Zetetiké

DOI: 10.20396/zet.v29i00.8661816

Hypothetical Learning Trajectory as a resource for teacher education

Trajetória Hipotética de Aprendizagem como recurso para a formação de professores

Julio Cezar Rodrigues de Oliveira[1]

Pamela Emanueli Alves Ferreira[2]

Abstract

The purpose of this article is to present a theoretical essay and discuss aspects of a Hypothetical Learning Trajectories (HLT) with the goal to subsidizing the practices of mathematics teachers in pre-service or in-service education. The study is a qualitative research and, in the light of Martin Simon's theoretical assumptions, we seek to argue about the elaboration of HLT as a strategic methodological resource that enables pre-service or in-service teachers to organize their school plans based on hypothetical processing. To achieve these objectives, the following are presented: theoretical aspects of HLT; an HLT on teaching Logarithms; a discussion about the process of its development as an education strategy. As a result, it is inferred that HLTs provide opportunities for teacher's education to analyze aspects that delimit their practices in an articulated way, to get involved in a process of reflection and decision-making about their professional practice. The analyzes and reflections are supported by the argument that HLT is revealed as a potential resource for teacher training because it has an instrumental, flexible, interactive, predictive, problematic and dynamic character.

Keywords: Mathematics Education; Mathematics teacher education; Hypothetical Learning Trajectories.

Resumo

O objetivo deste artigo é apresentar um ensaio teórico e discutir aspectos de uma Trajetória Hipotética de Aprendizagem (THA) com vistas a subsidiar práticas de professores de Matemática em formação inicial ou continuada. O estudo se trata de uma pesquisa qualitativa e, à luz de pressupostos teóricos de Martin Simon busca-se argumentar a respeito da elaboração de THAs como um recurso estratégico metodológico que possibilita aos docentes em formação (ou em exercício) organizarem seus planejamentos escolares a partir de um processamento hipotético. Para atingir estes objetivos são apresentados: aspectos teóricos da THA; uma THA sobre o ensino de Logaritmos; uma discussão a respeito do processo de sua elaboração como estratégia de formação. Como resultado infere-se que as THAs oportunizam aos docentes em formação analisarem aspectos que delimitam suas práticas de modo articulado, se envolverem em um processo de reflexão e tomada de decisão sobre seu exercício profissional. As análises e reflexões se amparam no argumento de a THA se revelar como um potencial recurso para a formação docente por apresentar carácter instrumentalizador, flexível, interativo, previsor, problematizador, dinâmico.

Sent on: 30/10/2020 – **Accepted on**: 01/02/2021 – **Published on**: 29/05/2021

[1] Master in Science Teaching and Mathematics Education from the State University of Londrina. Basic Education Teacher at Paraná State Network, Toledo, Brazil. E-mail: julioeconomit@hotmail.com. ORCID: https://orcid.org/0000-0002-1383-2477

[2] PhD in Science Teaching and Mathematics Education from the State University of Londrina. Professor at the State University of Londrina, Londrina, Brazil. E-mail: pam@uel.br. ORCID: http://orcid.org/0000-0002-9420-8536

DOI: 10.20396/zet.v29i00.8661816

Palavras-chave: Educação Matemática; Formação de professores de matemática; Trajetórias Hipotéticas de Aprendizagem.

Introduction

The option of presenting a Hypothetical Learning Trajectory (HLT) in this article is because it is believed that it has the potential for the teacher to go beyond the preparation of a lesson plan and reflect on their actions and the possible actions of students in the classroom. A HLT can be briefly defined as a planning that the teacher does based on his/her previous experiences as a teacher and as a student, considering, for example, the possible doubts that students may present in a class of a certain content. In HLT the teacher takes into account three elements: the goals he has for learning; the plan he devises with the learning activities; and the hypothetical learning process, which describes how students' thinking and understanding will evolve in the context of the learning activities (Simon, 1995).

In this sense, it is believed that the present study is pertinent, as it represents an alternative for the teaching of mathematics, especially for teacher training, through the elaboration of Hypothetical Learning Trajectory (HLT), which can provide subsidies for both professional training and teaching practice.

Methodological Procedures

This study is the result of a qualitative research and, in light of the theoretical assumptions of Simon (1995) and other authors, seeks to present a theoretical essay in order to argue about the development of HLT as a strategic methodological resource that enables teachers in training to organize their school planning from a hypothetical processing. To achieve these objectives, the following are presented:

- theoretical aspects regarding Simon's Hypothetical Learning Trajectory (1995);
- a proposed HLT on teaching logarithms[3] from the perspective of Problem Solving;
- a discussion about the elaboration of HLT as a training strategy.

This work aims to discuss the following issues: the HLT provide opportunities for teachers in training to analyze aspects that limit their practices in an articulated manner, engage in a process of reflection and decision-making on demands of their professional practice; the HLT can be configured as an instrumental resource, flexible, interactive, predictive, problematizing, dynamic; the HLT reveals itself as a potential instrument which can enable the development of knowledge of mathematics teachers and teacher educators.

[3] It is not the goal (in this article) to specifically discuss the teaching of Logarithms or the application of the THA that is presented, it was only used as a context to motivate the discussion.

DOI: 10.20396/zet.v29i00.8661816

As a result, it is believed that the elaboration and exploration of HLT through Problem Solving has potential for mathematics teaching and teacher education, as it provides both theoretical and practical subsidies for teachers to work with mathematics from this perspective.

Theoretical aspects regarding Simon's Hypothetical Learning Trajectory (1995)

Some dissertations of the Professional Master's Degree in Mathematics Teaching of PUC-São Paulo point out that Hypothetical Learning Trajectories in Simon's (1995) perspective represent an alternative for the teacher to plan his performance in the classroom, such as Angiolin (2009), Barbosa (2009), Lima (2009), Luna (2009), Menotti (2014), Mesquita (2009), and Rosenbaum (2010).

In an attempt to try to answer the question, "how can mathematics teachers promote the construction of mathematical ideas that the community of mathematicians has taken thousands of years to develop?" Simon (1995) considers teacher planning as an essential factor, since among his responsibilities is planning. Brousseau (1987) states that in his planning, part of the teacher's role involves making non-contextualized mathematical ideas that need to be taught and embedding them in a context so that students can investigate it. Such a context must present meaning to students, enabling them to solve problems in that context, and their solution may involve a new mathematical idea to be learned. The resolution to the problem may not be unique.

One of the fundamental responsibilities of teachers is to understand how their students' mathematical knowledge is constituted and how to articulate their teaching method to the nature of that mathematical knowledge. In this regard, Simon (1995) examines the role of different aspects of teacher knowledge and explores the ongoing and inherent challenge of integrating teacher goals and student learning through what he called the Hypothetical Learning Trajectory.

Simon (1995) presented the hypothesized learning trajectory through the Mathematics Teaching Cycle, which he developed as a model of the cyclical interrelationship of aspects involving the teacher's knowledge, his thinking, and decision making with respect to his planning.

Based on three episodes[4] from his study conducted on teaching area, Simon (1995) stated that two factors were considered for the purpose and structure of the lesson: the teacher's mathematical understanding and the teacher's assumptions about the students' knowledge. The author justifies that he uses the term "hypothesis" because the teacher does not have direct access to students' knowledge, but can infer the nature of students'

[4] In the research developed by Simon, the data were collected in an experimental classroom with 25 students. The researcher accompanied a mathematics teacher in tasks involving the concept of area, and after analyzing the data collected, he worked on a theoretical foundation in order to formulate a pedagogy of mathematics.

DOI: 10.20396/zet.v29i00.8661816

understanding from interpretations of their behaviors. For Oliveira et al. (2014), based on Simon (1995) this implies that the teacher can compare his understanding of a particular concept from his construction of students' understandings hypothetically, but it is not possible for him to know beforehand the actual understandings of the students.

From these considerations, Simon (1995) presents the teacher's learning objective as the starting point for designing a hypothetical learning trajectory. According to the author, the use of the expression "hypothetical learning trajectory" is to refer to the

> prediction as to the path by which learning might proceed. It is hypothetical because the actual learning trajectory is not knowable in advance. It characterizes an expected tendency. Individual students' learning proceeds along idiosyncratic, although often similar, paths. This assumes that an individual's learning has some regularity to it (cf. Steffe, et al., 1983, p. 118), that the classroom community constrains mathematical activity often in predictable ways, and that many of the students in the same class can benefit from the same mathematical task. A hypothetical learning trajectory provides the teacher with a rationale for choosing a particular instructional design; thus, I make my design decisions based on my best guess of how learning might proceed (Simon, 1995, p. 135).

The term "hypothetical learning trajectory" is used in order to emphasize aspects of teacher thinking that are grounded in a constructivist perspective and are common to both forward planning and spontaneous decision making (Simon, 1995).

The author justifies his choice of the word "trajectory" to refer to a path, and makes an analogy to a trip, considering that one intends to take a trip around the world, and does not intend to travel randomly, but also has not planned an itinerary to be followed. To this end, one seeks as much information as possible about each place one intends to visit, making a plan.

At first you can have the whole trip planned or just part of it. At the beginning of the trip you follow the plan, but during the trip, due to the conditions you encounter, you must constantly adjust this travel plan. So you may change the order in which the places are visited, or you may stay longer in some places and less time in others, or you may skip some places and visit others that you had not planned to. In short, the path along which one travels is the "trajectory," and the path one had planned is the "hypothetical trajectory" (Simon, 1995).

> The notion of HLT from Simon's perspective does not mandate that the teacher always pursue one goal at a time, or that only one trajectory should be considered, but it is worth noting the importance of having a goal, making a rational analysis for teacher decision making, and the hypothetical nature of such thinking (Oliveira, 2014, p. 4).

Simon (1995) considers that a hypothetical learning trajectory is composed of three components:

1) the learning objective, which sets a direction for the teacher's planning;

2) the plan that the teacher draws up with the learning activities;

DOI: 10.20396/zet.v29i00.8661816

3) the hypothetical learning process, which presents a prediction of how students' thinking and understanding will evolve in the context of the learning activities.

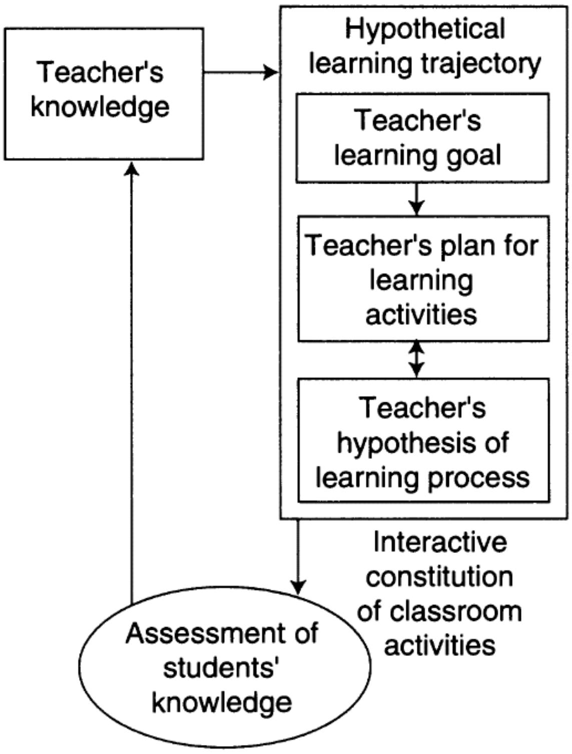

Figure 1 - Mathematics Teaching Cycle (abbreviated)
Source: Simon (1995, p.136).

From his knowledge, the teacher considers the objectives he plans to achieve through an activity plan, and in this plan he presents his hypotheses about how the lesson he has planned will be. Gómez, González and Lupiáñez (2007) assume that the teacher can choose a specific learning objective for which he is planning the lesson. This goal constitutes a frame of reference that delimits the conditions and procedures that the teacher expects to develop in order to formulate his hypotheses about the students' learning process. The information at the teacher's disposal needs to be organized into a systematic process to try to achieve the goals he has set.

> A learning goal is a complex notion. If the teacher wants to design tasks for promoting his students' achievement of that goal, then it is necessary to characterize it in such a way that he can conjecture how and to which extent a task (or a sequence of tasks) can contribute to it's attainment (Gómez, González & Lupiáñez, 2007, p. 3).
>
> .

Once the learning objective is defined, designing a hypothetical learning trajectory involves generating a hypothetical learning process of a particular set of tasks. Simon and Tzur (2004) present some questions that the teacher can think about for creating such a process: "what task, currently available to students, can be the basis for them to achieve the

DOI: 10.20396/zet.v29i00.8661816

learning objectives?" From this question, the teacher can look in his materials for which task brings these possibilities for him to achieve his goals.

While the learning objectives provide a direction for the development of the hypothesized learning trajectory, task selection and the hypotheses about the process of student learning are interdependent. Tasks are selected based on the assumptions the teacher has about the learning process, and the hypothesis of the learning process is based on the tasks that will be involved (Simon & Tzur, 2004). In this sense, some assumptions underlie these ideas

> 1. Generation of an HLT is based on understanding of the current knowledgeof the students involved.
> 2. An HLT is a vehicle for planning learning of particular mathematical concepts.
> 3. Mathematical tasks provide tools for promoting learning of particular mathematical concepts and are, therefore, a key part of the instructional process.
> 4. Because of be hypothetical and inherently uncertain nature of this process,
> the teacher is regularly involved in modifying every aspect of the HLT. (Simon & Tzur, 2004, p. 93).

Once the tasks for the hypothetical learning path have been selected, the teacher can consider his hypotheses for the learning process in more detail. To do this, he or she can consider the possible doubts that students present as they try to solve the tasks. By raising these doubts, the teacher also anticipates possible answers so that students can understand the details that have left them confused.

After establishing the learning objective(s) and selecting the tasks, the teacher evaluates the work he has done and has the possibility of reformulating his hypothesized learning trajectory (Simon, 1995). In Figure 1, the diagram indicates that student assessment is continuous and can bring adaptations to the teacher's knowledge, which can lead to a new or modified hypothetical learning trajectory.

The development of a hypothetical learning trajectory prior to classroom instruction is a process by which the teacher develops a plan of activities to be performed in the classroom by his students. However, by interacting with and observing students, the teacher and students constitute an experience, which by its social nature is different from that anticipated by the teacher. In this context, the teacher's ideas about the students' knowledge may change and he has the possibility to modify the hypothetical learning trajectory he had previously elaborated (Simon, 1995).

DOI: 10.20396/zet.v29i00.8661816

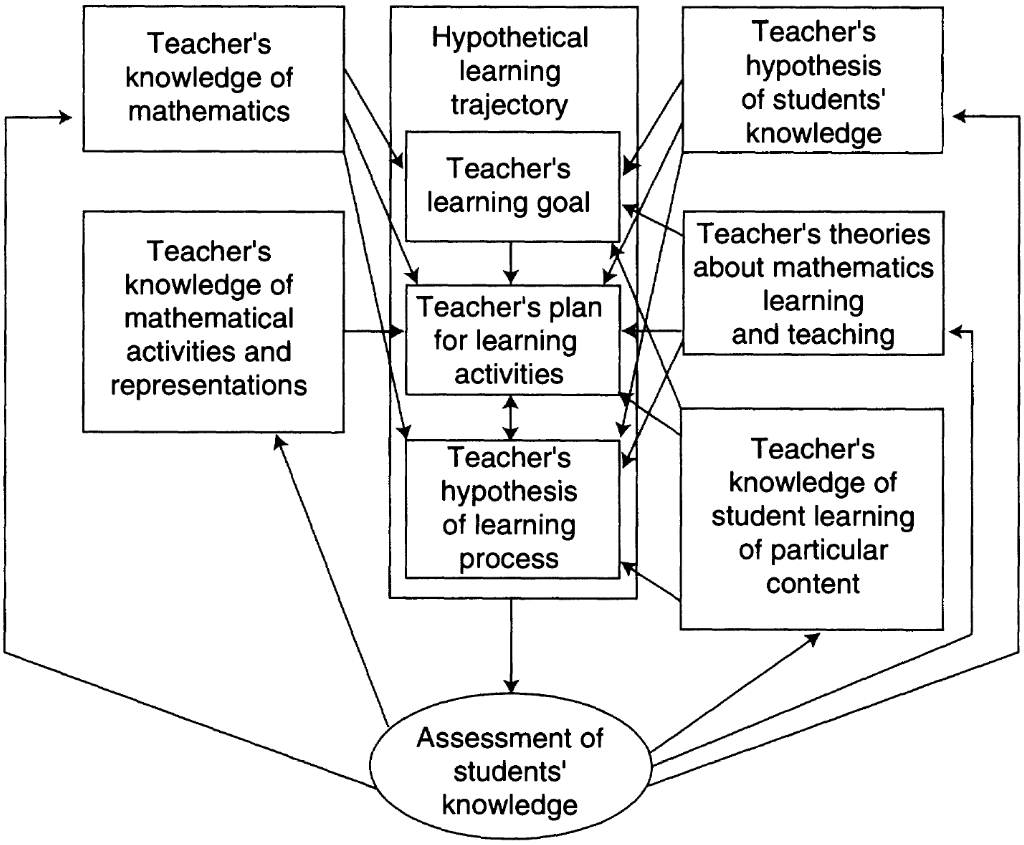

Figure 2 - Mathematics Teaching Cycle
Source: Simon (1995, p. 137)

Figure 2 describes the relationships between the domains of teacher knowledge, the hypothesized learning trajectory, and interactions with students. Simon (1995) explains the figure

> Beginning at the top of the diagram, the teacher's knowledge of mathematics in interaction with the teacher's hypotheses about the students' mathematical knowledge contribute to the identification of a learning goal. These domains of knowledge, the learning goal, and the teacher's knowledge of mathematical activities and representation, his knowledge of students' learning of particular content, as well as the teacher's conceptions of learning and teaching […] contribute to the development of learning activities and a hypothetical learning process. The modification of the hypothetical learning trajectory is not something that only occurs during planning between classes. The teacher is continually engaged in adjust- ing the learning trajectory that he has hypothesized to better reflect his enhanced knowledge. Sometimes fine tuning is in order, while at other times the whole thrust of the lesson must be discarded in favor of a more appropriate one. Regardless of the extent of modification, changes may be made at any or all of the three components of the hypothetical learning trajectory: the goal, the activities, or the hypothetical learning process. (Simon, 1995, p. 138).

DOI: 10.20396/zet.v29i00.8661816

According to Gómez, González and Lupiáñez (2007), the teacher's knowledge, experience, and available literature are the basic sources for the teacher to design a hypothetical learning trajectory to support his or her planning.

In devising a hypothetical learning trajectory, the teacher plans possible situations and routes by which learning can occur in the context of particular tasks. When a mathematical task is not sufficient for the teacher to have evidence that students have learned in the way he had planned, he adjusts his hypothesized learning trajectory, usually by modifying the task and sometimes by changing the interpretation he had of the students' concepts on which his trajectory is based (Simon & Tzur, 2004), or, also, by proposing new tasks in different contexts. According to Steffe (2004)

> The construction of learning trajectories of children is one of the most daunting but urgent problems facing mathematics education today. It is also one of the most exciting problems because it is here that we can construct an understanding of children’s mathematics and how we as teachers can profitably affect that mathematics. (Steffe, 2004, p. 130).

In this sense, we corroborate with Pires (2009, p. 164) when he states that "young teachers need knowledge about students' knowledge, to generate hypothetical learning trajectories and conceptual analysis so that they can teach mathematics".

Some studies have pointed to the development of Hypothetical Learning Trajectories as an opportunity for mathematics teacher education (Ivars et al., 2018; Sztajn et al., 2012; Wilson et al., 2014; Wilson et al., 2017).

In Sztajn et al. (2012) the authors argue that Learning Trajectories used in training provide four frameworks for teaching mathematics: mathematical knowledge for teaching, task analysis, discourse as facilitator action practices, and formative assessment. They present two frameworks in an attempt to: (1) conceptualize teaching from the perspective of Learning Trajectories, (2) define components of instruction based on Learning Trajectories. In this second schema, the goal is to associate, the elements of HLT with the four domains of "Mathematical Knowledge for Teaching" (Ball & Bass, 2003; Ball, Thames, & Phelps, 2008).

In the research by Wilson et al. (2014) an experiment is reported in which the authors examine the design of a program created to support teacher learning on a learning trajectory by performing a reinterpretation from the perspective of mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008). The results indicated that professional learning tasks focusing on pedagogical content knowledge present in learning trajectories enable teacher learning about "content knowledge".

Ivars et al. (2018) infer that using a hypothetical learning trajectory as a guide to interpret students' mathematical thinking can improve the discourse of teachers-in-training. In these authors' study, twenty-nine teachers-in-training participated in a learning environment in which they had to interpret students' thinking about the concept of fraction using a hypothetical learning trajectory as a guide. This training action helped the teachers develop

DOI: 10.20396/zet.v29i00.8661816

more detailed discourse when interpreting students' mathematical thinking, enhancing their "perceiving" skills, which were linked to the trainee teachers' mathematical content knowledge, according to these authors.

In an attempt to dialogue with these theoretical assumptions regarding HLT as a formative element, the next section presents a first version of a Hypothetical Learning Trajectory for teaching logarithms from the perspective of Problem Solving, considering some possibilities for the approach of this content. Based on the development of this trajectory, in the next section, we will seek to reflect on this activity as a teacher training tool.

A HLT on teaching Logarithms from the perspective of Problem Solving.

This section proposes the presentation under some clippings[5] of a "hypothetical learning trajectory" for the teaching of logarithms and their operative properties based on a problem. The HLT was developed from the perspective of Problem Solving, by Onuchic and Allevato (2011), so that it could be used to introduce the concept of logarithm and its operative properties in high school. As the purpose of this article is not to discuss the HLT itself, but rather what features of its elaboration process are potential and characterize the HLT as an instrument that can be an opportunism for the development of the knowledge of teachers who teach mathematics, we chose to perform some "clippings" of the original HLT in this section to support the discussions that will be promoted in the next section[6].

For the development of the HLT, it was considered the logarithm content and its operative properties because it is a high school content, as part of the topic functions, and specifically related to the exponential and logarithmic functions. However, the teaching of logarithms is still, in general, developed through the presentation of definitions, properties, and by solving exercises proposed in a list, and then working on problems involving this content. The intention is to work in the opposite direction, that is, to present a problem that could be solved by means of logarithms and its operational properties, so that students can understand the importance of studying this content by solving a problem that addresses it.

In the HLT proposed below, there is not a linear division of all the steps of a lesson presented in the script proposed by Onuchic and Allevato (2011) and inspired by the problem solving steps of Polya (1994). The problem solving stage is highlighted to describe a possible (hypothetical) solution that students may present, including possible doubts in the course of solving, as suggested by Simon (1995), as the teacher's hypotheses about the learning process. In the original HLT, emphasis is also placed on formalizing the content during the lesson, since problem solving and formalization represent essential aspects of the methodological teaching strategy adopted. This choice is justified by the reason that it is "a

[5] This cut is justified by the number of pages for this article, taking care not to interfere in its quality, considering the discussion it intends to promote.

[6] (Oliveira, 2015).

DOI: 10.20396/zet.v29i00.8661816

proposal", and in this case, there are some steps that are described in detail and others not so much due to the dependence/flexibility of how this proposal can be implemented.

The HLT presented in full in Oliveira (2015) aims to discuss each of the steps from the introduction of the problem to its formalization according to the Problem Solving perspective. It presents the following format: (1) Introducing the Proposed Activity; (2) Objectives for Learning; (3) Understanding the Problem; (4) Establishing a Plan; (5) Executing the Plan; (6) Retrospecting; (7) Formalizing the Content: knowing logarithms; (8) Studying the Consequences of the Definition; (9) Exploring Some Properties of Logarithms; (10) Consequences of the Change of Base Property; (11) Systems of Logarithms; (12) Returning to the Problem. The cut for the discussion in this article will rely on the presentation of items (1) through (6) presented below.

(1) Presentation of the Proposed Activity

To begin working with this HLT, it is suggested that the teacher contextualize the problem situation that will be presented, talking about natural resources, their scarcity, and what has been done to make the best use of them, considering the means of food production used today. Next, it is possible to relate the discussion to the world population, asking if the students know in what proportion the world population grows each year, what is the average percentage of annual world growth, and then the following problem situation will be presented to the class:

How long can the planet sustain us?[7]

It is estimated that 1350 m^2 of land is needed to provide food for one person. It is also assumed that there is 28 1350 billion m2 of arable land in the world and that therefore a maximum population of 28 billion people can be sustained if no other food sources are exploited. The world population at the beginning of 2013 was estimated to be approximately 7 billion. Given that the population continues to grow, at a rate of approximately 2% per year, determine in how many years, starting in 2013, the Earth would have the maximum population that could be sustained.

(Adapted from**: Unb – 1996**)

Fonte: http://www.estadao.com.br/ - access in 09/03/2014

Figure 3 - Problem Statement

The reading of the statement can be done individually or collectively, and the teacher can ask the students to explain what they have understood about the problem. If they have any doubts about the meaning of a word, they can check in a dictionary to understand this meaning.

[7] When presenting this problem to students, the teacher may or may not present the following data: (ln 1.02 = 0.02 and ln 2 = 0.70).

DOI: 10.20396/zet.v29i00.8661816

The problem can be proposed to the students to be solved individually or in small groups, and the teacher accompanies the resolutions of these groups at their desks, checking to see if the students have any doubts when solving it and raising questions so that they can reflect on their doubts and understand how they can develop strategies to solve this problem. In the sequence, a resolution of this problem and doubts that students may have when trying to solve it are presented, considering the four phases presented in Polya's model (1994).

(2) Learning Objectives

- Solve a problem situated in a context with reference to reality.
- Use proportionality concepts to establish relationships between different quantities.
- Describe the problem situation in a mathematical context.
- Develop procedures related to the use of logarithms and their operative properties to solve the problem.

(3) Understanding the Problem

Considering the presented statement, it is expected that students feel challenged to find a resolution that answers the problem's question. To this end, the teacher can investigate with the students what relevant information can be listed:

The world population in 2013 was 7 billion people, or, in scientific notation, was billion people, which will be considered as the initial population (P (0)).

Possible Doubt: Students may question regarding the notation used. Why use P(0) as the initial population?

The teacher can justify this choice by stating that this is the population considered at the initial time, which in this case is the year 2013, or they can also rework the question to the students expecting them to come to this conclusion. Another possibility is to let the students choose how to use the notation for the population of 2013, to which they can choose the notation of P (2013). In the discussion presented in this HLT, P (0) is considered, but nothing prevents students from using another notation to represent the initial population.

(4) Establishing a plan

In thinking about population growth, the statement says that the population continues to grow at a rate of 2% per year, so you know that the factor by which you multiply each year's population is to find out the next year's population.

Possible Doubt: Students may have doubts regarding the representation of the percentage. But can we say that 102% = 1.02? Why?

DOI: 10.20396/zet.v29i00.8661816

The teacher can pick up on the meaning of percentage by asking them about representations of 102%. If you think of 102% of any value, you can use the fractional representation of percentage: And when you divide 102 by 100, you get 1.02. In addition, the teacher can use this question to discuss with students the population of the following years, suggesting the preparation of a chart.

In this chart, the letter t will be used to represent time in years, considering that the year 2013 is the initial year, in which t = 0. Then, to determine the population of the following years, one can calculate as follows:

Table 01: Population as a function of Time

Year	t (in years)	P (population in the year t)
2013	0	$P(0) = 7.10^9$
2014	1	$P(1) = 7.10^9.1{,}02^1$
2015	2	$P(2) = 7.10^9.1{,}02^2$
2016	3	$P(3) = 7.10^9.1{,}02^3$
...	...	...
?	t	$P(t) = 7.10^9.1{,}02^t$

Source: from the authors.

This situation can be modeled by an exponential function defined by, where P(t) represents the number of inhabitants on the planet as a function of t years (starting in 2013), and time t is given in years. In this case, the domain of this function is given by the positive real numbers, but the domain can also be extended to negative real numbers, as per the possible doubt below.

Possible Doubt: But if 2013 is the initial year, how do we know the population of the year 2012?

In the statement, it is not stated that the 2% rate would be valid for before 2013, and in this sense, it is up to the teacher to make this information clear to the students. However, to answer the question, it is possible to use the same rate, and in this way one can consider the order of the exponent t, taking it as, see:

Table 02: estimation for $t = (-1)$

Year	t (in years)	P (population in years t)
2012	-1	$P(0) = 7.10^9.1{,}02^{-1} = \frac{7.10^9}{1{,}02} \cong 6{,}86.10^9$

Source: from the authors.

Note that to determine the population of any year y, it is sufficient to consider 2013 as year zero (t=0), and from there one can consider the following equality:

DOI: 10.20396/zet.v29i00.8661816

$$P(y) = 7.10^9.1{,}02^{y-2013}$$

In this case, simply substitute the value of y into the formula:

$$P(y) = 7.10^9.1{,}02^{y-2013},$$

and so one can estimate the population for any year y, considering that the rate could be used for both before and after 2013.

One can also use the idea of percentage and by means of the rule of three obtain the 2012 population. Being: $\begin{cases} 7.10^9 & = & 102\% \\ x & = & 100\% \end{cases}$

This information can be rewritten as the equality of two ratios:

$$\frac{7.10^9}{x} = \frac{102\%}{100\%}$$

And since you get a proportion, you can multiply mean by extremes, obtaining:

$$7.10^9.100 = x.102$$

$$x = \frac{7.10^9.100}{102} \cong 6{,}86.10^9$$

Therefore, the population in 2012 is $6{,}86.10^9$ approximately.

(5) Executing the plan

Our goal is to find out in how many years the planet's population will reach 28 billion, and to reach it we simply substitute this value into P(t).

Possible Doubt: Why substitute 28 in place of P(t)?

To answer this question, we will return to the idea of how equality is determined, detailing how we built the table, and remembering that the population is a function of time, and in this case, the unknown value is not the population, but the time. Therefore, this is why 28.109 will be substituted into P(t), in order to find out what the value of t will be. The following equation is presented:

$$P(t) = 7.10^9.1{,}02^t = 28.10^9$$

$$7.10^9.1{,}02^t = 28.10^9$$

$$7.1{,}02^t = 28$$

$$1{,}02^t = \frac{28}{7}$$

$$1{,}02^t = 4$$

DOI: 10.20396/zet.v29i00.8661816

This results in an exponential equation, and to solve this kind of equation, one can use approximations to find out to which exponent to raise "1.02" to get "4". With the aid of a scientific calculator, students can do some tests:

Table 03: Simulations for different values of t.

t	$1{,}02^t$
0	1
1	1,02
2	1,0404
3	1,061208
4	1,08243216
5	1,104080803

Source: from the authors**.**

For t = 5, you have a value even smaller than 2 for, so you can substitute larger values to get closer to 4:

Table 04: Simulations for different values of t to approximate 4.

t	$1{,}02^t$
10	1,21899442
20	1,485947396
30	1,811361584
40	2,208039664
50	2,691588029
60	3,281030788
70	3,999558223
71	4,079549387

Source: from the authors.

Note that t is between 70 and 71, but for t = 70 you have a more accurate approximation, so years. Thus the planet's population will reach 28 billion in approximately 70 years. Therefore, in 2083, the Earth will have the maximum population that can be sustained, considering that no other food sources will be exploited.

(6) Retrospect

It is believed that the resolution presented by the students is by means of approximations, and in the sequence the phases in solving this problem will be resumed.

1. reading and understanding the problem, analyzing the data that is relevant in order to come up with a plan to solve it.
2. Analyzing the population growth rate, first year by year, and realizing that this growth occurs in an exponential manner, allowing it to be modeled using a formula.
3. Construction of a formula that relates the world population growth rate, giving us its quantity as a function of time in years.

DOI: 10.20396/zet.v29i00.8661816

4. By means of trials, the number of years can be approximated until it is possible to find out approximately how long it will take for the population to become the one previously estimated, arriving at a solution to the problem.
5. Substituting 70 in the value of t in the formula, we have:

$$P(70) = 7.10^9.1{,}02^{70} = 7.10^9.\ 3{,}999558223 = 10^9\ .27{,}996907559 \cong 28.10^9$$

Thus, it can be seen that in 70 years the world population will be 28 billion, which means that in the year 2083 the Earth will have the maximum population that can be sustained, considering that no other food sources will be exploited.

Once this phase is over, students can be asked about the possibility of always performing these approximations, and also about the fact that it is often not so simple to perform these steps, stating that, to solve this problem, a considerable number of attempts were required, which can make the calculation laborious and exhaustive.

In order to simplify the work with these operations, new concepts can be introduced to the students, which can speed up the calculations and bring a new approach to the problem.

Next, we present other possible doubts that are expected to be proposed by the students subjected to the application of this trajectory and hypotheses about answers and ways in which teachers can conduct the doubts proposed to exemplify the quality of the mathematical treatment given to the initial problem.

Possible Doubt: Students may not understand the concept of logarithms just from the definition, a question that may arise is: *But what is the relationship of logarithm to the problem we are trying to solve?*

In considering this question, the teacher can take advantage of the context of the problem and rewrite the exponential equation: $1{,}02^t = 4$, in the form of a logarithm. In the case of the problem, the base is 1.02, t represents the exponent to which this base is raised so that the result is 4. In this sense, considering the definition presented, we have:

$$1{,}02^t = 4 \quad \text{is the same as} \quad a^x = b$$

Since *x* represents the exponent in the second equation and x is the logarithm, in the case of the first equation the logarithm is given by the unknown t, that is, one can conclude that the logarithm is the unknown, which is nothing more than the exponent of the exponential equation. Since in the definition we have $a^x = b$ that it is the same as $log_a b = x$, then, in the case of the problem $1{,}02^t = 4$, that can be rewritten as $log_{1,02} 4 = t$. In this case, 1.02 is the base of the logarithm, 4 is the logarithm and t is the logarithm.

ZETETIKÉ

DOI: 10.20396/zet.v29i00.8661816

Possible Question: Why in the definition of logarithm must the base value a be positive and different from 1? Why can't the value of a be negative?

The base of a logarithm represents the base of a power whose exponent is the logarithm, and the result obtained when calculating this power is equal to the logarithm. When the base is defined as greater than zero and different from 1, powers in real numbers are considered, and the base must be different from 1 since 1 to any exponent will result in one itself[8].

Possible Question: Why must the value of **b** be positive?

We know that *b* represents the value of the result of the power ax, so we resume the study of powers and consider that if, then ax > 0, that is, the result of a power when its base is positive and non-zero is always positive. Bringing these concepts to the logarithm, it is known that the result of a power is nothing more than the logarithm, therefore, the logarithm cannot be equal to zero and also cannot be negative.

If the teacher has chosen not to present the data (and) in the statement, he or she can use a calculator with the students and use the properties needed to solve the problem, including base change property.

Thus, in approximately 70 years, the Earth will have the maximum population that could be sustained. With the use of logarithms, students have one more tool to solve both this and other problems that may involve the same mathematical ideas and concepts.

This section ends here because it is believed that there are enough elements for the discussion that is intended to take place in the next section.

A discussion regarding the development of HLT as a training strategy.

In this section we will conduct an interpretive and descriptive analysis, based on the elements of HWT development presented in the previous section, which served as the basis for the discussion we intend to promote.

A first observation to be considered regarding the elaboration of a HLT is that the subject that will deal with it should keep in mind its three main elements, namely:

[8] In this case, you can display a counterexample to clarify this idea.

DOI: 10.20396/zet.v29i00.8661816

- the learning objective, which sets a direction for the teacher's planning;
- the plan that the teacher draws up with the learning activities;
- the hypothetical learning process, which presents a prediction of how students' thinking and understanding will evolve in the context of the learning activities.

These three elements are part of the daily routine of teacher planning, even if they are not formally documented. They are common tasks for the teacher to reflect on: what goals to focus on; through which tasks, learning activities and/or teaching materials; and under which methodological strategy and teaching procedures to put into practice. The action of putting these three main elements on paper makes the teacher (in initial or continuing education, or in service) reflect in a systematic way about all the necessary requirements for a well-organized and successful classroom practice.

In the cut of the HLT presented in the previous section, it is possible to observe these three elements presented in an articulated manner and that end up involving many other practices and decisions of the teaching work. Below are some examples that show, in part, the complexity of this proposition:

- the choice of the methodological strategy of teaching, which in this case was associated with Problem Solving[9]. As a result of the development of the presented HLT, it is believed that the planning and exploration of HLT through Problem Solving have potential for mathematics teaching and teacher training, as it provides both theoretical and practical subsidies for the teacher to work with mathematics from this perspective.
- to focus on the main objective of the lesson or didactic unit. In particular, the objective is to introduce concepts of logarithms and their operative properties. In the presented HLT, the authors take care to reflect on the "place" of Logarithms when associated with the concept of functions, exponential and logarithmic.
- care on how to articulate the desired content (logarithms) with a chosen reality problem, so that the methodological strategy adopted (Problem Solving) could be actually implemented. Moreover, the delimitation of the HLT objectives presented corroborate the ideas of Gómez, González and Lupiáñez (2007), which constitute a frame of reference that delimits the conditions and procedures that the teacher expects to develop in order to formulate his hypotheses about the students' learning process.
- the need to contextualize the problem situation that would be presented by bringing "extra" mathematical elements to the discussion with students, in

[9] In the THA presented, Problem Solving was the strategy adopted, but it could be from any of the trends in Mathematics Education.

DOI: 10.20396/zet.v29i00.8661816

order to ensure that they could engage with the proposed learning activity, which is suggested by talking about natural resources, their scarcity and what has been done to make the best use of them. This "ambiance" and "exploration of the topic" meet Simon and Tzur's (2004) questions about the teacher reflecting on the process: "what task, currently available to the students, can be the basis for them to achieve the learning objectives?".

- the decision about the organization of student groups in class, about the time allotted to solve the problem and other tasks that accompany "the nature" of Problem Solving as a methodological strategy, such as the help that the teacher should give to the groups and the way to conduct the discussion of the problem for its systematization.
- In particular, there is the potential exercise of making predictions (which Simon calls hypothetical) about what questions students can ask in the development of the planned action, about how to produce reflections about the mathematical content in question and, also, about what answers and conduct the teacher can give or make based on the questions produced.

In particular, the hypothetical processing of the didactic action unfolds with respect to these predictions highlighted in the last topic, as a strategy to process mentally and in writing the entire architecture of a lesson or teaching sequence. In this sense, the HLT provide opportunities for teachers in training to analyze aspects that delimit their practices in an articulated manner, as they engage in a process of reflection and decision-making on demands of their professional practice. In this perspective, we corroborate with Pires (2009, p. 164) when he states that "young teachers need knowledge about students' knowledge, to generate hypothetical learning trajectories and conceptual analysis so that they can teach mathematics".

Moreover, from the perspective of the authors of this article, HLT can be configured as an instrumentalizing, flexible, interactive, predicting, problematizing, and dynamic resource. In the case of the instrumentalizing feature, it can be emphasized that the teacher also reflects on how his class will be organized, taking into account which aspects to address at each stage, what content to address, what questions may arise and how he will answer them, how long each stage of the class can last.

The predictive character is present when highlighting the possible doubts that students may present. The teacher usually foresees situations that he has already experienced and the most common doubts that he has encountered, he knows the students and their context. It is worth noting that it is not possible to foresee all doubts precisely, but the work of systematizing and reflecting on them already provides a range of possible approaches that the teacher can put into practice based on these doubts.

The "flexible" characteristic involves thinking about what order the teacher can approach the different aspects of the content explored with the task being worked on, whether

DOI: 10.20396/zet.v29i00.8661816

or not this order interferes with student learning, or if the teacher can reverse it, depending on how students respond to the task. The teacher can lead the exploration of the problem from what the students interpret from the statement, and consider their justifications and argumentations to introduce the content to be explored. In addition, flexibility is considered, since the HLT does not have to be applied exactly as it was designed, the teacher has the autonomy to make different decisions as needed throughout the execution of the previously hypothesized action.

Regarding the interactive aspect, the teacher can promote student participation by means of questions previously established by the HLT, and also promote interaction among students, depending on how the teacher organizes the class dynamics (if students will work in pairs, trios, or individually).

Regarding the problematizing character, one can consider the teacher's intention when asking certain questions, so that these questions promote students' reflection about the theme and they are able to use their previous knowledge to rework their strategies, or even recognize which points they still have doubts so that they expose them to the teacher and also to the other students, so that they are able to work together in dealing with the problem at hand.

The dynamism of the hypothetical learning trajectory enables the teacher to rethink his class and modify aspects that he believes can be improved in order to favor student learning.

The arguments presented involve some aspects of teacher training that are routinely mobilized in teacher training courses, whether initial or continuing education. More specifically, the teaching actions highlighted in the hypothetical processing of a HLT (whatever it may be) concern the routine practice of a teacher in the classroom. The action of focusing on them is a complex and formative exercise insofar as it places the subject in training in a broad process of reflection.

In this sense, it is argued that the elaboration of HLT reveals itself as a potential instrument that can facilitate the development of the knowledge of mathematics teachers and teacher educators.

For teachers, the elaboration of hypothetical learning trajectories provides moments of reflection on how to teach and what possible questions may arise when a student is first faced with a problem. In particular, when the problem demands certain mathematical content that the student is unfamiliar with, for which he or she has no ways of dealing with this problem. The teacher is then responsible for doing his or her job in the classroom to create conditions for this student to be able to build his or her own knowledge from the previous knowledge he or she has already built up.

Another aspect worth mentioning about the development of a hypothetical learning trajectory is that it represents a continuous process that never ends, since it is always possible that a doubt or questioning arises that was not foreseen by the teacher in his or her HLT, for

DOI: 10.20396/zet.v29i00.8661816

which he or she must find ways to deal with this questioning during the lesson. This aspect reflects that the teacher's training is also a continuous process and he/she always needs to study in order to invest in his/her own training.

Some considerations

It should be noted that arguments were presented about the teaching work, however, for the students, the work with Problem Solving that was presented in the HLT on screen and through the implementation of other HLT can provide more autonomy, since the student is the one dealing with a problem that for him is usually new and different from those he has already solved, and he needs to find a way to solve it considering his previous knowledge. By doing this work, the student begins to understand his role in the classroom as a builder of his own knowledge and tends to become more critical because he can evaluate himself as a student by realizing whether or not he still has difficulties with another certain content that he has previously studied.

The Hypothetical Learning Trajectory allows the teacher to make assumptions about what he or she is about to teach, enables him or her, through learning activities, their objectives and hypothetical processes, to trace a hypothetical trajectory about his or her performance at certain times, thus thinking about the doubts he or she may encounter, what he or she believes the student knows, how they may react, and so on. This perspective meets the ideas of Simon (1995) when he justifies the use of the term "hypothetical" because the teacher does not have direct access to the students' knowledge, but can infer the nature of their understanding from the interpretations of the behaviors they present. When faced with reality, the teacher can verify whether his expectations were met, and can elaborate his hypotheses again, thus creating a cycle that does not end and puts the teacher in constant improvement of his practice.

The articulation movement of the three central elements of the HLT proposed by Simon (1995), besides being typical of the teaching task, can be presented as an action to glimpse the teaching process in consonance with the learning process. The richness of this articulation does not lie in the elements themselves, but in the movements between them, in the coherence, in the attention to the conception of a practice focused on the construction of competencies, in the argumentative power that the hypotheses may raise.

It is also worth considering that a teacher is also a researcher, because he is always assuming, verifying and changing himself, and in this sense the elaboration of Hypothetical Learning Trajectories can be a means for the teacher to manage his continuing education.

In future works, it is believed to be pertinent the elaboration of other HLT that address different mathematical contents, with diversified methodological strategies, with the objective of providing alternatives for teachers to implement in their classes, making the necessary adaptations, since they do not represent models to be followed, but a possibility for mathematics classes.

DOI: 10.20396/zet.v29i00.8661816

References

Angiolin, A. G. (2009). *Trajetórias Hipotéticas de Aprendizagem sobre Funções Exponenciais*. Dissertação de Mestrado Profissional em Ensino de Matemática. São Paulo: Pontifícia Universidade Católica de São Paulo.

Ball, D. L., & Bass, H (2003). Toward practice-based theory of mathematical knowledge for teaching. In B. Davis & Smith, E. (Eds), *Proceedings of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group* (pp.3-14). Edmonton. Proceedings… Edmonton: CMESG/GCEDM.

Ball, D., Thames, M. H., & Phelps, G. C (2008). Content Knowledge for Teaching: What make it special? *Journal of Teacher Education, 59 (*5), 389-407.

Barbosa, A. A. (2009). Trajetórias Hipotéticas de Aprendizagem relacionadas às razões e às funções trigonométricas, visando uma perspectiva construtivista. Dissertação de Mestrado Profissional em Ensino de Matemática. São Paulo: Pontifícia Universidade Católica de São Paulo.

Brousseau, G. (1987). Les differents roles du maitre. Colloque des P.E.N. Angers. In S. A. Martin (Ed.). *Reconstructing mathematics pedagogy from a constructivist perspective*. Washington, DC: National Science Foundation.

Gómez, P., González, M. J., & Lupiáñez, J. L (2007). *Adapting the Hypothetical Learning Trajectory Notion to Secondary Preservice Teacher Training*. Chipre: Universidade de Chipre.

Ivars, P., Fernández, C., Llinares, S., & Choy, B. H. (2018). Enhancing Noticing: Using a Hypothetical Learning Trajectory to Improve Pre-service Primary Teachers' Professional Discourse. Eurasia Journal of Mathematics, *Science and Technology Education*, 14(11), em1599. doi.org/10.29333/ejmste/93421

Lima, P. O. (2009). *Uma Trajetória Hipotética de Aprendizagem sobre Funções Logarítmicas*. Dissertação de Mestrado Profissional em Ensino de Matemática. São Paulo: Pontifícia Universidade Católica de São Paulo.

Luna, M. F. A. (2009). *Estudo das Trajetórias Hipotéticas de Aprendizagem de Geometria Espacial para o Ensino Médio na perspectiva construtivista*. Dissertação de Mestrado Profissional em Ensino de Matemática. São Paulo: Pontifícia Universidade Católica de São Paulo.

Menotti, R. M. (2014). *Frações e suas operações*: Resolução de Problemas em uma Trajetória Hipotética de Aprendizagem. Dissertação de Mestrado Profissional em Matemática. Londrina: Universidade Estadual de Londrina.

Mesquita, M. A. N. (2009). *Ensinar e Aprender funções polinomiais do 2° grau no Ensino Médio*: construindo trajetórias. Dissertação de Mestrado Profissional em Ensino de Matemática. São Paulo: Pontifícia Universidade Católica de São Paulo.

Oliveira, J. C. R (2015). *Uma Trajetória Hipotética de Aprendizagem para o Ensino de Logaritmos na Perspectiva da Resolução de Problemas*. Dissertação de Mestrado Profissional em Matemática em Rede Nacional. Universidade Estadual de Londrina, Londrina.

DOI: 10.20396/zet.v29i00.8661816

Oliveira, J. C. R., Frias, R. T., & Omodei, L. B. C. (2014). Uma Trajetória Hipotética de Aprendizagem para o Ensino de Função Afim em um curso de Formação Continuada. *Anais do Encontro Paranaense de Educação Matemátic*a. Campo Mourão: Unespar.

Onuchic, L. R., & Allevato, N. S.G. (2011). Pesquisa em Resolução de Problemas: caminhos, avanços e novas perspectivas. *BOLEMA - Boletim de Educação Matemática*, 25 (41),. 73-98.. Disponível em: < http://www.redalyc.org/pdf/2912/291223514005.pdf>. Acesso em: 15 jan. 2015.

Pires, C. M. C (2009). Perspectivas construtivistas e organizações curriculares: um encontro com as formulações de Martin Simon. *Educação Matemática Pesquisa*, 11 (1), 145-166.

Polya, G.(1994). *A Arte de Resolver Problemas*. Rio de Janeiro: Interciência.

Rosenbaum, L. S. (2010). *Uma Trajetória Hipotética de Aprendizagem sobre funções trigonométricas numa perspectiva construtivista.* Dissertação de Mestrado Profissional em Ensino de Matemática. São Paulo: Pontifícia Universidade Católica de São Paulo.

Simon, M. A (1995). Reconstructing Mathematics Pedagogy from a Constructivist Perspective. *Journal for research in Mathematics Education*, 26 (2), 114-145.

Simon, M. A.., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: an elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, 6 (2), 91-104.

Steffe, L. P. (2004) On the Construction of Learning Trajectories of Children: The Case of Commensurate Fractions. *Mathematical Thinking and Learning*, 6 (2), 129-162.

Sztajn, P., Confrey, J., Wilson, P. H., & Edgington, C. (2012). Learning Trajectory Based Instruction: Toward a Theory of Teaching. *Educational Researcher*, 41(5), 147–156.

Wilson, P. H., Sztajn, P., Edgington, C., & Confrey, J. (2014). Teachers' use of their mathematical knowledge for teaching in learning a mathematics learning trajectory. *Journal of Mathematics Teacher Education*, 17, 149-175. DOI 10.1007/s10857-013-9256-1

Wilson, P. H., Sztajn, P., Edgington, C., Webb, J., & Myers, M. (2017). Changes in Teachers' Discourse About Students in a Professional Development on Learning Trajectories. *American Educational Research Journal*, 54(3), 568-604. DOI 10.3102/0002831217693801