DOI: 10.20396/zet.v29i00.8661906

Kindergarten and Primary teachers' Specialized Knowledge on the topic of division

Conhecimento Especializado do professor que ensina matemática relativo ao tópico de divisão

Milena Soldá Policastro [1]

Miguel Ribeiro[2]

Abstract

Considering that teachers' knowledge plays a crucial role in their practice and in the students' learning, and because division is one of the most problematic topics for teachers and students, we focus our attention on the content of Specialized Knowledge, presented on a specialization course related to the subject, by teachers who work with early childhood education. The material collected from the teachers' interactions, when discussing a task for training, was analyzed according to the lens of Mathematics Teachers' Specialized Knowledge, in order to characterize the content of the teacher's Specialized Knowledge in the division topic. From this analysis, descriptors emerged from the Specialized Knowledge of the mathematics teacher related to concepts, procedures, properties, fundamentals, and representational systems within the scope of the division, contributing in such a way that, in formative contexts, effective instruments are intentionally implemented to develop such knowledge.

Keywords: MTSK; Division; Kindergarten and Primary school; Teacher Education.

Resumo

Assumindo que o conhecimento do professor exerce um papel crucial em sua prática e nas aprendizagens dos alunos e por ser a divisão um dos tópicos mais problemáticos para professores e para alunos, focamos nossa atenção no conteúdo do Conhecimento Especializado, revelado em um curso de especialização relativamente a este tópico, por professores que atuam desde a Educação Infantil. O material coletado das interações dos professores, ao discutirem uma tarefa para a formação, foi analisado segundo a lente do *Mathematics Teachers' Specialized Knowledge*, com intuito de caracterizar o conteúdo do Conhecimento Especializado do professor no tópico de divisão. Dessa análise, emergiram descritores do Conhecimento Especializado do professor de e que ensina matemática relacionado a conceitos, procedimentos, propriedades, fundamentos e sistemas representacionais no âmbito da divisão, contribuindo para que, nos contextos formativos, sejam intencionalmente implementados instrumentos eficazes para desenvolver tal conhecimento.

Sent on: 03/11/2020 – **Accepted on**: 01/02/2021 – **Published on**: 02/06/20211

[1] Master in Science and Mathematics Teaching (USP). Doctoral student at the Graduate Program of the Faculty of Education (UNICAMP), Campinas, São Paulo, Brazil. Email: mitapolicastro@gmail.com. ORCID: https://orcid.org/0000-0003-2437-2557

[2] PhD in Didactics of Mathematics (UHU). Professor of the Multi-Unit Graduate Program in Science and Mathematics Teaching (PECIM), Campinas, São Paulo, Brazil. Email: cmribas78@gmail.com. ORCID: https://orcid.org/0000-0003-3505-4431

DOI: 10.20396/zet.v29i00.8661906

Palavras-chave: MTSK; Divisão; Educação Infantil e Anos Iniciais; Formação de Professores.

Introduction

The specificities of the mathematics teacher's knowledge have been the focus of several studies in the last decades (e.g., Ball, Thames & Phelps, 2008; Carrillo et al., 2018; Rowland, Huckstep & Thwaites, 2005). International congresses of relevance in the area of Mathematical Education (e.g., ICME[3], PME[4], CERME[5]) have, each year, discussed more works on this theme. Although research with an emphasis on teacher knowledge presents certain divergences regarding central aspects for the characterization of the specificities of that knowledge, there is a consensus among them that a wide and deep knowledge of the content to be taught plays a crucial role in the teacher's practice (Dooren, Verschaffel & Onghena, 2002; Fennema & Franke, 1992; Ribeiro, 2011a). At the same time, another center of attention considered in the investigations relates the teacher's knowledge, in its most varied dimensions, acknowledging the impact that this knowledge has on students learning (Boyd, Grossman, Lankford, Loeb & Wyckoff, 2009).

Official documents from different countries, international exams, such as the case of PISA, and specific educational programs for the development of future professionals with capacities and skills focused on the areas of science, technology, engineering and mathematics (STEM), give mathematics a particularly relevant role. Thus, it is essential to broaden and deepen the understanding about the specificities of the knowledge of the mathematics teacher, in order to be able to discuss, for example, aspects related to the quality of teacher training, in particular, but not exclusively, in Brazil.

Among the most problematic mathematical topics to be explored with students, those related to the four operations stand out. The difficulties related to such topics are acknowledged (Fosnot & Dolk, 2001; Rizvi, 2007), traditionally faced by both students and teachers, and, in this context, division is considered particularly problematic (Correa, Nunes, & Bryant, 1998; Fischbein, Deri, Nello, & Marino, 1985). These difficulties are associated, among other aspects, with the prioritization of Knowledge of how to do the algorithm, to the detriment of understanding and attributing meaning to the operation; the (in)appropriate use of certain verbalizations associated with the procedures (Simon, 1993); failure to establish connections between division and other operations (Young-Loveridge & Bicknell, 2018); little (if any) attention given in this context to the principles of counting and grouping quantities in one unit - *unitizing* (Behr, Harel, Post & Lesh, 1994).

Research related to the teaching and learning of division has assumed the priority

[3]*International Congress on Mathematics Education*

[4]*Psychology on Mathematics Education*

[5]*The Congress of the European Society for Research in Mathematics Education*

DOI: 10.20396/zet.v29i00.8661906

focus of "identifying, describing and analyzing errors and resolution strategies in relation to the concepts of division" (Fávero & Neves, 2012, p. 36) on the part of students, without considering the teacher's role and knowledge. In a systematic review of ten years of research dedicated to the teaching and learning of division and rationals, Fávero and Neves (2012) point out a teacher practice, centered or "restricted in the exposure of rules, in detriment of the concept and, consequently, focused on memorization, to the detriment of reasoning "(p. 60). This type of approach is associated with a practice that has the ultimate objective of "Knowledge of-how", and not understanding it; that is, a type of instrumentalizing didactics not aimed at establishing relationships (Skemp, 1989).

Thus, for each of the mathematical topics and, in particular, for the division, it is essential to consider the aspects that make the teachers' knowledge specialized, taking into account not only what they know (or need to know), but essentially, the way this knowledge is structured to give coherence and cohesion to the teaching and learning processes. This specialization is understood here in the sense of *Mathematics Teacher's Specialized Knowledge* - MTSK (Carrillo et al., 2018), and is incorporated in the domain of both knowledge of content and pedagogical knowledge of content. Thus, aiming to broaden the understanding regarding the content of the teacher's mathematical knowledge within the scope of the division, here we seek the following question: *What elements characterize the knowledge of the topic of division on mathematics teachers participating in a teacher's training program?*

Theoretical perspectives within the division scope

Among the four operations, division is considered the most complex and problematic for students (eg, Bicknell, Young-Loveridge, Lelieved, & Brooker, 2015; Fischbein et al., 1985), which has led it to be the last one to be discussed (at least formally). This assumes a hierarchical organization of the teaching of these operations. However, such a hierarchy, assuming a disjunct approach, makes it possible to avoid the existing connections between the four operations (Young-Loveridge & Bicknell, 2018) and also between fundamental concepts present in different contexts of school mathematics, as the case of notions of groupings - *unitizing* - (Behr et al., 1994) and of relations between part and whole (Young-Loveridge, 2001).

When it comes to teaching operations, students' difficulties may be related, for example, to taking the algorithm as a starting point (Correa et al., 1998), focusing on procedures seen as meaningless rules (Fávero & Neves, 2012; Ribeiro, Policastro, Mamoré, & Di Bernardo, 2018). This lack of meaning can be related to the little (if any) attention given to the relations between the procedures and the underlying concepts (Robinson & LeFevre, 2012), such as the comparison between quantities - dividend and divisor -, the

DOI: 10.20396/zet.v29i00.8661906

reference whole or the meanings associated with the operation. The consequence is that students do not understand what they should do and why they should do it at each moment (Ribeiro et al., 2018).

The understanding of division is based on a constant comparison between the elements involved (dividend, divisor and quotient), which goes beyond the understanding of the distribution of elements expressed by numbers, traditionally present in problems and operations of the "build and calculate" type (Lautert, Oliveira & Correa, 2017). Although recognizing the importance of working with the algorithms (Brocardo, Serrazina & Kraemer, 2003), long before them, it is essential that students understand the meanings of the operation, paying special attention to the roles of elements involved in it (Correa et al. , 1998; Squire & Bryant, 2002) and the relationships between them (eg., divisibility, multiplicity and proportionality).

Among the meanings of the division (Fischbein et al., 1985), we assume the partitive and the measure as the ones we consider most appropriate to discuss with teachers and students of Early Childhood and Primary Education. The contexts of partitive division correspond to situations in which it is intended to distribute (or share) an equal number of elements in an initial set (dividend) among a certain number of sets (divisor) – so that, after partitioning, all sets (divisor) contain the same cardinality, which corresponds to the quotient. In the division in the sense of measurement, given an initial quantity (dividend), it is intended to identify how many times (quotient) another quantity (divisor) is needed to measure the first.

To understand the division, also associated with the usual verbalization of the traditional algorithm[6], where the answer to the question “what number multiplied by the divisor provides the dividend?” is sought (Ribeiro et al., 2018), it is essential to understand this sense of measurement. On the other hand, understanding the measure requires understanding some of the principles that underlie the activity of measuring, namely: i) choice of the unit of measurement; ii) partition of the whole to be measured; iii) iteration (repetition) of the unit of measurement over the whole to be measured; iv) quantity accumulation; v) assigning a numerical value, corresponding to the quantification of times that the unit was repeated until completing the quantity relative to the whole to be measured (Clements & Stephan, 2004).

The facts stand out that only from a problem (context), in association with a given verbalization and/or with a representation that is carried out, it is possible to identify which of the two meanings of the division is evoked (Fischbein et al., 1985) and realize that the representations that can be associated with the resolution of these problems make it possible

[6]We consider here the traditional algorithm formally known as the Euclidean division algorithm.

DOI: 10.20396/zet.v29i00.8661906

to assign meaning to each of the mathematical concepts involved (Golden & Shteingold, 2001; Lesser & Tchoshanov, 2005).

Teachers' Specialized Knowledge

Several investigations in Mathematics Education have sought to characterize and expand the understanding of the different domains of knowledge of the mathematics teacher. Among the conceptualizations that seek to understand the specificities of this knowledge, we assume the *Mathematics Teachers' Specialized Knowledge* – MTSK – (Carrillo et al., 2018), which considers these specificities, in the scope of both *Mathematical Knowledge* (MK) and *Pedagogical Content Knowledge* (PCK).

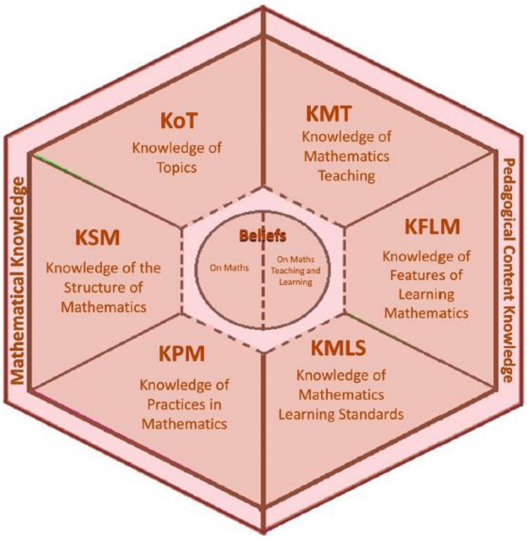


Figure 1 - The MTSK model
Source: Carrillo et al. (2018, p. 241)

The MK details the teacher's knowledge in terms of a "scientific discipline, within an educational context" (Carrillo et al., 2018, p. 240), while the PCK refers to this knowledge in terms of the teaching and learning processes of each of the mathematical topics. It is important to highlight that the PCK refers to the teacher's knowledge in which mathematical topics are conditioning factors for the teaching and learning of mathematics and, therefore, we exclude general pedagogical knowledge from this subdomain, even in contexts of mathematical activities. The model also includes the teacher's beliefs in relation to mathematics as a science and school discipline and in relation to their teaching and learning.

In each of the two domains of the teacher's knowledge, three subdomains are grouped, and in the context of this work (see the following section), we will discuss only the *Knowledge of Topics* (KoT) subdomain and the categories that contemplate the content of the teacher's knowledge included therein.

DOI: 10.20396/zet.v29i00.8661906

Knowledge of Topics (KoT) within the division

This subdomain refers, among other aspects, to the mathematics teacher's knowledge about concepts, foundations, procedures and the way in which the relationships between them occur, when they occur within the same topic. Knowledge of how these relationships are constituted and evidenced helps the teacher to work mathematics from a more structural point of view (Mason, Stephens & Watson, 2009), enabling learning (Hiebert & Grouws, 2007), as it contributes to the development of students' structural thinking in relation to mathematics. It refers to the teacher's knowledge of the different mathematical definitions associated with the same topic – when there is more than one definition – including its different forms of presentation (through symbolic and/or verbal language); of the properties and foundations of a mathematical object or entity; of the traditional or unconventional procedures (what, how, when and why they are done in a certain way) and the implications of the use of certain procedures, when trying to make sense of mathematical constructs or concepts; of the description of the meanings associated with a given concept or construct (phenomenology) and the association of contexts capable of evoking such meanings (applications); of the different systems of representations (pictorial, numerical, verbal, graphic and symbolic) and the relationships that can be established, within the same topic, between these different types of representations and certain procedures.

Within the scope of division, KoT includes, for example, knowledge associated with the mathematical meaning of what it is to divide; with the two meanings of division; with the different procedures associated with the division operation, including the Euclidean algorithm, but not only; with the different types of representation for a division and the relations between them (for example, pictorial, numerical, verbal), which contribute to give sense to the assumed meaning (partitive or measure); with the types of problems that can be formulated in correspondence with each of the meanings of the division, among other aspects. In this subdomain, four categories are considered (Carreño, Rojas, Montes & Flores, 2013), namely: (i) *Definitions, properties and foundations*; (ii) *Phenomenology and applications*; (iii) *Procedures;* (iv) *Registers of representation*.

(i) Definitions, properties and foundations

It includes the teacher's knowledge of the most elementary mathematics concepts, organized hierarchically and logically to shape the most complex mathematical concepts, or to characterize mathematical definitions of concepts. Also present are the mathematical properties related to each of the concepts, as well as the structural characteristics of the constructs and concepts related to the same topic.

Within the scope of the division, it refers to the teacher's knowledge about the dividing sub-construct as the number of sets among which the dividend will be distributed –

DOI: 10.20396/zet.v29i00.8661906

division as partitive –, or the unit of measurement to be considered to measure the whole (dividend) – division as a measure. It also encompasses the knowledge that the mathematical meaning of division is distinguished from that which refers to the decomposition of a natural number into an addition of not necessarily equal parcels, a notion associated with the semantic meaning of the term "divide". This knowledge of the (direct) non-congruence between these two notions helps to substantiate the mathematical meaning of division – particularly in the sense of partitive meaning.

(ii) Phenomenology and applications

This includes the teacher’s knowledge about the phenomena, contexts, applications of a topic, concept or problems to which an answer is sought (Gómez & Cañadas, 2016). In the case of division, aspects related to the phenomenology of this topic are considered, the knowledge of the two meanings attributed to the operation – partitive and measure (Fischbein et al., 1985), as well as the types of problems and contexts (ideas contained in them) that can be formulated in order to evoke each of these senses (Downton, 2009), which corresponds to this phenomenon applications.

(iii) Procedures

This category includes knowledge of the different procedures that can be used in the situations or contexts in which a given concept is at stake. Therefore, it involves the knowledge that the teacher has of each of the steps to be followed; of the different processes teachers can employ with the same mathematical objective – what, how, when and why it is done; and of the characteristics of the results obtained, whenever specific types of procedures are employed. In the topic of division, this category comprises the teacher’s knowledge of the different ways of proceeding to resolve a division operation (Bisanz & LeFevre, 1992), such as effecting the division by successive additions or subtractions and by decompositions and/or groupings; and processes (reasoning and steps) associated with the steps of the Euclidean algorithm (or any other).

(iv) Registers of representation

It is considered here the teacher's knowledge about the different representational systems (e.g., verbal, symbolic, pictorial, graphic) that contribute to give meaning to the concepts and/or the foundations of certain constructs, and also of the relationships between the different ways of representing a concept or construct, in order to enhance a fruitful navigation (Ribeiro, 2011b) between these systems. It is this teachers' knowledge that allows them to establish a correspondence between a mathematical object (e.g., concept, construct, phenomenon) and an adequate conceptual image (Golden & Shteingold, 2001; Timmerman,

DOI: 10.20396/zet.v29i00.8661906

2014).

In the case of division, it is also important to know the relationship and the suitability of a certain type of representation (pictorial, verbal, numerical) with each of the meanings of division. Some types of representation take on a central role in assigning meaning to the division as partitive or as a measurement and, in particular, to certain procedures associated with the calculation of the operation. And it is essential that the teacher knows these multiple representations and their mathematical correspondence with the evoked meaning, which necessarily includes the appropriate verbalization, associated with each of the mathematical meanings and senses of the constructs (Ainsworth, Bibby & Wood, 2002; Golden & Shteingold, 2001; Lesser & Tchoshanov, 2005).

Thus, for example, a verbalization such as "how many times three fits in six" is not adequately related to the sense of partitive, as it is usually understood, but associated with the notion of comparison of quantities and subsequent quantification – which corresponds to the measurement meaning. Therefore, when employing a certain verbalization, the teacher must be aware (Knowledge of with correspondence) of the meaning of the operation that is evoked and of the types of reasoning involved.

Context and Method

For this work we used information collected in a two-year Teacher's Training Program (TTP)[7], consisting of eight modules, in order to develop the specificities of the participants' knowledge in different mathematical topics. Here we consider the information from an eight-hour training session, in which a task for teacher's training (Ribeiro, Almeida & Mellone, 2019) on the division was discussed. In the training program 13 teachers participated, but only 9 were present at the session – 1 active in Kindergarten; 7 active in the Primary and 1 active in the Secondary. To identify them, each of them was assigned a number[8] and, therefore, P1 and P2 correspond to two different teachers, but without any hierarchy between them.

The entire session was recorded in audio and video, and the productions of the teachers' tasks were digitized. Typically, teachers solved tasks in a group and later a large group discussion took place. At the meeting analyzed, each group had three teachers[9]: Group

[7]Specialization Course in Mathematics offered by the State University of Campinas - Unicamp - in person, totaling 380 hours.

[8] The numbering associated with the pseudonyms of teachers P1, P2, P3, P4, ..., has an organizational character in the order of their speeches (moments when they talk) during the various training sessions that make up the greater research information set to which the cut for this work is linked.

[9] As it is not relevant to the discussions, we chose not to distinguish the gender of each participant and, thus, we refer to everyone as a "teacher".

DOI: 10.20396/zet.v29i00.8661906

1 – (P14) Secondary and (P11 and P19) Primary; Group 2 – three teachers from the Primary (P3, P4, P8); and Group 3 – a teacher of Kindergarten (P18) and two of the Primary (P20 and P2).

In the training task (Ribeiro et al., in press) two parts were considered: Part I, with a focus on the Teacher's Specialized Knowledge about the meanings, algorithms and associated representations on the topic of division; and Part II, focusing on the teacher's Interpretative Knowledge (Jakobsen, Ribeiro & Mellone, 2014) . In this work, we will discuss Part I of the task, whose implementation took place in two stages: resolution in groups of three, in three and a half hours; and plenary discussion, in 1 hour and 45 minutes.

The first part of the task contained seven questions and, given the focus here, we will present only two of them. The first question (Cf. Figure 2) aimed to access the teacher's knowledge about the mathematical concept of division and, in particular, the two meanings attributed to this operation.

Part I *What is dividing*? Answer for yourself, as a teacher who teaches mathematics and without considering a school context

Figure 2 - First question of the task presented to teachers
Source: authors (2018)

Starting from the expression "6 ÷ 3", the fourth question was composed of nine sub-questions regarding[10]: (a) possible procedures for resolving the operation; (b) the most appropriate school stage to start exploring the ideas of division; (c) the task considered most appropriate to introduce the division to students in the 3rd year of the Early Years; (d) how to discuss (types and focus of tasks) the division with 5-year-old students; (e) different types of problems involving the operation; (f) solving problems involving the operation, using different types of representations and establishing relationships between them; (g) types of resolutions and strategies of 2nd year students for solving problems involving the operation. With these sub-questions, the aim was to access the knowledge of teachers and develop it about the different meanings of the division operation; the types of tasks, procedures, representations; and the existing relations between each one, more appropriate to discuss the division with students from different school stages.

The data gathered was organized and analyzed considering its specificities (Ribeiro, Carrillo & Monteiro, 2012). The audio recordings were transcribed and supplemented with information from the video – in terms of the participants' actions, both at work in each of the groups and in the plenary. In these transcriptions, each line was indicated by a numbering of

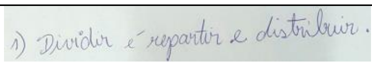
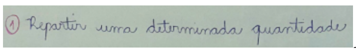
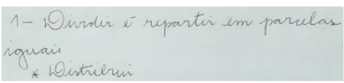
[10] Due to lack of space, we did not ask the complete question, but we refer to the main focuses associated with each of the points included in the question.

DOI: 10.20396/zet.v29i00.8661906

type (i.j), where "i" is related to the subgroup (1, 2 or 3) and "j", to the transcription line. In the case of the plenary discussion, the lines were indicated by (PLj), where "j" indicates the numbering of the line. All actions and gestures produced by the participants were duly indicated in bold in a line in the transcripts. In addition, the comments of each teacher are associated with their pseudonyms, indicated by PN, where "N" corresponds to the numbering assigned to each one, and the comments of the trainer, who is also the first author of this work, are indicated by F1. In fact, in the TTP, the two authors of this work acted as trainers at different moments in the program, having been responsible for preparing and implementing the training tasks, streamlining the meetings with the teachers.

After this stage of processing the data, the material, composed of the transcripts of the discussions in subgroup and plenary and the written productions, was analyzed.

Chart 1 - Example of the organization of information with associated analysis

	Teachers' productions
Group Discussion	2.1 P8: *Sharing... not necessarily in equal parts*. 2.2 **Teacher points to the symbol ÷ registered on the blackboard** 2.3 *Now, if she had placed the symbol, it would already be in equal parts*.
Group Discussion	3.16 P18: *"What is it to divide?". It is giving a little of what you have to the other.* 3.17 P20: *For the second year?* 3.18 P18: *"Teacher, he wants to share this with me!". "Give a little of what you have to him.* 3.19 *Share it with him! "* 3.20 P20: *Give a little of what you have to the other. That.*
Plenary discussion	PL5 P14: *For us, dividing is sharing and distributing.* [. . .] PL12 P2: *For us, we said that it is to share in equal portions and we also say that it is* PL13 *to distribute.* PL14 F1: *Okay. So is it "sharing and distributing"* (with **an emphasis on the word "and"**)*? Or is it "sharing* PL15 *or distributing"?* (**emphasizing the word "or"**) PL16 P2: *Yes ... because we put an asterisk in "distributing". So...* PL17 P18: *It is to share and distribute. This is what we say, as being another item.*
Written records	 1) *Dividing is sharing and distributing.*  *1) Dividing a certain amount.*  *1 - Dividing is sharing into equal portions * Distributing*
Analysis	Division associated with the semantic meaning of the term: "to share", without necessarily having the parts equivalent. The mathematical symbol of the division is that which allows it to be associated with dividing or distributing it in equivalent parts. They are related to the physical action of distribution.

DOI: 10.20396/zet.v29i00.8661906

Associated descriptor	**KoTd1** - Knowledge of that it is a necessary condition to effect a division that decomposes the dividend into parts, and that it is sufficient condition that the parts are equivalent.

Source: Authors (2020).

From the analysis, elements emerged that are characteristic of the specificities of the mathematics teachers' knowledge in relation to the topic of division and, from an exhaustive cyclical process, integrating all the information sources together, it was possible to compose a set of descriptors of the Specialized Knowledge related to KoT dimensions.

Chart 2 - Nomenclature associated with the descriptors related to the KoT categories

Subdomain *Knowledge of Topics* **(KoT)**	*Definitions, properties and foundations* (d)	*Phenomenology and applications* (ph)	*Procedures* (mp)	*Registers of representation* (rp)
	KoTd1; KoTd2; …	KoTph1; KoTph2; KoTph3, ...	KoTmp1; KoTmp2; KoTmp3, ...	KoTrp1; KoTrp2; KoTrp3, …

Source: authors (2020).

These descriptors were named according to the categories to which they are associated (Zakaryan & Ribeiro, 2018), and for each one they were assigned a number indicating the order in which they appear in the analysis. In the discussion, in some cases, its description is presented in a synthetic way, when associated with the identified evidence of the knowledge revealed by the Mathematics teacher and, in a detailed way, in the table that presents the synthesis of the results. (Table 1, in the following section).

Analysis and Discussion

When discussing the first question, two of the groups (2 and 3) revealed knowledge of the term "dividing" that does not consider the mathematical concept of the operation, as they exclusively adhered to the semantic sense of the term:

Chart 3 - Teachers' comments related to the first question of the task

Excerpt from the discussion in Group 2
2.1 P4: Come on... what is dividing? What do you think? *2.2* P8: *Sharing... not necessarily in equal parts.* *2.3* **Teacher writes the symbol ÷** *2.4 Now, if she had placed the symbol, it would already be in equal parts.*
Excerpt from the discussion in Group 3
3.3 P2: Dividing is sharing. 3.4 P20: Yes. 3.5 P18: I think so, too. 3.6 P20: And, for me, to divide is to share equally.

DOI: 10.20396/zet.v29i00.8661906

Excerpt from the plenary discussion
PL5 P14: *For us, dividing is sharing and distributing.*
[. . .]
PL13 P2: *For us, we said that it is to share in equal portions and we also say that it is*
PL14 *to distribute.*
PL15 F1: *Okay. So is it "sharing and distributing"* (**emphasizing the word "and"**)? *Or is it "sharing*
PL16 *or distributing"?* (**emphasizing the word "or"**)
PL17 P2: *Yes ... because we put an asterisk in "distributing".*
PL18 P18: *It is to share and distribute. This is what we say, as being another item.*

Source: Authors' archive (2018)

Teachers consider that the division must be related to the symbol "÷", so that it can be understood as an operation associated with obtaining (result) equivalent parts (2.3 - 2.4; 3.18–3.20). When associating the operation to be shared and/or distributed, they reveal to know that dividing is linked to the physical action of distributing elements of a group in subgroups, but not necessarily of the same cardinality, particularly when the division is interpreted as partitive (**KoTd1**: Knowledge of the condition for making a division). This knowledge illustrates the need to know the role of the dividend and the divisor in the division (Correa et al., 1998) **-** KoTd2: to know the roles of each element involved in the division.

In Group 3, they considered that dividing is an operation that results in subgroups with the same cardinality (**KoTmp2**: Knowledge of the nature of the result of the operation in the partitive meaning– that dividing is an operation that results in subgroups with the same cardinality). However, they emphasize the need to distinguish the notions of "sharing" and "distributing" (PL13 – PL18), which may be related to their knowledge about the typical distinction between the meanings of the division involving discrete quantities (**KoTph1**: Knowledge of the division phenomenon as partitive) or continuous quantities (**KoTph2**: Knowledge of the division phenomenon as a measurement), as well as the roles that represent dividend and divisor in a division operation (**KoTd2**: Knowledge of the role of the dividend and divisor in the division), as identified in the excerpt from Chart 4.

Chart 4 - Plenary discussion on the first question of the task

PL58	P4: *That's why I said, when you talk about dividing, they get into the concept of half.*
PL59	P18: *For 7-year-olds, around 6, 7, you tell them to divide, they get a ruler!*
PL60	P20: *Millimeter by millimeter, well divided.*

Source: Authors' archive (2018)

Indeed, although teachers seem to consider that there is a distinction in the way in which the division can be interpreted from its two meanings (Fischbein et al., 1985), in the course of the discussion, they show that this distinction is related to a type of language – verbal and symbolic – that they consider more or less adequate to discuss the division (**KoTrp1**: Knowledge of the role of the different representational systems in the attribution of meaning to concepts), exclusively as partitive (**KoTph1**).

DOI: 10.20396/zet.v29i00.8661906

Chart 5 **-** Plenary discussion focusing on verbalization to be used in the contexts of the division

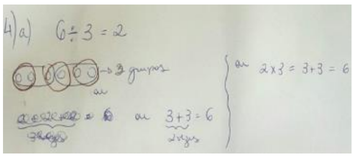
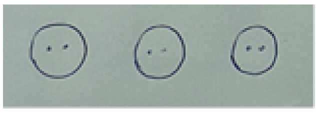
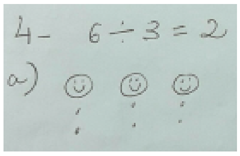
PL118 F1: *You said that, for the sixth year, you would say that to divide is to share equally.* PL119 *Do you all agree with that?* PL120 Ps: *Yes, for the sixth year, yes ...* PL121 **Trainer reads one of the records on the board.** PL122 F1: *Because it is expected that in the sixth year the algorithm has already been taught.* PL123 *So, my question is: what of the algorithm forces me, or allows me... provokes me to say that* PL124 *are in equal parts? Why is the algorithm associated with the idea of equal parts?* PL125 P19: *Because you left from the operation! And the operation has the symbol!*

Source: Authors' archive (2018)

For these teachers, this type of language must be used specifically in the 6th year of the Secondary, since, in this educational stage, *"it is expected that the algorithm has already been taught"* (PL122). In fact, by associating the mathematical symbol "÷" with one of the meanings of the division – **KoTrp1** – they revealed Knowledge of the role that symbolic language plays in the attribution of meaning to mathematical constructs (Golden & Shteingold, 2001; Lesser & Tchoshanov, 2005; Timmerman, 2014). However, they do not discuss the role that this symbol plays, when associated with each of the meanings of division (**KoTrp2:** Knowledge of the meaning of the symbol "÷", i.e., that in the case of partitive, it is related to the distribution of a whole in subgroups of equivalent cardinalities; and, for the measurement meaning, it must be interpreted as a comparison relationship between quantities – measurement).

Still regarding the knowledge of the different representational systems, during the discussion of item "a" of question 4, in which they were asked to determine the result of 6÷ 3 and describe the procedures used to find the solution, the three groups provided representations that were based on the pictorial and numeric type (**KoTrp1**).

Chart 6 **-** Written records of the three groups, related to the fourth question of the task

Group 1	Group 2	Group 3
		

Source: Authors' archive (2018)

Only Group 1 sought to establish some correspondence between the two types of records, by associating an addition ($\underbrace{3+3}_{2\ times} = 6$) to the representations of two groups of three elements. This type of record reveals a knowledge of different ways of navigating fruitfully between representations (Ribeiro, 2011b), in addition to indicating that teachers understand that it is important to establish relationships between the different forms of representation to give meaning to mathematical constructs (Ainsworth et al. , 2002; Golden & Shteingold,

DOI: 10.20396/zet.v29i00.8661906

2001; Timmerman, 2014) – **KoTrp3**: Knowledge of ways to establish relationships between different representations. However, these representations are associated with considering division with the sense of measure (**KoTph2**), and not as partitive **(**KoTph1), but this correspondence is not consciously established by teachers.

Chart 7 **-** Excerpt from the plenary discussion related to the most appropriate types of representation

PL326	P14: *We drew six balls and circulated every three.*
PL327	F1: *So, wait!*
PL328	**Trainer represents six circles on the blackboard**
PL329	*You drew six balls ...*
PL330	P14: *Yes, and I circulated every three.*
PL331	P19: *No!*
PL332	P18: *Did you do two groups, then?*
PL333	P4: *No, it was two at a time!*
PL334	P19: *No, every two.*
PL335	P14: *Every three!*
PL336	**Trainer circulates two groups of three elements** 
PL337	F1: *Did you do it like that?*
PL338	**Trainer draws six more balls and circles three groups of two elements** 
PL339	F1: *Or did you do it like that?*
PL340	P18: *But it was to be divided by three ...*
PL341	P14: *So, six balls ...*
PL342	P19: *It was to be divided by three!*
PL343	P14: *Divided by three ... every three. That's what I did!*
PL344	*Six balls, circulated every three.*
PL345	P4: *There will be two groups!*
PL346	P14: *How many groups will there be? Two!*

Source: Authors' archive (2018)

During the plenary, teachers' doubts about the types of pictorial representations most appropriate to evoke the meanings of the division are evident, in correspondence with the use of a specific verbalization that gives sense to the evoked meaning, when the operation is presented in the form of a numeric expression (6÷ 3). In fact, the verbalization that P14 employs (PL326) causes doubt in P19 (PL 334), one of the components of Group 1, which presented two types of pictorial grouping records (three in three and two in two) – cf. Chart 5 - **KoTrp4**: Knowledge of the role of verbalization. This fact may be related to the teachers' uncertainty about the most appropriate type of record to represent 6÷ 3 = 2 (**KoTrp3**).

In fact, P4 (PL330 – PL333), reveals knowledge of the pictorial representation with clusters of two in order to compose three groups is the most adequate for the resolution of the operation (**KoTrp5:** Knowledge of the types of pictorial representation more adequate associated to the meanings of the division). However, at the same time, it does not establish the correspondence that this grouping is the unit of measurement with which the whole is being measured (Behr et al., 1994), which highlights the need to discuss with these teachers the role of the divider in the operation – **KoTd2** –, and also to expand their knowledge about

DOI: 10.20396/zet.v29i00.8661906

the measure's sense of division – **KoTph2**.

In fact, only based on the pictorial representation (PL338), it is not possible to say whether it corresponds to (i) 6÷ 3 = 2 or (ii) 6÷ 2 = 3, because if (i) is interpreted with the meaning of partitive, the conceptual image (e.g., Golden & Shteingold, 2001; Timmerman, 2014) that is created is that of a distribution of six elements in three subsets with the same cardinality, two. In this case, the pictorial representation provided at the end of the operation would correspond to something as presented by Group 2., or even as presented by Group 3 , which shows the need for an adequate verbalization (**KoTrp4**), with correspondence between the evoked sense and the pictorial representation to assign meaning to that sense (**KoTrp5**).

Thus, it is the content of the Mathematics Teachers' Specialized Knowledge to know that  the representation can be associated with either 6÷ 3 = 2 or 6÷ 2 = 3 (**KoTrp5**), and therefore it is necessary to articulate more than one representation (numerical, for example), including adequate verbalization (Simon, 1993), to establish the correspondence between the expression and the elements involved in this operation (dividend, divisor and quotient) – **KoTd2**.

Also the verbalization of P14 (PL330; PL343-344), associated with the pictorial representation of groupings of three, evokes the meaning of measurement for 6÷ 3 = 2 (**KoTph2**), linked to the idea of "how many groups of three elements fit in a group of six elements?". However, despite these discussions, teachers do not explicitly recognize the division beyond the partitive meaning (**KoTph1**) or, at least, do not relate the verbalization of "how many times it fits" as an inappropriate association with such a meaning (**KoTrp4**). This is clear from the analysis of one of the sub-questions of the 4th question of the task, in which teachers should propose an introductory task on the division for 3rd year students, associated with the discussion of the different meanings of the division (Fischbein et al., 1985).

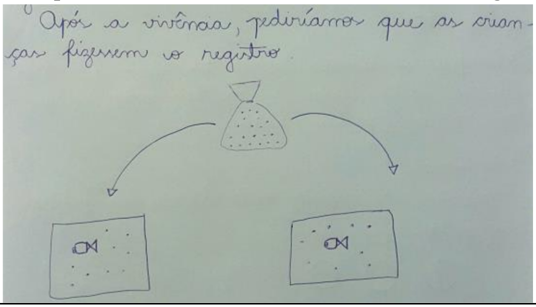
Chart 8 - Discussion and written record of Group 2, associated with the proposal for an introductory task for the 3rd year.

2.256	P8: *The problem, would we have to work something concretely? Is that it?*
2.257	P4: *And how would we introduce it?*
2.258	*Any proposal that needed to share some things and then exemplify what we did,*
2.259	*on the board with them, with a drawing of the sharing.*
2.260	P8: *In the third.*
2.261	P4: *Because it is introductory, right? We introduce with a drawing first.*
2.262	*We don't introduce it with the algorithm.*
2.263	[. . .]
2.286	P4: *So, there will be two aquariums in the room, we receive 20 fish.*
2.287	P8: *It could be.*
2.288	P4: (...) *I think we could also approach them*

DOI: 10.20396/zet.v29i00.8661906

2.289	*with the question, "is it right for one to have more and the other less? No.*
2.290	*So, the division, it's always fair when it's in equal parts", isn't it?*
[. . .]	
2.343	P3: *How can we divide so that the two classes have the same amount*
2.344	*of fish?*
2.345	P4: *I think that then we'll build with them the issue of the division being ... the concept of division.*
2.346	*So ok, now how can we do it?*
2.347	**P4 positions two paper sheets side by side**
2.348	*We have two aquariums.*
[. . .]	
2.355	P4: *Well, then there are 20... how can we share for the class? Hence it may be that some of them*
2.356	*say: "Ah, give them three and the rest stay with us!" They are sharing. "But you*
2.357	*you think this is a fair division? " then, "oh no!". I think the ideal, the right thing, is to make them*
2.358	*reach that concept.*

After the experience, we would ask the children to register

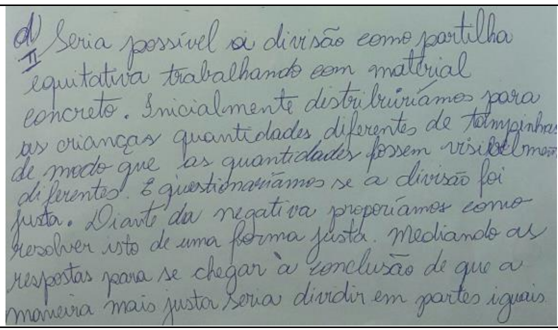


Source: Authors' archive (2018)

For teachers, introducing the division operation with 3rd year students means, on the one hand, developing work based on pictorial representations (2.256 - 2.262), which reveals that teachers know the role of this type of representation in the process of attributing meaning to concepts and mathematical constructs – **KoTrp2** – (e.g., Ainsworth et al., 2002; Golden & Shteingold, 2001; Timmerman, 2014). On the other hand, the notion of introducing the division operation is restricted to work strictly related to the sense of partitive (2.286 - 2.358) – **KoTph1** –, leaving aside the discussion of the sense of measure (Fischbein et al., 1985) – **KoTph2**.

Chart 9 - Group 1 record, associated with the introductory task for the 3rd year

The division would be possible as partitive working with concrete material. Initially we would distribute different amounts of bottle caps to the children, so that the amounts were visibly different. And we would question whether the division was fair. In the face of the negative, we would propose how to resolve this in a fair way, mediating the answers to arrive at the conclusion that the fairest way would be to divide it in equal parts.

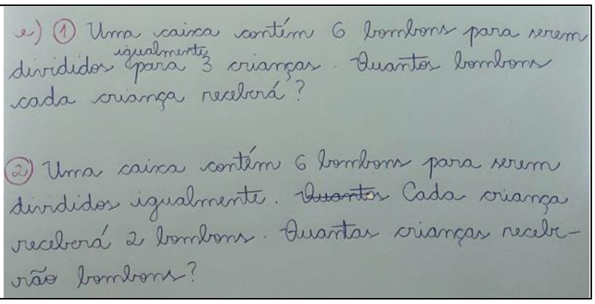

d) II Seria possível a divisão como partilha equitativa trabalhando com material concreto. Inicialmente distribuiríamos para as crianças quantidades diferentes de tampinhas de modo que as quantidades fossem visivelmente diferentes. E questionaríamos se a divisão foi justa. Diante da negativa proporíamos como resolver isto de uma forma justa. Mediando as respostas para se chegar à conclusão de que a maneira mais justa seria dividir em partes iguais

DOI: 10.20396/zet.v29i00.8661906

Source: Authors' archive (2018)

All groups recognize the importance of creating contexts (problems) to introduce the notion of division (Downton, 2009) – **KoTph3:** Knowledge of the types of problems to evoke the meanings of division. However, only Group 2 formulated problems with two mathematical contexts that evoked the different meanings of the division (**KoTph3**) – objective of sub-question 4 (e).

Chart 9 **-** Written record with the different problems elaborated by Group 2

e) 1) A box contains 6 candies to be shared equally for 3 children. How many candies will each box receive? *2) A box contains 6 candies to be divided equally. ~~How many~~ Each child will receive 2 candies. How many children will receive candies?*	 e) ① Uma caixa contém 6 bombons para serem divididos igualmente para 3 crianças. Quantos bombons cada criança receberá? ② Uma caixa contém 6 bombons para serem divididos igualmente. ~~Quantos~~ Cada criança receberá 2 bombons. Quantas crianças receb-rão bombons?

Source: Authors' archive (2018)

Although they presented a context evoking each of the meanings of the division, the second problem proposed (Chart 9), which evokes the measurement meaning, does not correspond with the expression of the statement 6 ÷3 = 2, but with 6÷2 = 3. The need for this correspondence is part of the Mathematics Teacher's Specialized Knowledge regarding the role of the dividend and the divisor in the division (Correa et al., 1998; Squire & Bryant, 2002), when the operation is interpreted from each of its two meanings – KoTd2 **-** , and also on the nature of the result of the division in the measurement meaning (**KoTmp3**: Knowledge of the nature of the result of the division as a measure).

During the discussion of the task, the need to deepen the distinction between the two meanings of the division became evident – in particular the types of contexts that evoke each of the meanings (**KoTph3**), which is also related to the fact that the use of a particular verbalization, especially in the traditional algorithm, does not suit the sense of partitive (**KoTrp4**).

Chart 10 **-** Excerpt from the discussion in which the division's *measurement* meaning is discussed

PL436	F1: *[...] I want the phrase that P20 says when she solves this algorithm*
PL437	*How many times is three within six?*
PL438	P20: *It's within six ...*
PL439	F1: *So, I ask again: What is dividing?*

DOI: 10.20396/zet.v29i00.8661906

PL440	P14: *See how many times something ...*
PL441	P20: *Sharing.*
PL442	P2: *Sharing.*
PL443	F1: *Wait a minute! If it is sharing, where is the idea of sharing with this sentence?*
PL444	*How many times is three within six?*
PL445	P8: *Nowhere!*
PL446	P2: *Wow!*
PL447	P19: *Wow!*
PL448	F1: *Because the idea of sharing, is the idea of me ...*
PL449	**Trainer simulates distributing quantities to each participant.**
PL450	P2: *Distributing!*

Source: Authors (2018)

In order to discuss the teachers' knowledge about the measure meaning associated with the division (**KoTph2**), it was necessary for the trainer (PL473) to take the notion of comparison of quantities as a starting point (Clements & Stephan, 2004) and to use a certain verbalization (Simon, 1993) associated with an image of quantity distribution (Ainsworth et al., 2002; Golden & Shteingold, 2001) - **KoTrp3**. This discussion is linked to the knowledge associated with the type of context (idea contained therein) that can be evoked to give meaning to the division phenomenon (**KoTph3**).

All teachers revealed that they knew more than one way of proceeding to resolve an operation (Bisanz & LeFevre, 1992). An example of this is the discussion in Group 2 regarding item a) of the fourth question related to the expression 6÷3 (Determine the result of this operation and describe the procedures and steps taken to find the answer):

Chart 11 - Excerpt from the discussion in Group 2 related to item a) of question 4

2.185	F1: *Anyway, and if it were any other value, if it were any other*
2.186	*numbers that you imagine there, I don't know, 36 divided by 3. How would you do*
2.187	P4: *With the algorithm!*
2.188	F1: *The algorithm?*
2.189	P3: *Yeah!*
2.190	P4: *I would do three divided by three, one.*
2.191	F1: *Why is the ...*
2.192	P4: *And six divided by three, two.*
2.193	P8: *I think I would take 30 first and divide by three. Then I would take six and divide*
2.194	*by three.*
2.195	F1: *Okay. Describe what ...? What image do you form in your head when*
2.196	*you divide six by three?*
2.197	P4: *I make three ... three parts with ... I mean, two parts with three.*
2.198	P8: *Two!*
2.199	P4: *No! Three parts with two (elements) in each.*
[. . .]	
2.209	P8: *Yeah, but I think that's really what we think. You end up adding 2 + 2 + 2.*
2.210	*And, if we were to think about the mental calculation with which she spoke, 36, we could*
2.211	*get the tens first ... then the units ...*

Source: authors (2018)

To refine the discussions, it was necessary to suggest a reflection on the division,

DOI: 10.20396/zet.v29i00.8661906

involving larger quantities, because for teachers operation 6÷3 "*is already an automatic answer*" (production written on the answer sheet). In addition, they reveal knowledge (**KoTmp1**: Knowledge of different strategies for solving a division) associated with recognizing the procedures associated with the Euclidean algorithm (2.187–2.192) as the possibility of resolving the operation; the breakdown of the dividend into parcels corresponding to tens and units (2.193–2.194; 2.210–2.211); the groupings (2.197–2.199); and successive additions (2.209–2.211).

In a summary form of the results obtained, which focus on the specificities of the KoT mobilized (and revealed) by Mathematics teachers within the scope of the division in a CEP, we present the categories and descriptors of the knowledge obtained.

Table 1 - Categories and descriptors related to the *Knowledge of Topics* subdomain in the division topic

Categories	Descriptors
Definitions, properties and foundations (KoTd)	**KoTd1**: Knowledge of that it is a necessary condition to dividing that the dividend is decomposed into parts, and that it is a sufficient condition that these parts are equivalent. **KoTd2**: Knowledge of the role of the dividend and the divisor in the division: in the sharing, dividend corresponds to the whole to be distributed and divisor corresponds to the number of subgroups among which the whole will be distributed; in the measure, dividend corresponds to the whole to be measured; and divisor, to the unit of measurement.
Phenomenology and applications (KoTph)	**KoTph1**: Knowledge of the phenomenon of division from its interpretation as partitive. **KoTph2**: Knowledge of the division phenomenon from its interpretation as a measure. **KoTph3:** Knowledge of the types of problems and contexts (ideas contained in them) that contribute to evoke each of the meanings of the division, i.e., in the sense of sharing, the idea of distributing elements of a set in subsets; in the sense of measurement, the idea of comparing quantities of the same magnitude ("how many times does one quantity fit inside the other?").
Procedures (KoTmp)	**KoTmp1:** Knowledge of different strategies for solving a division: a division can be solved by the Euclidean algorithm; by decomposing the dividend into parcels corresponding to tens and units; by grouping; and by successive additions. **KoTmp2**: Knowledge of the nature of the operation's result in the sense and partitive: the numerical value obtained – quotient – corresponds to the cardinality of each of the sets among which the whole was distributed. **KoTmp3**: Knowledge of the nature of the result when the division is understood as a measure: the numerical value obtained (quotient) corresponds to the number of times that the reference unit fits into the whole (grouping the whole into parts of the same magnitude).

DOI: 10.20396/zet.v29i00.8661906

Systems of representations (KoTrp)	**KoTrp1**: Knowledge of the role of different representational systems in the attribution of meaning to concepts. **KoTrp2**: Knowledge of the meaning attributed to the symbol "÷": it relates to the distribution of a whole in subgroups of equivalent cardinalities, when the operation is taken as partitive; and the symbol "÷" is associated with the establishment of a comparison relationship between quantities (measures). **KoTrp3**: Knowledge of ways to establish relationships between different representations to assign meaning to concepts, properties and/or procedures. **KoTrp4**: Knowledge of the role of the use of a given verbalization to give meaning and/or correspond to each of the meanings of the division. **KoTrp5:** Knowledge of the relationship and the adequacy of the types of pictorial representation, associated with each of the meanings of division: in partitive, indicating distribution; in the measure, indicating groupings.

Source: Compiled by the authors

It is possible to observe a greater emergence of descriptors related to the categories *Definitions, properties and foundations* and *Systems of representations*. In fact, KoT is, among the subdomains that make up the conceptualization of MTSK, the one that most relates to the Mathematics Teacher's Specialized Knowledge associated with Knowledge of how to do – which is considered knowledge at the level of the students – and with attributing meaning to concepts and procedures, from mathematical foundations and properties. This greater emergency is also justified by the nature and focus of the questions included in the training task, discussed here and by the typical practice of teachers.

Final comments

In this work, we focus on explaining the aspects that characterize the Mathematics Teacher's Specialized Knowledge related to the topic of division. This characterization aims at explaining the specificities and particularities of the teacher's knowledge in this topic and also aims at representing a way of structuring and organizing the understandings that are held about the specificities and particularities of the Mathematics Teacher's Specialized Knowledge, in each of the mathematical topics – which constitutes an advance in the way of facing this specialization (Ball et al., 2008).

Carrying out a discussion that leads to the emergence of this structuring, organization and description of specificities, from the teacher's point of view, a way for them to become aware of the nature of one of the dimensions of their professional knowledge, which also allows them to understand the role of research to improve training and practice. On the other hand, from the point of view of teacher training, such structuring and organization contribute by providing new focuses of intentionality, for example, in the conceptualization of Formative Tasks (Ribeiro et al., In press) to develop these specificities of knowledge.

Special attention to the nature and content of this knowledge and to the focus of training is essential whenever it is intended to develop the specificities of the knowledge of

DOI: 10.20396/zet.v29i00.8661906

teacher of and who teaches mathematics, as these specificities do not develop only with practice (Jakobsen et al. , 2014). Thus, the results of this work contribute to also (re)orient the focuses and objectives of the training programs (initial and continuing).

At the same time, the results obtained and the analytical processes involved, in addition to having potentialities in terms of research and training, leave open some important discussions to consider – as this is also a way of contributing to the advancement of the field – and which refer, for example, to what will happen with these descriptors of the content of the mathematics teacher's specialized knowledge when confronted (stressed) with a broader analysis of the participants' responses to the entire task and with the responses of other group(s) to the same task.

Another important aspect to be considered relates to the connections between different topics that can be identified in the knowledge of the participants (*Knowledge of Structures of Mathematics*), if we change the focus of the analytical lens – in particular, connections in the scope of division and measurement. Such problems support some research questions that emerge from analysis and discussion, such as: (i) what aspects characterize the content of the teacher's Specialized Knowledge about elements that make up the mathematical structures, within the scope of the division? (ii) what relationships exist (if any) between the knowledge descriptors associated with each of the KoT and KSM subdomains, within the scope of the division? (iii) what connections between the topic of division and the topics of Measure do teachers make, when discussing formative tasks?

In addition, these questions are also considered as one of the results that allow us to reflect on the directions that we can take in research in Mathematical Education with a focus on knowledge, practice and training of teachers of and who teach mathematics.

References

Ainsworth, S., Bibby, P., & Wood, D. (2002). Examining the effects of different multiple representational systems in learning primary mathematics. *Journal of the Learning Sciences*, *11*(1), 25–62.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407. DOI: https://doi.org/10.1177/0022487108324554

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1994). Units of quantity: A conceptual basis common to additive and multiplicative intents. In G. Harel & J. Confrey, *The development of multiplicative reasoning in the learning of mathematics* (pp. 121–176). Albany, NY: State University of New York Press.

Bicknell, B., Young-Loveridge, J., Lelieved, J., & Brooker, J. (2015). Using multiplication and division contexts with young children to develop part-whole thinking. *Journal Issue*, 2, 53–59. DOI**:** http://dx.doi.org/10.18296/set.0018

DOI: 10.20396/zet.v29i00.8661906

Bisanz, J., & LeFevre, J.-A. (1992). Chapter 3 Understanding Elementary Mathematics. In J. I. D. Campbell (Org.), *Advances in Psychology* (Vol. 91, pp. 113–136). North-Holland. DOI: https://doi.org/10.1016/S0166-4115(08)60885-7

Boyd, D. J., Grossman, P. L., Lankford, H., Loeb, S., & Wyckoff, J. (2009). Teacher Preparation and Student Achievement. *Educational Evaluation and Policy Analysis*, *31*(4), 416–440. DOI: https://doi.org/10.3102/0162373709353129

Brocardo, J., Serrazina, L., & Kraemer, J.-M. (2003). Algoritmos e sentido do número. *Educação e Matemática*, *75*(Nov/Dez), 11–15.

Carreño, E., Rojas, N., Montes, M. Á.., & Flores, P. (2013). Mathematics teacher's specialized knowledge. Reflections based on specific descriptors of knowledge. *Proceedings of CERME 8*, 2976–2984. Antalya, Turquia: METH.

Carrillo, J., Climent, N., Montes, M., Contreras, L. C., Flores-Medrano, E., Escudero-Ávila, D., … Muñoz-Catalán, M. C. (2018). The mathematics teacher's specialised knowledge (MTSK) model. *Research in Mathematics Education*, 20(3), 236–256. DOI: https://doi.org/10.1080/14794802.2018.1479981

Clements, D., & Stephan, M. (2004). Measurement in pre-K to grade 2 mathematics. In D. Clements, J. Sarama & A.-M. DiBiase (Orgs.), *Engaging Young Children in Mathematics: Standards for Early Childhood Mathematics Education* (pp. 299–317). New Jersey: LEA.

Correa, J., Nunes, T., & Bryant, P. (1998). Young children's understanding of division: The relationship beteween division terms in a noncomputational task. *Journal of Educational Psychology*, *90*(2), 321–329. DOI: https://doi.org/10.1037/0022-0663.90.2.321

Dooren, W. V., Verschaffel, L., & Onghena, P. (2002). The impact of preservice teachers' content knowledge on their evaluation of students' strategies for solving Arithmetic and Algebra words problems. *Journal for Research in Mathematics Education*, *33*(5), 319–351. DOI: https://doi.org/10.2307/4149957

Downton, A. (2009). It Seems to Matters Not Whether it is Partitive or Quotitive Division When Solving One Step Division Problems. In R. Hunter, B. Bicknell & T. Burgess, *Crossing divides: Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 161–168). Palmerston North, NZ: MERGA.

Fávero, M. H., & Neves, R. S. P. (2012). A divisão e os racionais: Revisão bibliográfica e análise. *Zetetiké*, *20*(37), 33–67. DOI: https://doi.org/10.20396/zet.v20i37.8646635

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 147–164). New York: National Council of Teachers of Mathematics.

Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Reaserch in*

DOI: 10.20396/zet.v29i00.8661906

Mathematics Education, *16*(1), 3–17. DOI: https://doi.org/10.5951/jresematheduc.16.1.0003

Fosnot, C. T., & Dolk, M. (2001). *Young mathematics at work: Constructing multiplication and division*. Portsmouth, NH: Heinemann.

Golden, G., & Shteingold, N. (2001). Systems of representation and the development of mathematical concepts. In A. A. Cuoco & F. R. Curcio, *The role of representation in school mathematics* (pp. 1–23). Boston, Virginia: NCTM.

Gómez, P., & Cañadas, M. C. (2016). Dificultades de los profesores de matemáticas en formación en el aprendizaje del análisis fenomenológico. *Revista Latinoamericana de Investigación en Matemática Educativa*, *19*(3), 311–334. DOI: https://doi.org/10.12802/relime.13.1933

Hiebert, J., & Grouws, D. (2007). The effects of classroom mathematics teaching on students' learning. In F. Lester (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 371–404). New York: Information Age Publishing.

Jakobsen, A., Ribeiro, M., & Mellone, M. (2014). Norwegian prospective teachers' MKT when interpreting pupils' productions on a fraction task. *Nordic Studies in Mathematics Education*, *19*, 135–150.

Lautert, S. L., Oliveira, D. C. A.., & Correa, J. (2017). Compreensão de crianças sobre relações inversas sem explicitação numérica. *Arquivos Brasileiros de Psicologia*, *69*(1), 73–89.

Lesser, L. M., & Tchoshanov, M. A. (2005). The effect of representation and representational sequence on students' understanding. In G. M. Lloyd, M. Wilson, J. L. M. Wilkins & S. L. Behm, *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 813–819). Roanoke, Virginia: Virginia Tech.

Mason, J., Stephens, M., & Watson, A. (2009). Appreciating Mathematical Structure for All. *Mathematics Education Research Journal*, *21*(2), 10–32. DOI: https://doi.org/10.1007/BF03217543

Ribeiro, M. (2011a). A importância do conhecimento do conteúdo matemático na prática letiva de uma professora: Discutindo um modelo de análise. *Zetetiké*, *19*(35), 71–102. DOI: https://doi.org/10.20396/zet.v19i35.8646646

Ribeiro, M. (2011b). Abordagem aos números decimais e suas operações: A importância de uma "eficaz navegação" entre representações. *Educação e Pesquisa*, *37*(2), 407–422.

Ribeiro, M., & Amaral, R. (2015). Early years' prospective teachers' specialized knowledge on problem posing. *Proceedings of 39th Psychology of Mathematics Education Conference*, *4*, 81–88. Hobart, Australia: Beswick, K., Muir, T. & Wells, J.

Ribeiro, M., Carrillo, J., & Monteiro, R. (2012). Cognições e tipo de comunicação do Professor de matemática. Exemplificação de um modelo de análise num episódio

DOI: 10.20396/zet.v29i00.8661906

dividido. *Revista latinoamericana de investigación en matemática educativa*, *15(1)*, 93–121.

Ribeiro, M, Almeida, A. R., & Mellone, M. (2019). Desenvolvendo as especificidades do conhecimento interpretativo do professor e tarefas para a formação. In V. Giraldo, J. Viola, & H. R. Elias, *Problematizações sobre a Formação Matemática na Licenciatura em Matemática* [s.l.]. SBEM. No prelo.

Ribeiro, M, Policastro, M., Mamoré, J., & Di Bernardo, R. (2018). Conhecimento Especializado do professor que ensina Matemática para atribuir sentido à divisão e ao algoritmo. *Educação Matemática em Revista – RS*, *1*(19), 152–167.

Rizvi, N. F. (2007). Prospective teachers' knowledge: Concept of division. *International Education Journal*, *8*(2), 377–392.

Robinson, K. M., & LeFevre, J.-A. (2012). The inverse relation between multiplication and division: Concepts, procedures, and a cognitive framework. *Educational Studies in Mathematics*, *79*(3), 409–428. DOI: https://doi.org/10.1007/s10649-011-9330-5

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, *8*(3), 255–281. DOI: https://doi.org/10.1007/s10857-005-0853-5

Simon, M. A. (1993). Prospective Elementary Teachers' Knowledge of Division. *Journal for Research in Mathematics Education*, *24*(3), 233–254. DOI: https://doi.org/10.5951/jresematheduc.24.3.0233

Skemp, R. R. (1989). *Mathematics in the primary school*. London: Routledge Falmer.

Squire, S., & Bryant, P. (2002). From sharing to dividing: Young children's understanding of division. *Developemental Science*, *5*(4), 452–466. DOI: https://doi.org/10.1111/1467-7687.00240

Timmerman, M. A. (2014). Making Connections: Elementary Teachers' Construction of Division Word Problems and Representations. *School Science and Mathematics*, *114*(3), 114–124. DOI: https://doi.org/10.1111/ssm.12059

Young-Loveridge, J., & Bicknell, B. (2018). Making connections using multiplication and division contexts. In V. Kinnear, T. Muir & M. Y. Lai (Orgs.), *Forging Connections in Early Mathematics Teaching and Learning*, *Early Mathematics Learning and Development*, (pp. 259–272). Brisbane, Autralia: Lyn D. English.

Young-Loveridge, J. M. (2001). Helping children move beyond counting to part-whole strategies. *Teachers and Curriculum*, *5*, 72–78.

Zakaryan, D., & Ribeiro, M. (2018). Mathematics teachers' specialized knowledge: A secondary teacher's knowledge of rational numbers. *Research in Mathematics Education*, *21*(3), 1–19. DOI: https://doi.org/10.1080/14794802.2018.1525422