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Covariational approach to functions: aspects of teaching, learning, and possibilities of digital technologies

Abordagem covariacional de função: aspectos do ensino, aprendizagem e possibilidades das tecnologias digitais

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Abstract

Covariation involves focusing on how variables or quantities vary together. This paper describes a systematic literature review that aims to analyze recent research on the covariational approach to functions and the possibilities of digital technologies to support this approach. Data were collected on Periodicos Capes and Eric databases, resulting in 26 studies, 11 involving digital technologies. The results showed: cognitive processes and learning difficulties associated with covariational reasoning; specificities of the epistemology of each function; didactic influences on the covariational approach, from curriculum to task design and teachers' knowledge; and finally, aspects of digital technologies that can support or limit covariational reasoning.

Keywords: Systematic Literature Review; Functions; Covariation; Digital Technologies

Resumo

A covariação envolve o foco em como as variáveis ou quantidades variam em conjunto. Este artigo descreve uma revisão sistemática de literatura que teve por objetivo analisar um quadro recente de pesquisas sobre a abordagem covariacional de função e as possibilidades das tecnologias digitais nessa perspectiva. Os dados foram coletados nas bases Periódicos Capes e Eric, resultando em 26 estudos, dos quais 11 envolveram o uso de tecnologias digitais. Os resultados apontaram: processos cognitivos e dificuldades de aprendizagem associadas ao raciocínio covariacional; especificidades da epistemologia de cada tipo de função; influências didáticas na abordagem de covariação, do currículo ao design de tarefas e o conhecimento de professores; e, por fim, aspectos das tecnologias digitais que podem dar suporte ou limitar o raciocínio covariacional.

Palavras-chave: Revisão Sistemática de Literatura; Funções; Covariação; Tecnologias digitais

Introduction

The covariational perspective on functions focuses on the relationship between variables or quantities. This approach has gained greater emphasis since the work of Confrey and Smith (1994) on the exponential function and rate of change, and Thompson (1994) on

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how students conceive of relationships between quantities whose values vary and the rate of change. Covariation is considered a critical aspect in constructing concepts such as function, rate of change, derivative, and integrals (Thompson & Carlson, 2017).

However, this aspect does not seem to receive appropriate attention in research or in the mathematics classroom, particularly in the Brazilian context. The National Common Core Curriculum[3] (Ministério da Educação, 2018) makes only a modest reference to the development of skills such as understanding and interpreting the variation of quantities involved in contextualized situations of exponential and logarithmic functions.

Digital technologies have been identified as an important aid for learning functions (Kaput, 1992; Ferrara, Pratt & Robutti, 2006; Lagrange, 2014). According to Kaput (1992), dynamic computational media are well-suited for representing variables, making it a natural medium for representing variation. This and other characteristics suggest that digital technologies can support the covariational approach, which emphasizes joint variation between variables.

This article presents a research framework for teaching and learning functions, with a focus on covariation, covariational reasoning, and the potential of digital technologies in these areas. The framework's description was included in a larger study that examined the impact of a computational artifact on students' covariational reasoning (Silva, 2022). Its purpose was to highlight important aspects of the covariational approach and aid in the planning and analysis of the teaching experiment conducted with the study participants.

A systematic literature review model (Ramos, Faria & Faria; 2014) was utilized to address the following questions: (i) What are the aspects involved in teaching and learning functions from a covariational perspective? and (ii) What are the possibilities, contributions, and limitations of the covariational approach using digital technologies as indicated in the studies?

The sections below outline the covariational approach framework, its concepts and theoretical advancements, the methodological aspects that guided the systematic review, the review results, and, finally, the main findings of the systematic review and research prospects in the covariational approach in the concluding remarks.

Covariational approach to functions

Confrey and Smith (1994) described two general approaches by which functional relations can be conceptualized: correspondence and covariation. While the correspondence approach emphasizes the association of a single value of x with a single value of y through a rule, the approach based on covariation implies "being able to move operationally from y_m to y_{m+1} coordinating with movement from x_m to x_{m+1}". (ibid, p. 33). In a table, for example, this

[3] The Common National Curriculum Base (BNCC) is a normative document that defines the learning to be developed by basic education students in Brazil.

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approach involves "the coordination of the variation in two or more columns as one moves down (or up) the table." (ibid, p. 33)

For them, the covariation approach is often more powerful than the correspondence approach and gives visibility and centrality to the concept of rate of change, which the authors use to characterize the exponential function from the construction of the idea of a multiplicative unit.

Thompson's covariation approach is based on his studies of how students conceive of situations as composed of quantities and relationships between quantities whose values vary and the rate of change. The notion of 'quantity' is defined by him as "someone's conceptualization of an object such that it has an attribute that could be measured." (Thompson & Carlson, 2017, p. 425). Based on this notion, the author constructs the idea of quantitative reasoning as a conceptualization of a situation in terms of quantities and relationships between quantities.

The notions of variation and covariation became necessary in Thompson's Theory of Quantitative Reasoning, to "explain the reasoning of students who conceptualized a situation quantitatively and at the same time took it as dynamic—they envisioned quantities in their conceptualized situation as having values that varied." (Thompson & Carlson, 2017, p. 425). According to Thompson's construction, covariational reasoning occurs when a person "envisions two quantities' values varying and envisions them varying simultaneously" (Thompson & Carlson, 2017, p. 425)

Furthermore, according to Saldanha and Thompson (1998), reasoning covariationally implies the conception of a multiplicative object, which according to the authors is a conceptual object created from the mental union of the attributes of two quantities:

> our notion of covariation is of someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value. (Saldanha & Thompson, 1998, p. 299)

According to Carlson, Jacobs, Coe, Larsen, and Hsu (2002), covariational reasoning involves "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354). The authors also developed a framework for analyzing students' reasoning, which is structured into levels of mental coordination of the variation of variables and focuses on the concept of rate of change.

Castillo-Garsow (2010, 2012) contributed to the covariational reasoning framework by describing ways in which variation can be conceived. The author explains that students can conceive of the value of a quantity varying discretely or continuously. Continuous variation can be further distinguished between smooth continuous variation and chunky

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continuous variation. In chunky variation, the student imagines the variation occurring in 'chunks', while in smooth variation, it is thought of as a progressing variation.

> Chunky continuous variation is a way of thinking that is similar to thinking that values vary discretely, except that the student has a tacit image of a continuum between successive values. (...) This image of variation is like laying rulers end to end and marking the endpoints. (Thompson & Carlson, 2017, p. 427)

Thompson and Carlson (2017) revisited the covariational reasoning chart (Carlson et al.,2002) and added the theoretical contributions of Confrey and Smith (1994), Saldanha and Thompson (1998), and Castillo-Garsow (2010, 2012).

Chart 1 - Levels of covariational reasoning

Level	Description
Smooth continuous covariation	The person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly and continuously.
Chunky continuous covariation	The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation.
Coordination of values	The person coordinates the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y).
Gross coordination of values	The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values.
Precoordination of values	The person envisions two variables' values varying, but asynchronously—one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects.
No coordination	The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values.

Source: Thompson and Carlson (2017, p. 441)

Thompson and Carlson (2017) also presented what they consider to be a meaning of function based on covariational reasoning:

> A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other. (Thompson & Carlson, 2017, p. 444)

In the following sections, we present and analyze the systematic review.

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Methodological aspects of the systematic review

This systematic review is based on the model proposed by Ramos et al. (2014). The model is operationalized according to the following protocol:

(i) objectives, which define the problem and research problem;

(ii) search equations, which consist of search terms associated with Boolean operators;

(iii) scope, which defines the search bases, considering their specificities.

(iv) inclusion criteria define the characteristics of studies that qualify them as acceptable based on the research objectives.

(v) exclusion criteria define the characteristics of studies that are excluded from the results based on the research objectives.

(vi) methodological validity criteria ensure the objectivity of the research.

(vii) results.

(viii) data processing.

A selection of studies involving the use of digital technologies by subjects was made after drawing a general picture of studies on covariation. Research articles were collected from two databases, ERIC (Education Resources Information Center) and Portal de Periódicos Capes (Coordination for the Improvement of Higher Education Personnel), between May 10 and 20, 2019. Both platforms are online digital libraries that gather scientific resources and productions at an international level.

The search equations were based on the combination "covariation OR covariational AND Mathematics Education" (translated into English, Portuguese, and Spanish). The inclusion and exclusion criteria are described in Chart 2. Checking the criteria ensured the methodological validity of the review.

Chart 2 - Inclusion and exclusion criteria

Inclusion criteria	**Exclusion criteria**
Peer-reviewed articles reporting research in Mathematics Education on teaching and learning of functions from a covariational perspective	Inaccessible or unavailable articles at the hosting address
Publication period: 2014 to 2019	
Languages: Portuguese, English and Spanish	

Source: Prepared by the authors.

After collection, studies were listed separately by database and then combined into a single list of articles, excluding duplicates. The texts were then read and analyzed in their entirety.

The results were organized into two parts to address each of the research questions. General study data were categorized as follows: year of publication, database of origin,

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language, subject profile, and study objective. The first research question ("What aspects are involved in teaching and learning functions from a covariation perspective?") was subdivided into the following specific questions:

- What cognitive aspects and difficulties are identified as important in studies of covariational reasoning?
- What epistemological aspects of functions are highlighted by research and how are they related to the development of the concept of function from a covariational perspective?
- What aspects of teaching are identified as having an impact on the covariational approach to functions?

In the second part, the data were structured to allow an analysis of the possibilities of digital technologies for a covariational perspective. The aim was to answer the following question:

- What are the possibilities, contributions, and limitations of the covariational approach using digital technologies as identified in the studies?

The studies were analyzed to determine how the use of digital technologies specifically affected the results. This impact was not always explicit in the texts because many of them, although involving the use of technologies, did not seem to have considered their role as critical in the research. In these cases, we analyzed the discussion of the data in the studies and tried to identify the moments when the use of technologies and the analysis of this use were made explicit by the authors, to construct an analysis of the impact of technologies in these studies.

Results

The results are presented and discussed in this section, based on the research questions and theoretical constructs of the covariational perspective of function.

General aspects of the studies

After applying the inclusion and exclusion criteria, the ERIC search yielded 14 studies, all in English; the Capes Portal search yielded 21 studies, 20 in English and 1 in Spanish. The search in Portuguese yielded no results, suggesting that the covariation approach was not given due attention in Brazilian research until the period covered by this study.

Excluding duplicate studies in the two databases, the review resulted in 26 studies, 11 of which dealt with the use of digital technologies by research subjects. Table 1 details the process of filtering the studies according to the criteria. The list of selected studies can be found in Appendix A.

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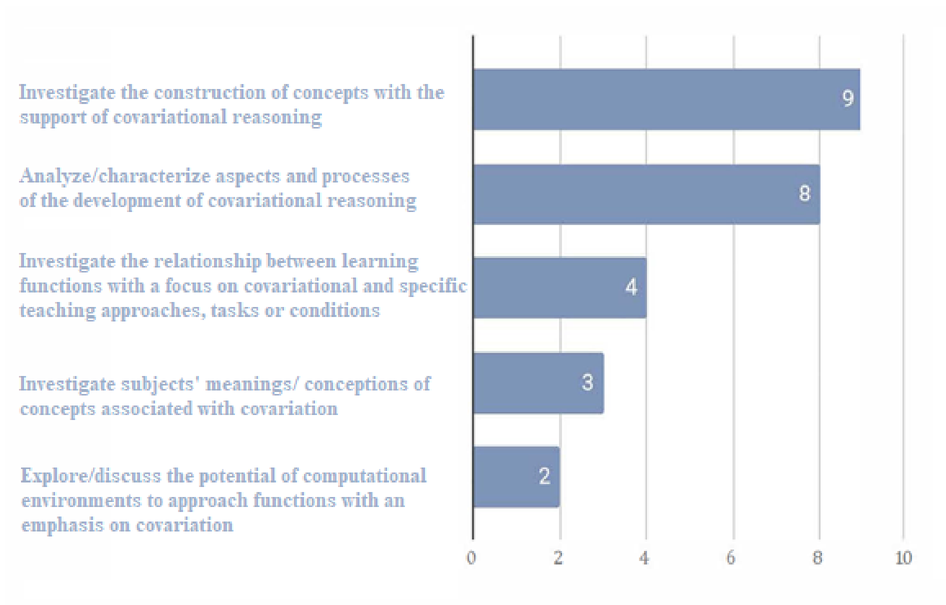
Table 1 - Number of studies

Database	Number of studies (initial)	Excluded studies (inclusion criteria)	Excluded studies (exclusion criteria)	Number of studies (final)	Total number of studies	Digital tech.
CAPES	25	3	1	21	26	11
ERIC	49	31	4	14		

Source: Prepared by the authors.

When analyzing and categorizing the research objectives of the studies, it was found that most of them investigated how concepts related to function are constructed by subjects and the role of covariation in this construction. Another considerable number of studies analyzed or characterized aspects of covariational reasoning itself, and only two studies aimed to investigate the potential of technologies to address covariation. Graph 1 shows this categorization:



Graph 1 - Number of studies by objective category
Source: Prepared by the authors.

Figure 2 shows the profiles of the subjects involved in the studies, who were students and teachers. The students were distinguished according to their level of education: secondary and tertiary. Secondary education generally corresponds to the period between the ages of 11 and 18 in different countries. Most of the covariate studies were applied to secondary school students, most of whom were in their final years. Among students in higher education, three of the nine studies were with students in initial mathematics teacher training courses and the others with other courses that include calculus or pre-calculus in the curriculum. The studies with teachers include two with practicing teachers and one with postgraduate teachers.

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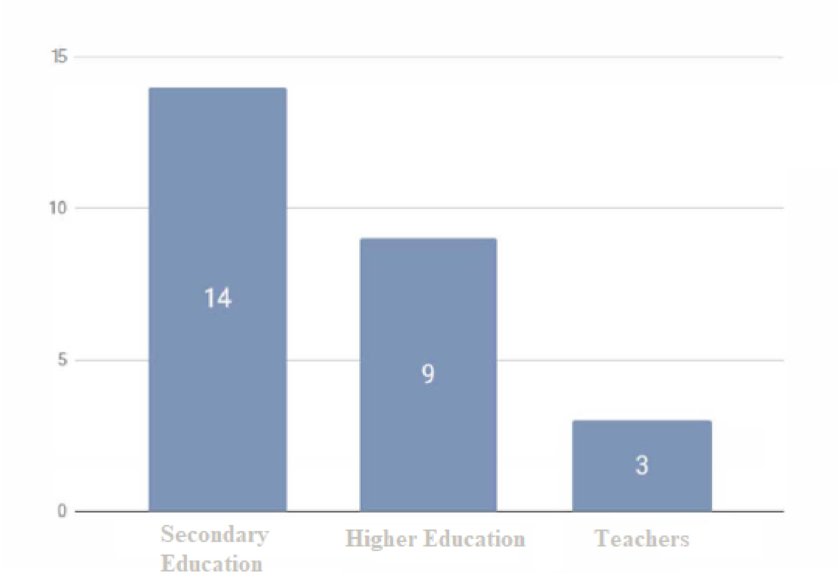

Graph 2 - Profiles of research subjects
Source: Prepared by the authors.

In the following sections, we will analyze and discuss the cognitive aspects involved in approaching functions from a covariational perspective. We will also address the contribution of digital technologies to this perspective.

Cognitive aspects involved in approaching functions from a covariational perspective

The results indicate the significance of cognitive processes and support for covariational reasoning, such as creating multiplicative objects, quantifying, and using smooth images of variation. The challenges of covariation are linked to quantifying variation, conceptualizing how variables or quantities differ from one another, modeling functional relationships covariationally, and representing and interpreting covariation in various forms of representation.

Quantification is crucial for quantitative and covariational reasoning (Thompson, 1994; Thompson & Carlson, 2017). This process involves conceptualizing an object with a measurable attribute. Studies E2, E11, and E17 emphasize the significance of conceiving attributes as measurable and variable for effective covariational reasoning.

In study E2, Johnson and McClintock (2018) investigated conditions that might foster students' discernment of variation in unidirectional change. In this investigation, students explored the functional relationships between the length and area of flat figures. The authors concluded that students who discerned variations in the increase or decrease in the variables, such as a 'decreasing' increase, had conceived the quantities involved as measurable and variable. In study E17, Moore (2014) characterized an undergraduate precalculus student's progress exploring angle measure and trigonometric functions. The author considered that quantifying the angle measure was a critical step for students to reason covariationally when approaching the sine function.

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Another important process identified was creating a multiplicative object of the attributes of the quantities in covariation (Saldanha & Thompson, 1998). In E9 study, Thompson, Hatfield, Yoon, Joshua and Byerley (2017) examined the covariational reasoning of 487 teachers. After watching a dynamic animation that displayed the values of two variable magnitudes, the teachers were asked to sketch a graph that expressed the relationship between the two. The study found that the teachers who created multiplicative objects of the values of the two quantities constructed the most accurate graphs.

The importance of using smooth images to foster covariational reasoning and support students in discerning variation in the intensity of variation was highlighted in the E11 and E19 studies (Castillo-Garsow, 2012; Thompson & Carlson, 2017) (see figure 1). In E11 study, Johnson, McClintock and Hornbein (2017) investigated how a student's covariational reasoning on Ferris wheel tasks, influenced a student's covariational reasoning on filling bottle tasks. In E19 study, Johnson (2015) investigated secondary students' quantification of ratio and rate.

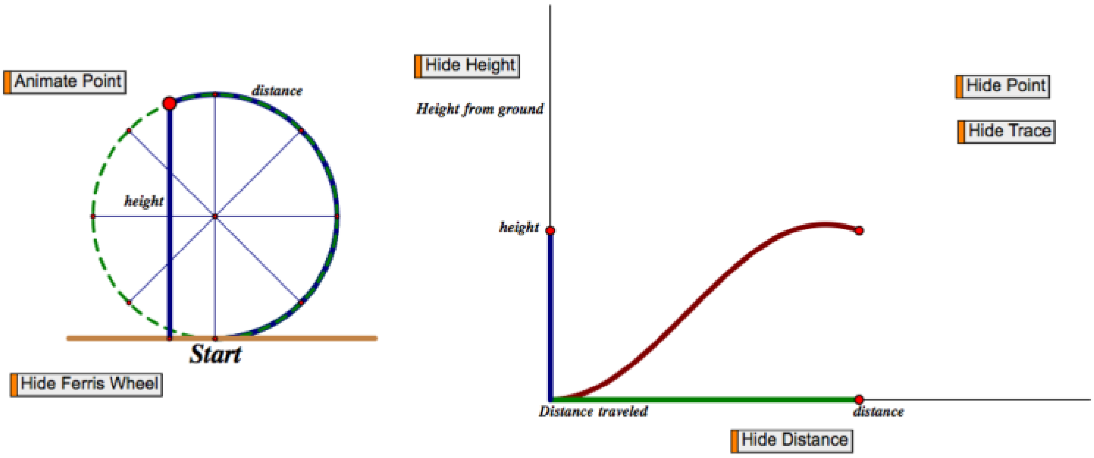

Figure 1 - A task from study E11 involving variation in the intensity of variation in the functional relationship between distance traveled on a Ferris wheel and relative height.

Source: (Johnson, McClintock, & Hornbein, 2017, p.854)

The relationship between students' understanding of concepts related to covariation and their ability to reason covariationally may vary. Studies E19 and E25 highlight the significance of the concept of ratio in students' understanding of rate of change and how they apply it to covariational reasoning. In E25 study, Johnson (2015) investigated students' reasoning about quantities involved in rate of change when working on tasks incorporating multiple representations of covarying quantities.

Conversely, studies E5 and E23 reveal that mathematical misunderstandings regarding limits can hinder students' covariational reasoning. In E5 study, Jones (2015) examined Calculus students' understanding of limits at infinity and infinite limits. In E23 study, Nagle, Tracy, Adams and Scutella (2017) investigated outcomes of building students' intuitive understanding of a limit as a function's predicted value.

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Regarding the subjects' difficulties in covariation (question 2), studies E1, E4, and E17 identified a limitation to covariational reasoning due to the difficulty in quantifying variation. Although the students visualized the variation, they were unable to quantify how it occurred.

In study E4, Ellis, Ozgur, Kulow, Dogan and Amidon (2016) investigated students' understanding of exponential growth within the context of covarying quantities. In this study, students explored a computer simulation involving a plant whose height grew exponentially as a function of time. At the beginning of the experiment, the students provided a qualitative description of exponential growth, but were unable to quantify the plant's growth. In the E1 study, Lagrange (2014) discussed the contribution that dynamic software can bring to the learning of functions. The author analyzed the interactions of students who explored covariation in software and noted that some students perceived the variation but did not comprehend that it could be quantified.

One difficulty highlighted in the studies was the challenge of establishing a relationship between variations in different variables. For instance, study E23 examined the understanding of limits among Calculus students and discovered that many students described the values in one variable approaching a specific value, but failed to describe the corresponding changes in the values of the other variable. In E7 study, Aranda and Callejo (2017) investigated how high school students build the integral function concept using applets. In both studies E4 and E7 students failed to establish a connection between the growth of the dependent variable and the growth of the independent variable in exponential and integral functions.

Studies E1 and E13 investigated the use of computational environments to the teaching learning of functions (Lagrange, 2014; Lagrange & Psycharis, 2014). In both E1 and E13, students encountered difficulties in identifying variables that covaried when modeling functional relationships in software.

Interpreting function graphs from a covariational viewpoint can be challenging for both students and teachers in initial training. Study E17 analyzed the quantitative reasoning of Pre-Calculus students in the sine function and found that, at the beginning of the experiment, they interpreted the graph based on non-quantitative aspects such as the shape and physical movements involved in the situation, without considering the joint variation of the variables involved. According to Moore (2014, p. 26), "to the students, the graph was smooth because the ride is smooth both in motion and shape". However, interpreting the rate of change from the function graph was challenging.

In E16 study, Yemen-Karpuzcu, Ulusoy and Işıksal-Bostan (2017) investigated the covariational reasoning abilities of prospective mathematics teachers in a task about dynamic functional events. The study participants in E16 and E15 had difficulty interpreting the rate of change from graphs that included an independent variable of 'time' in situations where it was not involved, such as graphs of volume as a function of height.

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Difficulties were observed in the representation of covariation. In E21 study, Habre (2017) examined how students coordinated covariation in the polar coordinate system with covariation in the Cartesian system (see Figure 2). The study revealed that students' understanding of graphs was based on the Cartesian system, which posed challenges in interpreting covariation in the polar system, particularly in cases with negative radial distances.

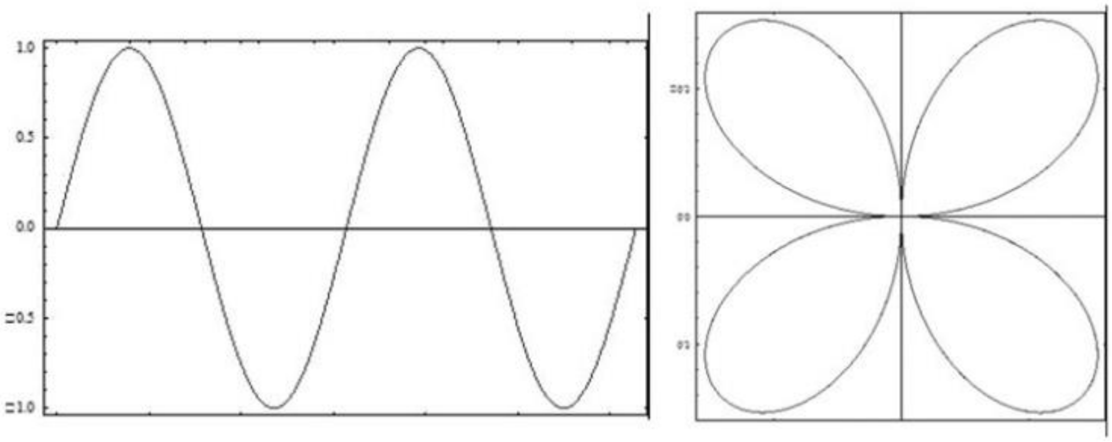

Figure 2 - Graphs of y = sin (2x) and r = sin (2θ)
Source: Adapted from Habre (2017, p. 61)

Reasoning covariationally from an algebraic model or formula was identified as a challenge. In E3 study, Jones (2017) investigated ways of understanding and ways of thinking that students exhibited when working with applied, non-kinematics derivatives. The author revealed that students often misinterpreted expressions of the rate of change and the derivative as amounts for the original quantity, such as interpreting $dF/fm = GM/r^2$ as if it were $F = GM/r^2$. Even expressions that did not involve an equation were interpreted as representing an amount rather than a rate of change or an expression that defines a relationship between variables.

Epistemological Aspects of Functions and Covariation

Different types of functions have distinct covariational aspects that are linked to their mathematical epistemology. Studies have shown the importance of considering the intrinsic characteristics of each function for a covariational understanding. Difficulties are closely related to these characteristics.

The E4 study examined students' reasoning about exponential growth and found that using an image of growth as 'repeated multiplications' limited to small intervals on the line made it challenging to generalize to arbitrarily large or small intervals. This way of reasoning is naturally related to the way the exponential function is defined. The authors note that the evolution of students' covariational reasoning began with the transition from images of multiplicative repetition to the ability to coordinate the ratio of y values to variations in x by multiple units. This process of reunification is also identified as fundamental by Confrey and Smith (1994).

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Regarding trigonometric functions, study E17 emphasizes the importance of covariational reasoning not being solely supported numerically, as these functions cannot be calculated through arithmetic operations. The author highlights the quantification of angle measure as a crucial step towards a covariational approach to the sine function (see Figure 3).

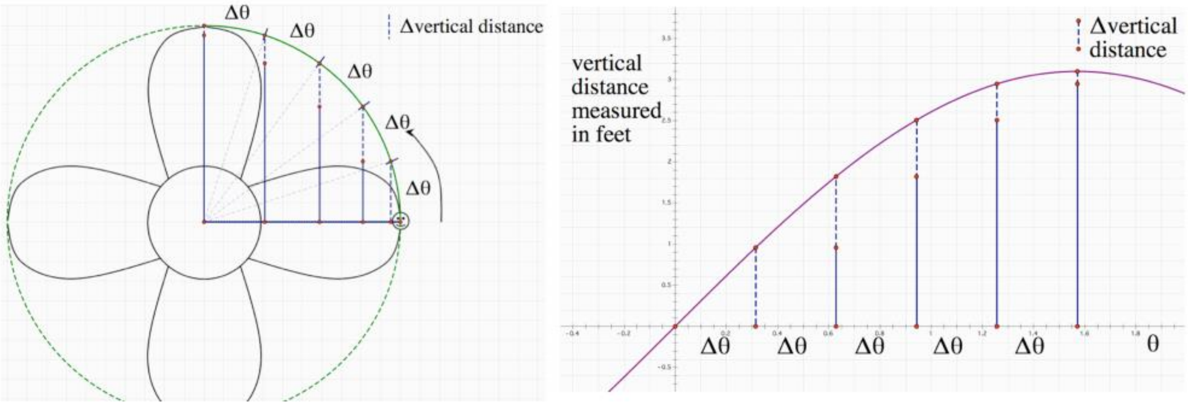

Figure 3 - Representation of covariation in the fan problem (study E17)
Source: (Moore, 2014, p.11)

Polynomial functions exhibit a pattern of variation that is interrelated (Lima et al., 2005). In E20 study, Hohensee (2016) examined how students perceive covariational aspects in the linear function after receiving instruction on the quadratic function. The results indicated that most students were able to perceive the covariational characterization in the linear function after having perceived the covariation in the quadratic function. The authors concluded that, under certain conditions, the perception of aspects in new concepts, such as the rate of change in the quadratic function, can influence the perception of the same aspects in previously encountered concepts, such as the rate of change in the linear function.

The highlighted epistemological aspects demonstrate the importance of considering certain characteristics for a covariational understanding in this perspective of function.

Teaching Functions from a Covariational Approach

Studies have revealed the influence of didactic choices and aspects on the development of covariational reasoning in the context of teaching.

For instance, studies E6, E9, and E15 demonstrate how different teaching contexts and curricula can affect the differences between students' and teachers' strategies, errors, and conceptions. In E6 study, Watson, Ayalon and Lerman (2018) examined how students develop concepts related to understanding functions. The study pointed to a relationship between student problem-solving strategies and two different curricula, one with a more intuitive approach and the other with a more formal approach to functions, in two different national contexts. In the first context, students showed a greater understanding of variability, while in the second context, they demonstrated greater success in generalizing through algebraic expressions and the correspondence approach.

The tasks' design is directly related to more explicit and effective covariational reasoning in several studies (E2, E4, E6, E11, E12, E14, and E15). In E12 study, this

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relationship was pointed out by Ayalon, Watson and Lerman (2015) in an investigation into the ways in which students deal with linear sequence data and the relationships between this data and the design of the tasks. The relationship between covariational reasoning and task design was also highlighted by Wilkie and Ayalon (2018) that investigated evidence of functional thinking in secondary school students exploring linear functions across different contexts and representations.

Study E6 highlights that the presentation and experience of school tasks can significantly impact the use of reasoning based on covariation or correspondence. Studies have shown the need to integrate different representations (E1, E5, E7, and E13) and use various contexts (E3 and E13) to approach functions from a covariational perspective.

Research related to the mathematical knowledge and meanings of practicing teachers and teachers in initial training has revealed that they face similar difficulties to those of students, indicating a problematic cycle in the teaching of covariation.

Teachers' meanings for central concepts such as rate of change were found to be limited and fragile, as reported in studies E26, E10, and E18. In E26 study, Musgrave and Carlson (2017) investigated graduate student teaching assistants' meanings for average rate of change. In E18 study, Byerley and Thompson (2017) investigated 251 high school mathematics teachers' meanings for slope, measurement, and rate of change. In E10 study, Zengin (2018) investigated how university students construct the relationship between the concepts of differential and derivative. Specifically, problems were identified with interpreting graphs of functions covariationally and with teachers' understanding of rate of change, which was often reduced to simply 'computing' a value or linked to speed.

Study E16 found that teachers can interpret function graphs based on their shape, without considering how the variables change. However, they struggle with interpreting the role of each variable and fail to include time as a variable, even in relationships where time is a factor. Study E10 revealed that teachers in initial training often misinterpret the concepts of differential, derivative, tangent, and slope. Specifically, they attribute movement to the objects themselves when exploring relationships between these concepts, such as interpreting points and lines tangent to the graph. This leads to a mistaken understanding of variables, as they incorrectly state that 'the secant becomes tangent.'

In general, studies focusing on didactic aspects have shown that several variables are involved in the covariational approach and how students' difficulties can be rooted in didactic choices or teachers' misconceptions. This may point to limitations in teacher training to deal with the covariational approach.

The role and possibilities of digital technologies for a covariational approach

To answer the second question, a cut was made in the results, which included only the 11 studies that used digital technologies (table 1).

It is important to note that even in these 11 studies, digital technology was not necessarily the object of the research or a central element in the analysis. Less than half (five)

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of these studies mention the technologies in the title or abstract of the article, and only two studies had the technologies themselves as the main focus of their objectives. This suggests that technology may be viewed as a mere adjunct in the construction of knowledge.

The authors analyzed the data and identified instances where the use and analysis of technologies were explicitly mentioned. The results were categorized into two topics: (i) the aspects and contributions of digital technologies to a covariational approach and (ii) the difficulties and limitations in using digital technologies for a covariational approach.

Aspects and Contributions of Digital Technologies to a Covariational Approach

The studies highlight the aspects and contributions of digital technologies, including the representation of variation in a dynamic and continuous way, which allows for the manipulation of variables. Additionally, digital technologies allow for the dynamic and simultaneous connection between representations/notations, the possibility of action on the representations/notations, the automatic and continuous scaling of the graph, and the tools for testing hypotheses and invariance.

Studies E4, E10, and E13 have highlighted the importance of representing and dynamically manipulating variables. In E4, the authors utilized a plant growth simulation to explore covariation in exponential growth (see figure 4). The results demonstrated that the ability to manipulate quantities in a continuous and dynamic manner enhanced students' capacity to coordinate multiplicative growth in y with additive growth in x, a crucial aspect of exponential growth.

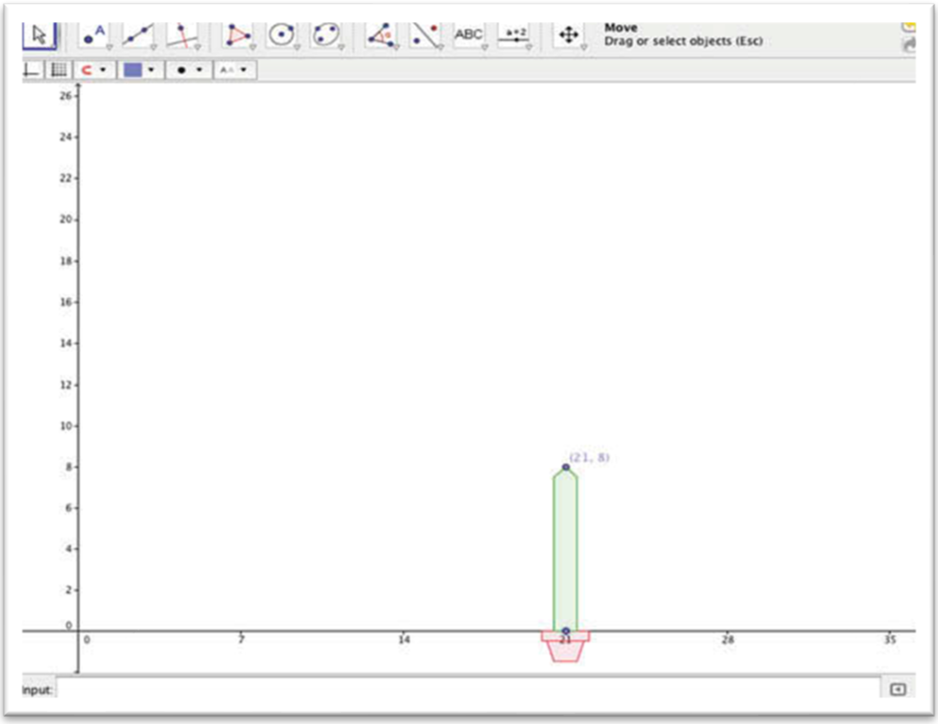

Figure 4 - The jactus script in the Geogebra environment, used in study E4.
Source: (Ellis, Ozgur, Kulow, Dogan & Amidon, 2016, p.157)

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According to the authors of study E13, students were able to identify and characterize types of dependencies (additive, multiplicative) between quantities in a situation where they were building a model of the letter 'N' with the help of dynamic exploration in a LOGO environment (figure 5). Additionally, manipulating the values of the variables using sliders allowed the students to see whether their constructions were successful or not through the deformations of the figure. In study E10, the use of sliders and the ability to drag points on the graph aided in connecting the tangent to a curve with the derivative.

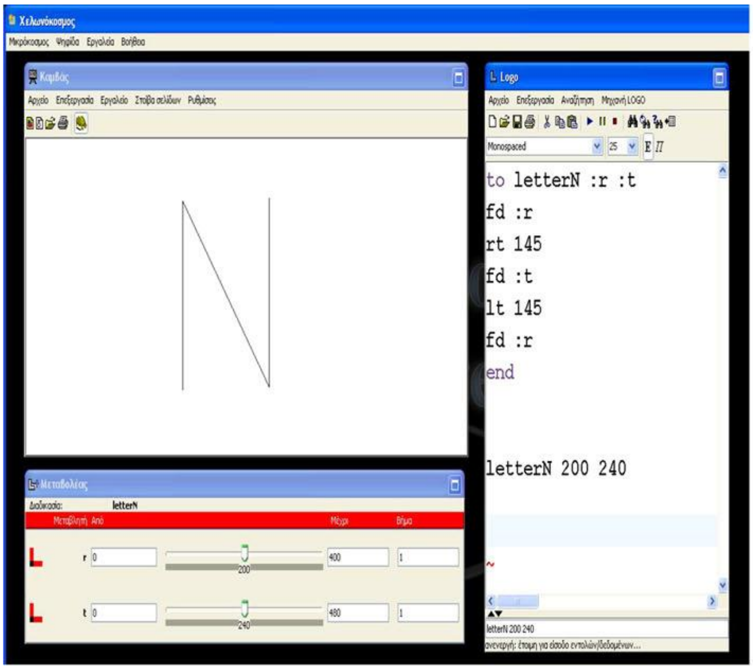

Figure 5 - Building a model of the letter N in the LOGO environment
Source: (Lagrange & Psycharis, 2014, p.266)

Another aspect of the computer environment highlighted as important by E1, E2, E4, E7 and E11 is the dynamic and simultaneous connection between representations/notations, which makes it possible to represent dynamic variations simultaneously and in multiple notations, thus allowing different conceptual aspects to be articulated. According to Kaput (1992), complex ideas are rarely well represented when only one notation system is used, and furthermore, the connection of notations is justified by "to expose different aspects of a complex idea, and to illuminate the meanings of actions in one notation by exhibiting their consequences in another notation" (Kaput, 1992, p. 542). In study E2, an environment used to articulate the variation of the area of polygons (triangles and rectangles) with a coordinate graph (figure 6) was pointed out as supporting the students' discernment of variable variation.

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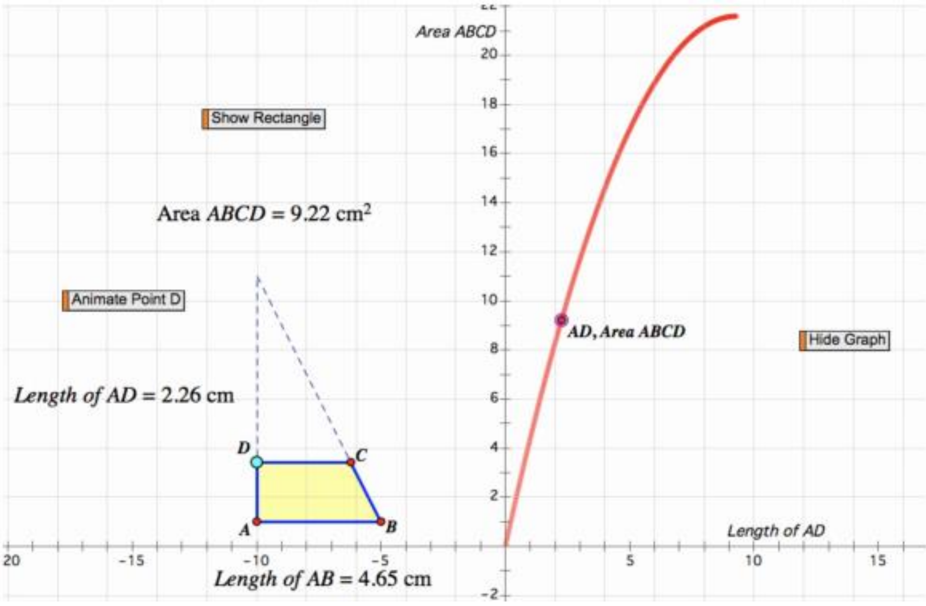

Figure 6 - Filling in the area of the triangle and covariation in study E2
Source: (Johnson & McClintock, 2018, p.8)

Study E1 highlighted the potential of software that combined symbolic forms (graphs, algebraic formulas, etc.) with dynamic manipulations of geometric objects so that students could make connections with magnitudes in covariation situations (figure 7).

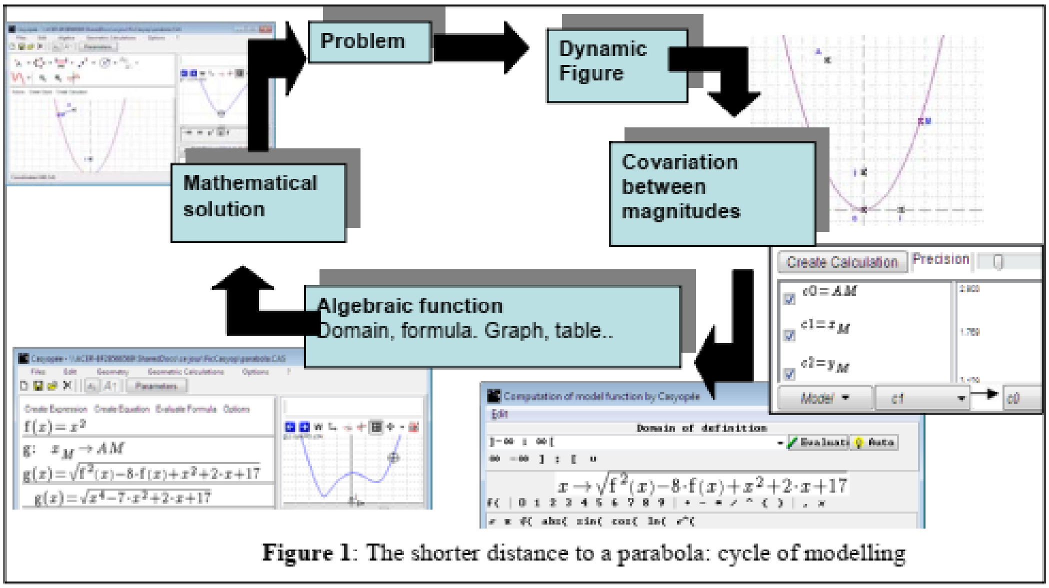

Figure 7 - Modeling a covariation relationship with articulation of representations (study E1)
Source: (Lagrange, 2014, p.3)

From the point of view of the interaction between the subjects and the technologies, we highlight some aspects that suggest a contribution to the exploration of covariation. The possibility of action on the representations/notations was highlighted in studies E7, E10, E13 and E17 as one of these aspects. Kaput (1992) differentiates an action notation from a display notation: systems used only to display information are referred to as display notations, while

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those used as a basis for transformations are referred to as action notations. Thus, a graph that only displays the curve of a function is distinguished from a graph that allows, in addition to display, actions such as varying the variables, the interval, and the scale in the graph itself, with simultaneous alteration of the curve. These possibilities are present in the studies cited.

The possibility of manipulating the variables directly on the graph, rather than just passively viewing the variation, seems to contribute to more effective reasoning, since the user can vary and observe the behavior of the variable directly and simultaneously. In the E7 study, this possibility was used to support the construction of the covariation between a variable, a function *f* defined on the given interval and its integral (figure 8). The authors of study E13 stated that the various representations and the opportunity to act on these representations established a rich milieu[4], offering multiple opportunities for generating meaning.

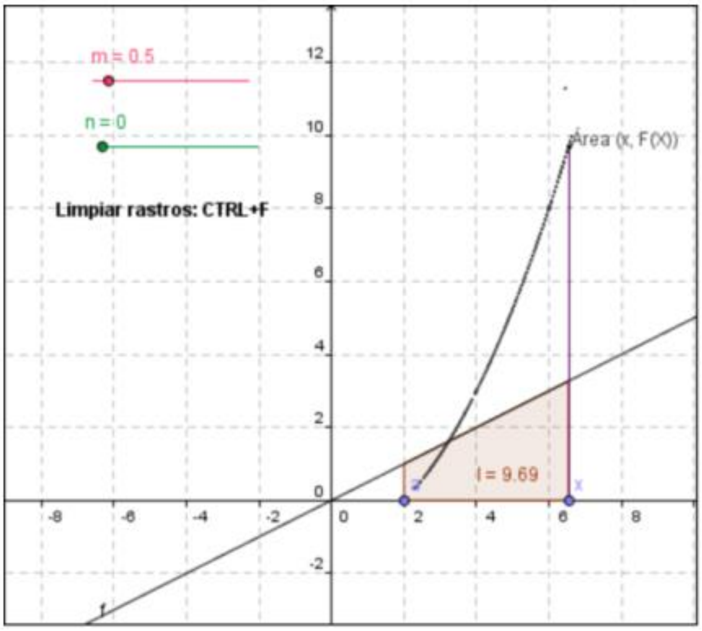

Figure 8 - Manipulation of variables directly on the graph (study E7)
Source: (Aranda & Callejo, 2017, p.786)

The E4 study highlighted another possibility for supporting students in building images of smooth variation in the exploration of exponential growth: the automatic and continuous scaling of the graph. Castillo-Garsow (2012) and Thompson & Carlson (2017) have also explored this approach. Previously, students had an image of this growth as being limited to repeated multiplication.

Studies E13, E17, and E22 provide examples of how students can use digital technologies as tools to test their hypotheses about covariation and test invariance. In Study E13, one of the software programs tested whether the graphical object was deformed by varying the sliders (Figure 5). In the other study's software program, students observed that

[4] From Brousseau's Theory of Didactic Situations: autonomous subsystem, antagonistic to the subject.

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moving a point horizontally in a window did not affect the graph, allowing them to make inferences about the relationship between the variables.

In study E22, Weber e Thompson (2014) investigated how students might generalize their understanding of graphs of one-variable functions to graphs of two-variable functions. In this study, students used a graphing calculator to plot graphs and test their hypotheses about function behavior.

The computational aspects mentioned have the potential to aid in the construction of covariation meanings by subjects. However, it is important to consider other factors for effective contribution, such as the teacher's role, task design, technology usage conditions and configurations, and specific uses.

Difficulties and Limitations of Digital Technologies for a Covariational Approach

While digital technologies can aid in the exploration of covariation, their use, combined with the characteristics of the computational environment and the design of the artifact itself, can lead to difficulties and limitations in this exploration.

For instance, the computer's ability to offer a dynamic environment with numerous possibilities can complicate reasoning in certain situations. In study E10, teachers in initial training explored a dynamic environment and misinterpreted the behavior of variables and lines by attributing movements to the same object. They used expressions such as 'the secant becomes tangent', which is a misconception that can arise more easily in a dynamic environment. In studies E1 and E13, students reported difficulties in identifying and relating variables in situations where the covariation between them was not explicit.

In study E13, students encountered challenges in connecting their understanding of function and covariation with the software's semantics. They found it confusing when a length on the y-axis (in the geometric construction of figure 9) was identified as an independent variable, as the functions approach typically represents the independent variable on the x-axis. The study revealed that students encountered challenges in linking the knowledge acquired from software exploration with conventional mathematical concepts. For instance, in one instance, the absence of mathematical notation to represent covariation (as shown in figure 5) posed a challenge.

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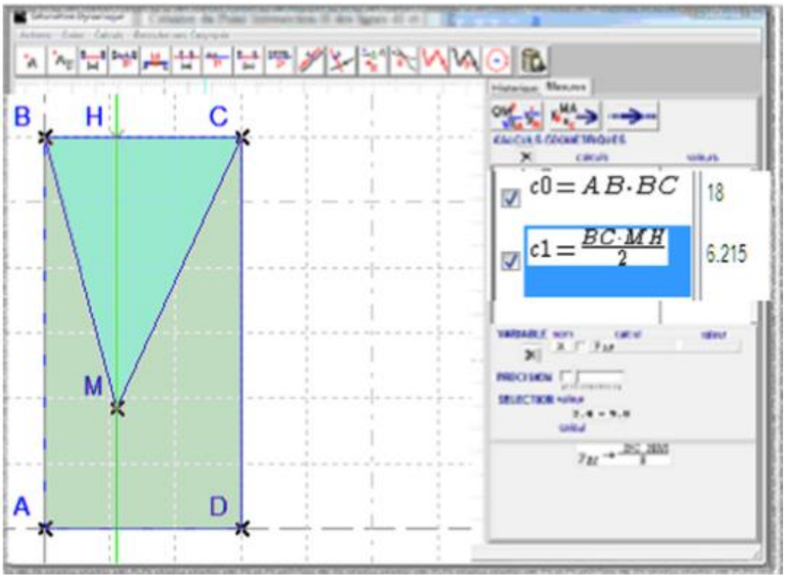

Figure 9 - Modeling a function from a geometric context (study E13)
Source: (Lagrange & Psycharis, 2014, p.274)

The difficulties and limitations that can arise with the use of digital technologies can be related both to the characteristics of the computational environment itself and to the design decisions of the technologies and the tasks applied with their use. Therefore, specific attention needs to be paid to these phenomena, which most research has not emphasized sufficiently, which reveals part of the gaps in research on the use of digital technologies to explore covariation.

Considerations and reflections

The purpose of this literature review was to analyze current research on covariation from cognitive, didactic, and epistemological perspectives, as well as the potential of digital technologies.

Important cognitive processes and aspects, such as the creation of multiplicative objects, quantification, and the use of smooth images of variation, were identified as supporting covariational reasoning. The studies found a link between the approach to covariation and students' difficulties in understanding joint variation between variables, quantifying variation, modeling functions covariationally, and representing and interpreting covariation in different registers of representation.

It is important to note that different types of functions have distinct covariational aspects that characterize them. Difficulties in covariational reasoning may be related to the intrinsic characteristics of each function. Therefore, it is important to take these aspects into account.

In the didactic context, various factors can influence the approach to covariation, from curricular choices to the ways in which covariation is taught in the classroom, such as the design of tasks and ways of representing functions. These aspects can influence whether

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students mobilize their covariational reasoning. The studies on teachers' knowledge, meanings, and conceptions generally indicate insufficient covariational reasoning and weak understanding of concepts related to covariation.

Few studies analyzed the use of digital technologies as a central object, but those that did show how this focus enriches the analysis and sheds light on important phenomena that would otherwise go unnoticed.

The studies highlight the contributions of digital technologies, including the dynamic and continuous representation of variation, which enables the manipulation and coordination of variables. Additionally, the dynamic and simultaneous connection between representations and notations allows for the emergence and articulation of different conceptual aspects of covariation. The opportunity of active coordination of variation is also made possible through the ability to act on the representations and notations. Furthermore, the automatic and continuous scaling of the graph supports the construction of images of smooth variation and provides tools for testing hypotheses and invariance.

However, the use of these technologies also presents challenges. For instance, students may struggle with connecting the characteristics and semantics of computational environments to their understanding of covariation. Additionally, students may face difficulties in conceptualizing specific features of the computational environment, such as the dynamic nature of objects.

Research from a covariational perspective is a recent development that requires further exploration. However, in the Brazilian research scenario, the databases searched during the surveyed period did not yield any results. This suggests that the covariational approach to function has not received adequate attention as a research problem, at least until the indicated period. More attention should be given to this object, both in research and in teacher training and the mathematics curriculum in Brazilian basic education.

Thompson and Carlson (2017) listed topics that require investigation, such as ways of conceptualizing aspects of covariation, relationships between curricula and covariational reasoning, and teacher practices that support covariational reasoning. This list should include the use of digital technologies to support the development of covariational reasoning. Research into mathematics learning supported using digital technologies should examine the specific aspects and phenomena of this context.

References

Aranda, C. & Callejo, M. L. (2017) Construcción de la Función Integral y Razonamiento Covariacional: dos Estudios de Casos. *Bolema*, 31(58), 777-798. https://doi.org/10.1590/1980-4415v31n58a13

Ayalon, M., Watson, A. & Lerman, S. (2015). Functions represented as linear sequential data: relationships between presentation and student responses. *Educ Stud Math,* 90, 321–339. https://doi.org/10.1007/s10649-015-9628-9

DOI: 10.20396/zet.v31i00.8664258

Ayalon, M., Watson, A. & Lerman, S. (2016) Progression Towards Functions: Students' Performance on Three Tasks About Variables from Grades 7 to 12. *Int J of Sci and Math Educ* 14, 1153–1173. https://doi.org/10.1007/s10763-014-9611-4

Byerley, C., & Thompson, P. (2017). Secondary mathematics teachers' meanings for measure, slope, and rate of change. *Journal of Mathematical Behavior*, *48*, 168-193. https://doi.org/10.1016/j.jmathb.2017.09.003

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. In A.H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Issues in mathematics education*: Research in collegiate mathematics education III (Vol. 7, pp. 114–162). Providence, RI: American Mathematical Society.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33,* 352–378. Available in: https://www.jstor.org/stable/4149958

Castillo-Garsow, C. C. (2012). Continuous quantitative reasoning. In R. Mayes, R. Bonillia, L. L. Hatfield, & S. Belbase (Eds.), *Quantitative reasoning:* Current state of understanding, WISDOMe Monographs (Vol. 2, pp. 55–73). Laramie: University of Wyoming.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educ Stud Math* 26**,** 135–164. https://doi.org/10.1007/BF01273661

Ellis, A. B., Ozgur, Z., Kulow, T, Dogan, M. F., & Amidon, J. (2016). An Exponential Growth Learning Trajectory: Students' Emerging Understanding of Exponential Growth Through Covariation, *Mathematical Thinking and Learning*, 18(3), 151-181. https://doi.org/10.1080/10986065.2016.1183090

Ferrara, F., Pratt, D., & Robutti, O. (2006). The Role and Uses of Technologies for the Teaching of Algebra and Calculu". In *The Role and Uses of Technologies for the Teaching of Algebra and Calculus*. Leiden, The Netherlands: Brill | Sense. doi: https://doi.org/10.1163/9789087901127_010

Habre, S. (2017). Students' challenges with polar functions: covariational reasoning and plotting in the polar coordinate system, *International Journal of Mathematical Education in Science and Technology*, 48(1), 48-66. https://doi.org/10.1080/0020739X.2016.1220027

Hitt, F., & González-Martín, A.S. (2015). Covariation between variables in a modelling process: The ACODESA (collaborative learning, scientific debate and self-reflection) method. *Educ Stud Math,* 88**,** 201–219. https://doi.org/10.1007/s10649-014-9578-7

Hohensee, C. (2016). Student noticing in classroom settings: A process underlying influences on prior ways of reasoning. *The Journal of Mathematical Behavior*, 42, 69-91. https://doi.org/10.1016/j.jmathb.2016.03.002.

Jones, S. R. (2015). Calculus limits involving infinity: the role of students' informal dynamic reasoning, International Journal of Mathematical Education in Science and Technology, 46(1), 105-126. https://doi.org/10.1080/0020739X.2014.941427

DOI: 10.20396/zet.v31i00.8664258

Jones, S. R. (2017). An exploratory study on student understandings of derivatives in real-world, non-kinematics contexts. *The Journal of Mathematical Behavior*, 45, 95–110. https://doi.org/10.1016/j.jmathb.2016.11.002

Johnson, H. L. (2015). Secondary Students' Quantification of Ratio and Rate: A Framework for Reasoning about Change in Covarying Quantities, *Mathematical Thinking and Learning*, 17:1, 64-90. https://doi.org/10.1080/10986065.2015.981946

Johnson, H. (2015). Together yet separate: Students' associating amounts of change in quantities involved in rate of change. *Educational Studies in Mathematics, 89*(1), 89-110. Disponível em: https://www.jstor.org/stable/43590240

Johnson, H.L., McClintock, E., & Hornbein, P. (2017). Ferris wheels and filling bottles: a case of a student's transfer of covariational reasoning across tasks with different backgrounds and features. *ZDM Mathematics Education,* 49**,** 851–864. https://doi.org/10.1007/s11858-017-0866-4

Johnson, H.L., & McClintock, E. (2018). A link between students' discernment of variation in unidirectional change and their use of quantitative variational reasoning. *Educ Stud Math,* 97, 299–316. https://doi.org/10.1007/s10649-017-9799-7

Kaput, J. (1992). Technology and mathematics education. In D. A. Grouws (Ed.), *Research on mathematics teaching and learning* (pp. 515–556). New York: Macmillan.

Lagrange, J-B. (2014). A functional perspective on the teaching of algebra: current challenges and the contribution of technology. *The International Journal For Technology in Mathematics Education*, hal-01740456, Version 1. Available in: https://hal.archives-ouvertes.fr/hal-01740456

Lagrange, J., & Psycharis, G. (2014). Investigating the Potential of Computer Environments for the Teaching and Learning of Functions: A Double Analysis from Two Research Traditions. *Tech Know Learn*, 19, 255–286. https://doi.org/10.1007/s10758-013-9211-3

Lima, E. L., Carvalho, P. C. P., Wagner, E., & Morgado, A. C. (2005). *A Matemática do Ensino Médio,* v.1, 8 ed. Rio de Janeiro: SBM.

Ministério da Educação. (2018). *Base Nacional Comum Curricular*. Brasilia: MEC. Disponível em: http://basenacionalcomum.mec.gov.br/

Monk, S. (1992). Students' understanding of a function given by a physical model. In G. Harel & E. Dubinsky (Eds.), The concept of function: Aspects of epistemology and pedagogy (pp.175–193). Washington, DC: Mathematical Association of America.

Monk, S., & Nemirovsky, R. (1994). The case of Dan: Student construction of a functional situation through visual attributes. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Issues in mathematics education: Research in collegiate mathematics education I* (Vol 4, pp. 139–168). Providence, RI: American Mathematical Society.

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education*, 45(1), 102-138. Available in: http://www.jstor.org/stable/10.5951/jresematheduc.45.1.0102

Musgrave, S., & Carlson, M.P. (2017). Understanding and advancing graduate teaching assistants' mathematical knowledge for teaching. *The Journal of Mathematical Behavior*, 45, 137-149. https://doi.org/10.1016/j.jmathb.2016.12.011

DOI: 10.20396/zet.v31i00.8664258

Nagle, C., Tracy, T., Adams, G., & Scutella, D. (2017). The notion of motion: covariational reasoning and the limit concept. *International Journal of Mathematical Education in Science and Technology*, 48(4), 573-586. https://doi.org/10.1080/0020739X.2016.1262469

Paoletti, T., & Moore, K. C. (2017) The parametric nature of two students' covariational reasoning, *The Journal of Mathematical Behavior*, 48, 137-151. https://doi.org/10.1016/j.jmathb.2017.08.003.

Ramos, A., M. Faria, P., & Faria, A. (2014). Revisão sistemática de literatura: contributo para a inovação na investigação em Ciências da Educação. *Revista Diálogo Educacional, 14*(41), 17-36. doi:http://dx.doi.org/10.7213/dialogo.educ.14.041.DS01

Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berenson & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education North America* (Vol. 1, pp. 298–304). Raleigh: North Carolina State University. Disponível em http://bit.ly/1b4sjQE

Silva, C. T. J. (2022). O uso de um artefato computacional como suporte ao desenvolvimento do raciocínio covariacional em função. (Tese de Doutorado, Universidade Federal de Pernambuco, Recife, Brasil). Repositório Digital da UFPE. https://repositorio.ufpe.br/handle/123456789/45772

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179–234). Albany, NY: SUNY Press.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston: National Council of Teachers of Mathematics.

Thompson, P.W., Hatfield, N., Yoon, H., Joshua, S., & Byerley, C. (2017). Covariational reasoning among U.S. and South Korean secondary mathematics teachers. The Journal of Mathematical Behavior, 48, 95-111. https://doi.org/10.1016/j.jmathb.2017.08.001

Watson, A., Ayalon, M., & Lerman, S. (2018). Comparison of students' understanding of functions in classes following English and Israeli national curricula. *Educ Stud Math* 97, 255–272. https://doi.org/10.1007/s10649-017-9798-8

Weber, E., & Thompson, P.W. (2014). Students' images of two-variable functions and their graphs. *Educ Stud Math*, 87, 67–85. https://doi.org/10.1007/s10649-014-9548-0

Wilkie, K.J., & Ayalon, M. (2018). Investigating Years 7 to 12 students' knowledge of linear relationships through different contexts and representations. *Math Ed Res J*, 30, 499–523. https://doi.org/10.1007/s13394-018-0236-8

Yemen-Karpuzcu, S., Ulusoy, F., & Işıksal-Bostan, M. (2017). Prospective Middle School Mathematics Teachers' Covariational Reasoning for Interpreting Dynamic Events During Peer Interactions. *Int J of Sci and Math Educ* 15, 89–108. https://doi.org/10.1007/s10763-015-9668-8

Zengin, Y. (2018). Examination of the constructed dynamic bridge between the concepts of differential and derivative with the integration of GeoGebra and the ACODESA method. *Educ Stud Math*, 99**,** 311–333. https://doi.org/10.1007/s10649-018-9832-5

DOI: 10.20396/zet.v31i00.8664258

APPENDIX A

Chart 3 – Selected studies

Id	Title	Database	Year	Digital tech
E1	A Functional Perspective on the Teaching of Algebra: Current Challenges and the Contribution of Technology	Eric	2014	Yes
E2	A link between students' discernment of variation in unidirectional change and their use of quantitative variational reasoning.	Capes	2018	Yes
E3	An exploratory study on student understandings of derivatives in real-world, non-kinematics contexts	Capes	2017	No
E4	An Exponential Growth Learning Trajectory: Students' Emerging Understanding of Exponential Growth Through Covariation	Capes/ Eric	2016	Yes
E5	Calculus Limits Involving Infinity: The Role of Students' Informal Dynamic Reasoning	Eric	2015	No
E6	Comparison of students' understanding of functions in classes following English and Israeli national curricula	Capes/ Eric	2018	No
E7	Construcción de la Función Integral y Razonamiento Covariacional: dos Estudios de Casos	Capes	2017	Yes
E8	Covariation between variables in a modelling process: The ACODESA (collaborative learning, scientific debate and self-reflection) method	Capes/ Eric	2015	No
E9	Covariational reasoning among U.S. and South Korean secondary mathematics teachers	Capes	2017	Yes
E10	Examination of the constructed dynamic bridge between the concepts of differential and derivative with the integration of GeoGebra and the ACODESA method	Capes	2018	Yes
E11	Ferris Wheels and Filling Bottles: A Case of a Student's Transfer of Covariational Reasoning across Tasks with Different Backgrounds and Features	Capes/ Eric	2017	Yes
E12	Functions Represented as Linear Sequential Data: Relationships between Presentation and Student Responses	Capes/ Eric	2015	No
E13	Investigating the Potential of Computer Environments for the Teaching and Learning of Functions: A Double Analysis from Two Research Traditions	Capes	2014	Yes
E14	Investigating Years 7 to 12 students' knowledge of linear relationships through different contexts and representations	Capes	2018	No

Source: Prepared by the authors

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Chart 3 – Selected studies

E15	Progression towards Functions: Students' Performance on Three Tasks about Variables from Grades 7 to 12	Capes/ Eric	2016	**No**
E16	Prospective Middle School Mathematics Teachers' Covariational Reasoning for Interpreting Dynamic Events during Peer Interactions	Capes/ Eric	2017	**No**
E17	Quantitative Reasoning and the Sine Function: The Case of Zac	Eric	2014	**Yes**
E18	Secondary mathematics teachers' meanings for measure, slope, and rate of change	Capes	2017	**No**
E19	Secondary Students' Quantification of Ratio and Rate: A Framework for Reasoning about Change in Covarying Quantities	Capes/ Eric	2015	**No**
E20	Student noticing in classroom settings: A process underlying influences on prior ways of reasoning	Capes	2016	**Yes**
E21	Students' Challenges with Polar Functions: Covariational Reasoning and Plotting in the Polar Coordinate System	Eric	2017	**No**
E22	Students' images of two-variable functions and their graphs	Capes/ Eric	2014	**Yes**
E23	The Notion of Motion: Covariational Reasoning and the Limit Concept	Eric	2017	**No**
E24	The parametric nature of two students' covariational reasoning	Capes	2017	**No**
E25	Together yet separate: Students' associating amounts of change in quantities involved in rate of change	Capes	2015	**No**
E26	Understanding and advancing graduate teaching assistants' mathematical knowledge for teaching	Capes	2017	**No**

Source: Prepared by the authors