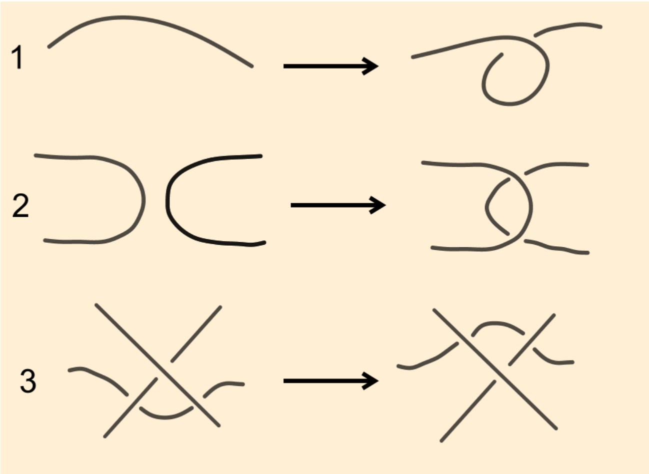

Figure 6 – Reidemeister moves.
Source: authors.

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The others answered simply *I'd like to try* without risking to figure it out.

B: *When moving the top which passes over, we find the following figures (Figure 7):*

Figure 7 – Answer from Student B.
Source: research data.

C: *It was impossible to undo the knot without breaking the rope*.

F: *I could not untie the knots either by passing over or trying to untwist it. Always go back to the starting position or the knots overlap and there are two circles, or there is only one knot and a circle.*

H: *I could only move into two overlapping circles, but couldn't make one with zero knots* (figure 8).

Figure 8 – Drawing by Student H.
Source: research data.

We observed that all participants had the same kind of perception. We may call this intuition in the same sense treated by Hirza *et al*. (2014) that puts intuition as an immediate effort without reference use and the results considered as truth, so that the person using their intuition feels that there is no need to prove or justify their thinking. In this sense, mathematical knowledge and the justification for such statements is important. Such concern is reflected in the records of participants I and Y.

I: *I'm afraid it is not possible. We had to 'pass' the top line to the other side, which is not possible at this knot*.

Y: *I believe it is impossible to undo the knots this way, because in every attempt I returned to the same knot*.

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According to Fischbein (1987, p.90), "experience plays a fundamental role in shaping our intuitions, and this at least partially explains its impact on any productive, theoretical or practical endeavor. On the other hand, experience is always restricted to a limited system of circumstances and this contributes to limiting the domain of the reliability and effectiveness of intuitions".

These participants used the words "I'm afraid" and "I believe" in their records, which shows that intuition alone was not enough to determine the possibility of turning one knot into another. But by experimenting and visualizing, they were able to argue. Therefore, from discussions held in the group, it was possible to realize that, from the experienced situations, the participants could intuit that it was impossible to undo the knot just by rearranging the rope. That is, the discussions corroborate Janos (2009) in stating that if we have a trefoil knot on a rope, it will not be possible to undo it, that is, to turn it into a trivial knot. However, this does not necessarily constitute proof that these two nodes are not topologically equivalent, as perhaps there was not enough skill to turn one knot into another.

We observed in this experiment that the objective was not to prove the impossibility of transforming one knot into another, but to verify whether individuals would choose criteria to determine whether or not the two given curves were equivalent. After the discussions, we came to the conclusion that the given knots were not equivalent, because only by cutting the rope it would be possible to transform one knot into another, as shown in Figure 9.

Figure 9 – Knot transformation.
Source: authors.

In order to prove that two knots are topologically distinct, it is necessary, according to Stewart (2012), to find certain invariants, that is, properties that do not change with equivalence. The first and simplest of these properties is the tricolorability that allows us to distinguish trivial knot from trefoil knot. We observe that a knot is tricolorable if the arcs in its diagram can be colored with exactly 3 different colors, so that each intersection would be the meeting of 3 different colors or the same color, as illustrated in Figure 10. Obviously, trivial knots are not tricolorable.

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Figure 10 – Knot tricolorability.
Source: authors.

In the second activity, the researchers projected once more the trefoil knot image and the inverted or reflected trefoil: same number of trefoil knots (or as denominated in topology, left-handed trefoil and right-handed trefoil).

Similarly to the previous activity, in this experiment the objective was to investigate if participants could construe / intuit if the given knots would be equivalent. We emphasize that at no time the participants were given the name inverted knot.

Students were asked to record on the sheet provided an answer to the following question with alternatives, in which more than one could be chosen.

Are the knots represented (figure 11) equal to each other?

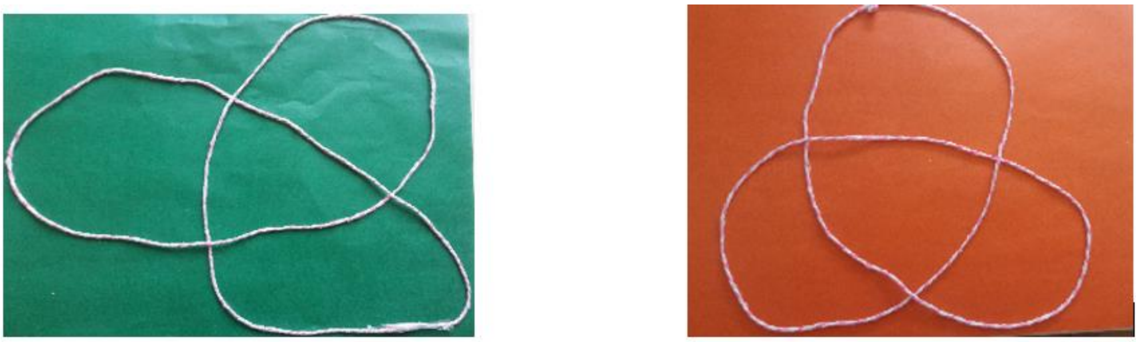

Figure 11 – Trefoil knots.
Source: research data.

( ) yes ( ) no (..) maybe ( ) I can't even imagine ( ) I'd like to

Chart 2 shows answers distribution.

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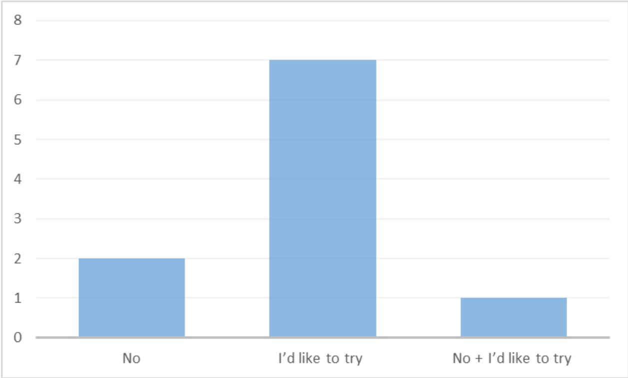

Chart 2 – Answers distribution to activity 2.
Source: research data.

We reiterate that none of them answered *yes*, as was the case in activity 1. Only two answered *no*, in contrast to only one in the first activity, and one answered *no* and *I'd like to try*, unlike in the first case when three of them had answered that way. The others answered *I'd like to try*. From this response, the following request was made:

If you have also answered 'I'd like to try' take the rope given and confront your initial hypothesis. What was the result? Is it possible to pass each other without breaking the rope?

Participant A, who have answered *no* and *I'd like to try*, offered both alternatives as follows:

A: *Could not perform the transformation without breaking the rope.*

It is not possible to verifiy if this participant did not see or understand what would happen or was unsure of offering the answer since she could have explored the material resource to ponder over her answer.

The two participants who only answered *no* made the following entries:

D: *As the knots are reflected, when trying to transform them into each other, the same problem occurred as in the previous activity.*

G: *The knots are opposite to each other, in the same arrangement, the knot that is above one, is below the other.*

Unlike the previous participant, it seems that both explored visualization as a mental construct (Leivas, 2009) and intuition (Fischbein, 1987) to provide answers to researchers.

The others, who had only opted for the alternative *I'd like to try*, expressed themselves as follows.

B: *When moving the part of a reflective knot, we have the following figures and either by rotation or translation, we do not get to the figure, so it is not possible to transform the reflected knot into a trefoil knot (figure 12).*

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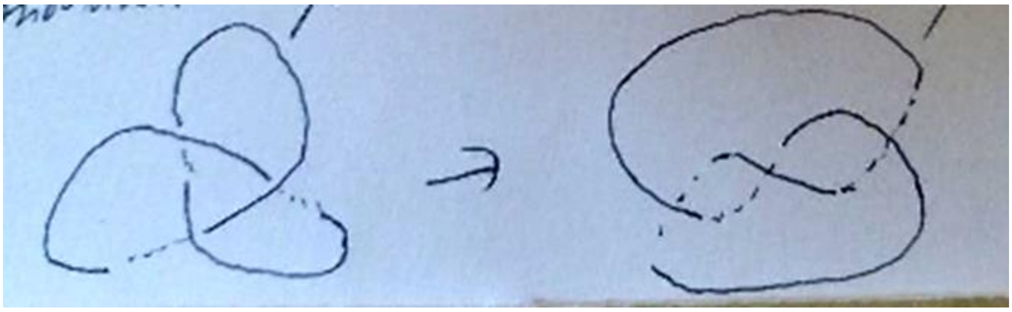

Figure 12– Drawing by Student B.
Source: research data.

C: *Again, knot number two could not be transformed into knot zero. Knots 1 and 2 are reflected.*

E: *It resulted in the same structure as the previous knot, but all knots are inverted, using rotation, translation among others moves, knots located in the same place of knot 1 and node 2, are always inverted to each other, no matter other's location*.

F: *The rose-colored seems to be the inverse of the green-colored, but I can't move from one to the other, nor turn it all backwards*.

H: *No, it seems that the knots are inverted "horizontally".*

I: *I'm afraid it is not possible.*

Y: *They are inverted, and it is also impossible to move from knot to knot.*

We verify, since most students chose to explore the didactic resource available to draw their conclusions and this does not mean that they did not intuit from visual skills, what could possibly be happening. In this sense, it is important that the investigator also explore, in some situations, the resource of the interview, in order to elucidate participants thinking.

There are actually two trefoil knots, right-handed and left-handed, which are mirror images of each other, as illustrated in figure 13.

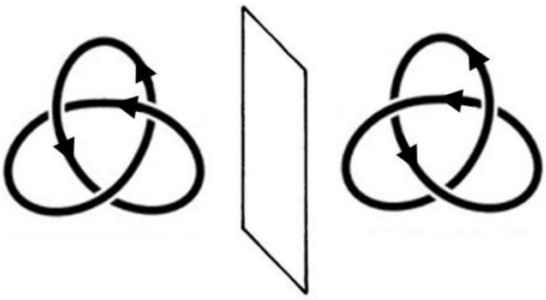

Figure 12– Right-handed and left-handed.
Source: authors.

It was demonstrated by Max Dehn, using group theory in 1914, that trefoil knots are not equivalent to each other.

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Final considerations

This article presented a qualitative research, carried out with a geometry study and research group with teachers in continuous action aiming to investigate how the participants would interpret intuitively, imaginatively and creatively 'knots' obtained by loosening ropes randomly on a flat surface.

Activities adapted from Conway *et al*. (1991) were proposed exploring knots originating from the release of a bicycle belt at random after being removed. These works aimed at geometry and imagination solely, to which the authors of this article added intuition.

Initially the group focused on its own belt and, later on, used strings as a didactic resource in an attempt to search for answers to the questions raised. Moreover, we came across the mathematical concepts involved in the process which led us to some aspects of knot theory; the pathway from topology; homotopic spaces and cover theory. However, However, we do not delve into theoretical-mathematical concepts since it is an investigation with teachers who work at different educational stages. The intention was to adapt activities of these advanced contents, not always studied in initial training courses, in a way that can be directed to elementary students, in an intuitive and visual exploratory way.

Research results initially showed that individuals have difficulty following intuition as a process that can evidence knowledge. This issue, however, needs proof of what visualization may contribute to the help people imagine situations that can occur concretely.

Initially, it was asked if the participants had the mental construct (visualization) of what was indicated in a certain situation; what they could intuit (conjecture) about what they had envisioned and, finally, exploit the educational resource provided by the researchers to prove or reject the hypothesis raised.

The results showed that it is possible to explore creativity and imagination to develop visual skills, since the theme approached is not often explored in the early years of math teachers training courses. Therefore, we consider that the research objective was achieved.

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