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The discipline Real Analysis and the future Mathematics teacher: a rethink

A disciplina Análise Real e o future professor de Matemática: um repensar

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Abstract

The purpose of this article is to promote a discussion about the Real Analysis discipline to understand why this discipline has its permanence in Mathematics degree courses, considering its use in Basic Education, and at the same time it is so criticized for its rigor and formality. To achieve this goal, we bring a survey of this discipline in undergraduate courses in Mathematics in Brazil, perspectives of national and international work on the topic, situations related to Elementary and High School that would justify discussions in analytical classrooms, as well as a look at this discipline based on the Horizon of Content Knowledge (HCK) and on the Specialized Content Knowledge (SCK) belonging to the theoretical framework of Mathematical Knowledge for Teaching by Deborah Ball et al. As a result, we list some justifications for the permanence of this discipline in the Mathematics Degree curriculum.

Keywords: Mathematics Education; Real Analysis; Mathematics Teacher Education.

Resumo

O objetivo desse artigo é promover uma discussão a respeito da disciplina Análise Real com vistas a entender por que essa disciplina tem sua permanência em cursos de licenciatura em Matemática, considerando sua utilização na Educação Básica, e ao mesmo tempo é tão criticada por seu rigor e formalidade. Para atingir esse objetivo, trazemos um levantamento dessa disciplina em cursos de licenciatura em Matemática no Brasil, perspectivas de trabalhos nacionais e internacionais a respeito do tema, situações relacionadas ao Ensino

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Fundamental e Médio que justificariam discussões em salas de aula de análise, bem como um olhar para essa disciplina com base no Conhecimento do Conteúdo no Horizonte (HCK) e no Conhecimento Especializado do Conteúdo (SCK), pertencentes ao quadro teórico do Conhecimento Matemático para o Ensino de Deborah Ball e colaboradores. Como resultado elencamos algumas justificativas para a permanência dessa disciplina em currículos de licenciatura em Matemática.

Palavras-chave: Educação Matemática; Análise Real; Formação de Professores de Matemática.

Introduction

Interested in investigating and justifying the importance of the subject Real Analysis in the training of future mathematics teachers, we carried out a survey on the supply of mathematics degree courses, which take place in person, at public institutions in Brazil. Although we organized the data considering Federal Institutes, Federal Universities and State Universities, in this article we chose to present the panorama without making this subdivision, since the focus is on the subject of Real Analysis. This overview is shown in Table 1, according to the offer by region of Brazil.

Chart 1 - Mathematics degree programs offered by public institutions in Brazil

Region				
Midwest	North East	North	South East	South
36	101	44	75	49

Source: The authors

In this scenario, the Northeast region has the largest number of degree courses in mathematics, accounting for almost a third of the total, i.e., 101 of the 305 courses. Regarding the subject Real Analysis, in our readings of the course documents, we noticed the presence of eight different nomenclatures: Analysis; Mathematical Analysis; Analysis on the Straight Line; Analysis for Undergraduates; Real Analysis; Fundamentals of Analysis; Introduction to Analysis and Topics in Analysis. The name Real Analysis was the most representative, accounting for more than a third of the names of the subjects in the courses surveyed, which is why we are using it.

We would like to point out that Real Analysis is not a compulsory subject in only two of the mathematics degree courses surveyed: at the Universidade Federal do Ceará (UFC) in the city of Fortaleza and at the Universidade Estadual do Norte do Paraná (UENP) on its campus in the city of Jacarezinho. Table 2 shows the average workload of the subject Real Analysis in the courses surveyed by institution and in each region of Brazil.

Chart 2 - Average number of hours of Real Analysis in the courses surveyed in hours

Institution	Average Actual Analysis Load in hours per Region		
Instituto Federal	Midwest: 91	Northeast: 78	North: 87
	Southeast: 71	South: 76	Total: 81
Universidade Federal	Midwest: 106	Northeast: 77	North: 79
	Southeast: 83	South: 93	Total: 88
Universidade Estadual	Midwest: 107	Northeast: 74	North: 83
	Southeast: 78	South: 110	Total: 90
Overall Average	Midwest: 101	Northeast: 76	North: 83
	Southeast: 77	South: 93	Total: 86

Source: The authors

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The highest regional average belongs to the Central-West region, with 101 (one hundred and one) hours of real analysis. On the other hand, the courses surveyed at the state universities in the South have an average of 110 (one hundred and ten) hours devoted to this subject. Thus, even though certain regions have a higher average workload than others, as well as some institutions have a higher average workload than others, it is possible to observe the marked presence of the subject of Real Analysis in the curricular matrices of mathematics courses that take place in person, within the framework of public institutions in Brazil.

This is in line with what is proposed in the National Curriculum Guidelines for Mathematics Degree Courses, which suggest that the content of "Fundamentals of Analysis" should be included in the curriculum proposed by higher education institutions, and can be distributed throughout the course.

Based on the panorama presented, in which it is possible to clearly perceive the marked presence of the discipline of Real Analysis in mathematics degree courses in Brazil, in this article we try to understand the importance of its presence in practically all of these courses, considering its use in basic education. This work corresponds to bibliographical research, in which, according to Severino (2007, p. 122), "data from theoretical categories already worked on by other researchers and duly recorded are used. The texts become sources of the topics to be researched. The researcher works from the contributions of the authors of the analytical studies contained in the texts".

This article is a narrative review. According to Gonçalves and Fiorentini (2019, p. 229), this type of review "[...] differs from systematic reviews in that it brings together for review publications that do not have a defined and common problem of study, but that address or reflect on an issue or topic of interest to the researcher". In this methodological context of investigation, we sought to answer the following question: "Why do teachers need to study real analysis in undergraduate courses to teach mathematics in elementary education?". In order to find answers to this question, we first addressed the view of Real Analysis according to some authors, such as Dysman and Dysman (2021) and Moreira and Vianna (2016), as well as presenting some results of published research that reveal the opinions of groups involved in the subject.

Next, understanding that "teachers with deep conceptual knowledge of the subject establish more connections and relationships with other topics and can translate this knowledge into teaching" (Marcelo-García, 2009, p. 119), and that "superficial knowledge harms students by limiting their understanding of concepts and leading them to erroneous representations of the subject" (p. 119), and thus justifying our view of the importance of the subject for the training of future mathematics teachers, we relied on the Horizon Content Knowledge (HCK), which is part of the Mathematical Knowledge for Teaching (MKT). 119), and to justify our view of the importance of the subject for the training of future mathematics teachers, we relied on Horizon Content Knowledge (HCK), which is part of the theoretical map of Mathematical Knowledge for Teaching (MKT) by Deborah Ball et al.

Finally, to further justify the importance of this subject for the training of mathematics teachers, we developed two situations that could occur in secondary school classrooms and that could show how fragmented and superficial content knowledge can affect the way teachers teach, making them "unable to connect students' comments and questions to other topics" (Marcelo-García, 2009, p. 119) or limiting students' activities to procedural aspects.

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Opening discussions

In order to discuss the importance of teaching real analysis to undergraduate mathematics students, we believe it is essential to understand the positions of some groups involved in this process: mathematics and mathematics education researchers, teacher educators, elementary school mathematics teachers, and undergraduates.

The research carried out by Almouloud, Silva, Miguel & Fusco (2008) has already pointed out the difficulties that primary school teachers have in solving problems involving mathematical demonstrations and proofs, mainly because they use a mathematical language that lacks meaning and, consequently, creates barriers for these teachers to teach their students to reason, argue, prove and demonstrate mathematically.

In order to highlight the opinions of one group of mathematicians and another group of mathematics educators, we consider relevant the data provided by Cury, Moreira and Vianna (2005) and Moreira and Vianna (2016). These studies used very similar questionnaires with questions about the content that should be taught in real analysis classes for undergraduate mathematics courses, the supporting bibliography for teaching the subject, and the mandatory presence of the subject in the courses. In our analysis, we focused on the reasons given by the respondents to justify making real analysis a compulsory subject in mathematics courses.

About 90% of the mathematicians interviewed considered the subject to be significant for the training of mathematics teachers and that it should be a compulsory part of the curriculum, while for mathematics teachers in the 2016 survey, this was the opinion of just over 80% of the interviewees. It is worth noting that in both the 2005 and 2016 surveys, no one expressed the opinion that real analysis should not be part of the course syllabus, but there was a small percentage of respondents who did not answer or expressed less conclusive opinions. Although the professions of these two groups are wholly unique, the justifications they gave were very similar and mostly based on three main axes: The study of real analysis is a good (and perhaps the best) opportunity for students to develop a mathematical culture and to understand what it means to think mathematically; the study of real analysis provides a more solid foundation for future elementary school teachers, who become more confident when teaching and find it easier to make connections between the subjects they teach; and finally, real analysis has applications in other sciences, and its study provides the means to understand phenomena that occur in other areas of knowledge.

Another group whose opinion should be considered, since it is closely related to future elementary school teachers, is made up of teachers of the subject Real Analysis on mathematics degree courses. Martines (2012) concluded, through the responses to an interview with four teachers who teach this subject in undergraduate courses and four course coordinators, that, among the reasons that justify its importance in the course, some deserve to be highlighted, namely the role of grounding the mathematical knowledge of future teachers, even considering that Real Analysis has no direct application in Basic Education, consolidating and formalizing content and, finally, grounding knowledge about the set of real numbers.

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Finally, there is another group whose opinion on Real Analysis should be mentioned. Gomes (2013) concluded, following a survey of three elementary school teachers and three mathematics undergraduates, that the presence of this subject on the course is still surrounded by some uncertainties, and even contradictions, for the research subjects.

They believe that

> [...] the subject of Analysis is not significant for the mathematics teacher, but at the same time it provides the undergraduate/graduate with an in-depth view of what is covered in basic education. [...] In the same way that it appears to be useless and could be excluded from the curriculum of a mathematics degree course, it opens up possibilities to strengthen and support the teacher's attitudes (Gomes, 2013, p. 256).

Thus, Gomes (2013) shows that although the research subjects consider Real Analysis to be a difficult subject, which generates suffering and aversion, and which is often considered too formal, they understand its important role for future primary school teachers. According to the author, the responses of the participants in his research seem to point to two parallel paths taken by teachers or future teachers of mathematics: *"the subject of analysis is not relevant for the mathematics teacher, but at the same time it provides the undergraduate/graduate with an in-depth view of what is covered in basic education*" (p. 256, emphasis added).

It is worth noting that, despite the divergences and controversies surrounding the subject, all of these studies point to a consensus regarding the importance of real analysis in the curriculum of mathematics majors. We would like to emphasize that the data presented here reflect the opinions of a limited group of people, and therefore should not be generalized or taken as the opinion of the whole group.

Gomes, Otero-Garcia, Silva and Baroni (2015) report on research in real analysis, with various discussions ranging from rote learning to the arithmetization of this discipline, *freeing* it from geometric reasoning and intuition. For these authors,

> [...] the formality of analysis is a cultural legacy of its development; throughout history, as a body of knowledge - and the techniques employed, from the Greek analytical method to the arithmetization of analysis were based on the idea of rigor characteristic of each era" (Gomes et al, 2015, p. 2).

According to these researchers, "there is a close relationship between analysis (in all its aspects) and demonstration" (Gomes et al., 2015, p. 2).Some characteristics of real analysis are formal, difficult, theoretical and abstract. According to the authors, it is a discipline that requires content from other disciplines, that the student uses intuition, and that one must go beyond it.

Gomes et al. (2015), based on Souza, Perez, Bicudo, Bicudo, Silva, Baldino and Cabral (1991), argue that the basic educational content is located in the domains of counting and measurement, discrete numerical and continuous geometric, respectively. In addition, the authors, based on the work of Souza et al. (1991), outline a path for the distribution of subjects in a degree course in mathematics, so that when the student arrives at real analysis, he/she can follow the subject as it is presented, which would lead the student to build differential and algebraic thinking and complement his/her view of the various "contents covered in basic education from the perspective of mathematics" (Gomes et al., p.

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1258). Therefore, they ensure that it should be introduced when differential thinking is being developed.

Considering the importance of deepening mathematical concepts in the process of training mathematics teachers, Gomes et al. (2015) also make two points that they consider important: "[...] the subject of analysis provides a certain autonomy for teachers working in basic education" and "[...] places the subject of analysis among those that raise the level of thinking of the graduate/future teacher of basic education to a higher level" (p. 1260).

From this perspective, Dysman and Dysman (2021, p. 356), when discussing the problem of real analysis in undergraduate courses in relation to epistemic colonization and the logic of abyssal thinking and Boaventura Santos[5]' ecology of knowledge, point out that

> [...] the fact that if, on the one hand, we tend to believe in the importance of teaching mathematics (and Analysis in undergraduate courses) because of its emancipatory potential, on the other hand, when this teaching becomes an imposition (of formulas, methods, theorems, demonstrations, etc.) legitimized because of hierarchies (between knowledge or subjects), it can be transformed into an instrument that contributes to the emergence of the modern subhumanity mentioned by Santos, an instrument that prunes self-esteem, hijacks autonomy and produces submission. This is the paradox of mathematics education as formulated by Paul Ernest (2004)[6]: "mathematics is obvious and coherent, but when reasoning is not understood it becomes the most irrational and authoritarian of subjects" (p. 356).

Of course, there are still many doubts to be answered and countless questions can be asked, but as our aim in this article is to present some arguments that show the importance of Real Analysis in undergraduate courses, we will stick to a few main questions. Moreira and Vianna (2016, p. 527) ask

> [...] why do future school teachers need to acquire the mathematical culture that Real Analysis provides? Why do they need to understand what it means to think mathematically, as the development of this discipline leads them to understand? And why, as a future professional teacher in basic education, do they need to understand the nature of this mathematical knowledge that Real Analysis offers them?

According to Moreira and David (2005), there is a distinction between the approach to and production of mathematical knowledge developed in higher education (especially in mathematics degree courses) and the mathematical knowledge taught in the context of basic education. For the authors, the former corresponds to academic mathematics, which includes the entire scientific field of knowledge that is verified and constituted by professional mathematicians (here with emphasis on real analysis), while the latter is school mathematics, which consists of mathematical knowledge that has already been formally validated by academic mathematics and that is related or connected to the process of teaching and learning mathematics in basic education.

We understand that the knowledge of academic mathematics studied in the subject of

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[5] Santos, B. S. (2018). *Construindo as Epistemologias do Sul*: Para um Pensamento Alternativo de Alternativas. Volume 1. Buenos Aires: CLACSO. Available at: https://www.boaventuradesousasantos.pt/media/Antologia_Boaventura_PT1.pdf

[6] Ernest, P. (2004). *Postmordernism and the Subject of Mathematics.In Walshaw, M. Mathematics Education whithin the Postmodern*. Greenwich: IAP.

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real analysis provides, among other things, the theoretical and formal validation of the knowledge of school mathematics present in basic education. In the case of definitions and demonstrations, according to Moreira and David (2005), the deductive and rigorous mathematical proof (present in real analysis) that leads to mathematical demonstrations, within the framework of school mathematics, in the midst of the teaching practice of the mathematics teacher, can be worked with less formal argumentation that are more convenient in the context of school mathematics. In this sense, Lima (2007) points out that proving and demonstrating, even in basic education, is a way of convincing by reason rather than by authority, but the author emphasizes that it is necessary to balance proving and demonstrating, respecting the intellectual level of the students, prioritizing certain important mathematical facts, such as the Pythagorean Theorem, which have simple and elegant demonstrations.

In the process of teaching and learning mathematics, the use of applications with contextualized problem-solving, involving the use of teaching methods linked to historical facts and figures in the field, has been a source of research in the field of mathematics education. However, as Lima (2007) points out, we need to formulate and work correctly, precisely and objectively with the mathematical concepts and definitions involved in the educational process and the statements of propositions, as well as promote the practice of deductive reasoning, establishing connections between different concepts and the multiple possibilities of interpretation and reformulation of ideas that emerge from discussions with students.

Collaboration with the debate, according to Morais Filho (2016, p. 157, emphasis added),

> It's no exaggeration to say that expressing ideas clearly and precisely, and knowing how to write a demonstration is just as important as inventing them; it's not enough just to solve complicated exercises, have brilliant insights or understand mathematical theories in depth. You need to write down your ideas. Writing improves thinking, strengthens convictions in arguments, sharpens reasoning and should become a practice.

According to Ávila (2011, p. 1-2), "one of the main objectives of undergraduate Analysis is to practice demonstrations. Enunciating and demonstrating theorems is one of the central occupations of every teacher or scholar of mathematics". Thus, we understand that the study and development of mathematical demonstrations that cover the disciplinary scope of Real Analysis seeks to improve the future teacher's ability to conceptualize mathematically and provide them with mathematical knowledge support for discussions with their students. After all, mathematics "is not only useful for the concrete things of real everyday life, but it is a formidable tool for understanding the world in its many forms" (D'Amore, 2011, p. 23).

In this context, we understand that the subject of Real Analysis is fundamental to the initial training of mathematics teachers so that they can possibly have confidence and conviction in the art of conceptualizing mathematically, writing down their ideas and constructing mathematical demonstrations. This does not imply that the presence of the subject Real Analysis in the compulsory mathematics degree curriculum, and its attendance by future teachers, is enough to guarantee good professional training in mathematical conceptualization and demonstrations. However, Real Analysis in some way closes a cycle of mathematical maturation and, as highlighted in the works by Wasserman, Fukawa-Connelly,

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Villanueva, Mejia-Ramos & Weber (2017) and Wasserman, Weber, Fukawa-Connelly & Mcguffey (2019), it is up to the teacher trainer responsible for this curricular component to promote dialogues in order to establish connections between the contents of Real Analysis and the mathematical contents present in the Basic Education Curriculum. In the next section, we present the intertwining of HCK and SCK as a theoretical contribution to our discussions.

Between Knowledge of Content on the Horizon and Specialized Knowledge of Content to be Taught

For a long time, it was believed that mastery of the content to be taught was sufficient for a teacher to practice. Thus, technical knowledge of content was considered necessary and sufficient for the proper development of the profession. In the mid-1980s, however, a movement began in educational research that pointed to the importance of a type of knowledge that went beyond technical mastery of content. Schulman (1986) then introduced pedagogical content knowledge, which encompasses the knowledge necessary for teaching practice and includes various examples, illustrations, analogies, and different ways of representing ideas. In this context, Schulman states that pedagogical content knowledge is a "way of representing and formulating content that makes it understandable to others. [It also includes an understanding of what makes learning certain topics easy or difficult" (p. 9). The author distinguishes pedagogical content knowledge from purely technical content related to the subject being taught.

Ball, Thames and Phelps (2008), in an attempt to build on the work begun by Schulman and to more clearly delineate the types of knowledge in mathematics, then began to use the term mathematical knowledge for teaching to refer to the set of knowledge, skills, and techniques that are part of the mathematics teacher's practice in addition to content knowledge. Based on the work developed by Schulman, they propose a division of mathematical knowledge for teaching into two subdomains: Content Knowledge and Pedagogical Content Knowledge, which are then subdivided into other categories of knowledge. One of these, which is the focus here, stems from Ball's (1993) concern with how school mathematics can relate to academic mathematics. In this context, knowledge on the horizon emerges, a kind of "peripheral view of mathematics, a view of the wider mathematical universe that teaching requires" (Ball & Bass, 2009, p. 1).

For Ball, Thames and Phelps (2008), Horizon Content Knowledge (HCK) functions as an awareness of how mathematical content relates to each other in the school curriculum, as well as presenting possibilities for connections with Academic Mathematics. According to Ball and Bass (2009), HCK concerns the relationships between what is currently being taught and a deeper and broader knowledge of mathematical structures, ideas, and principles. Jakobsen, Thames, Ribeiro and Delaney (2012) define Content Knowledge on the Horizon as a

> [...] familiarity with the discipline (or disciplines) that contribute to the teaching of school content, providing teachers with a sense of how the content being taught is situated and connected to the wider disciplinary territory. [...] It allows teachers to 'listen' to students, to make judgments about the importance of particular ideas or

DOI: 10.20396/zet.v31i00.8667533

issues, and to treat the subject with integrity [...] (p. 4642).

It is worth noting, however, that this does not mean that teachers should teach such content to their students, which could be irresponsible, since it is not part of the curriculum and is part of a body of knowledge that students are not prepared to access at the moment. The content on the horizon acts as "an awareness - more like an experienced and appreciative tourist than a tour guide - of the great mathematical universe in which the present instructions and experiences are situated" (Ball & Bass, 2009, p. 6).

On the other hand, according to Silva, Andrade & Santos (2018, p. 216, emphasis added), the Specialized Content Knowledge (SCK) to teach

> [...] it is unique for teaching, in the sense that only math teachers (theoretically) need it. Often, in the course of teaching, teachers have to do a kind of mathematical work that others don't have to do. This work involves a kind of unpacking of mathematics that is not necessary in other areas. The pedagogical intention and objective proof that this kind of knowledge is much more than a solid understanding of mathematical content.

The subject of Real Analysis is a set of academic mathematical knowledge that (theoretically) lies on the horizon of the mathematics teacher, and which should not be taught to primary school students. The teacher trainer responsible for the subject of Real Analysis in the Mathematics degree course needs to establish dialogues and interlocutions with the contents of School Mathematics, in the sense of seeking to problematize (formulate and reformulate problems), give other meanings to school contents, make and remake connections between these contents, seeking to promote didactic transpositions between the systematic and formal knowledge of Real Analysis (Academic Mathematics) and the mathematical content present in Basic Education (School Mathematics), in pedagogical actions that are not dominant, but supportive of academic knowledge (HCK) in relation to the knowledge of school practice (SCK).

Some situations that justify the importance of Real Analysis in mathematics teacher training

Situation 1

According to Ávila (2011), the contact that primary school students have with irrational numbers is limited to the study of the number π, associating it with the ratio of the length of a circle to its diameter, and "the student is only informed that the decimal expansion of this number is infinite and not periodic" (p. 35). For the author, another critical moment in relation to irrational numbers occurs when studying calculations with radicals, and students are generally only told that numbers like $\sqrt{2}$ and $\sqrt{3}$, for example, are irrational numbers. According to Ávila (2011, p. 35), "This 'learning' of irrational numbers can leave students with the impression that irrational numbers are π and some radicals, and they may even form the idea that the set of these numbers is minimal, at most enumerable".

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Given the scenario of this fragile relationship between primary school students and irrational numbers, in Chart 3 we present a supposed situation in secondary school in order to illustrate the importance of primary school mathematics teachers having knowledge of Real Analysis, even if their work is only at this level of education.

Chart 3 - Description of situation 1

After a math teacher defines, in one of his high school lessons, a rational number as "the number that can be expressed as a quotient of two integers", an irrational number as "the number that is not rational" and $\mathbb{R}$ as the set of numbers that are rational or irrational, a question that may arise from a student is: ___Teacher, in our daily lives and in the subject of Mathematics more rational numbers appear than irrational ones. Is it true that there are more rational numbers than irrational numbers? Then, intending to answer his colleague, another student might say: ___Of course not, the set of rational and irrational numbers are both infinite, so they have the same number of numbers.

Source: The authors

According to Klein (1932), most elementary school students tend to be satisfied with calculations that give results with limited accuracy, and in the case of irrational numbers, they generally understand them through examples, which is what math teachers usually do. In other words, most elementary students understand irrational numbers as a set of numbers separate from rational numbers, consisting of a few specific numbers. For the author, mathematics teachers are expected to teach in a traditional and watertight way, but they need to be able to promote connections between the mathematics developed in basic education and the mathematics studied in higher education in order to influence their practice.

In this sense, regarding the teaching of real analysis in undergraduate mathematics courses, teacher trainers responsible for this subject can apply the teaching model developed by Wasserman et al. (2017). According to the authors, the model begins with the selection of the undergraduate mathematical content to be taught and the observation of possible mathematical teaching practices that undergraduate teachers usually perform in the process of teaching the content discussed. Next, the teacher trainer responsible for the subject of real analysis conducts discussions with his students in order to identify the real analysis content that might be suitable for developing and, in a way, validating the necessary mathematical knowledge by formal proof and, with the teaching practices previously observed, acts to promote the process of teaching and learning the mathematical content of school mathematics under study.

In this process of teaching and learning the content of academic mathematics present in the curriculum of the subject of Real Analysis of the degree in Mathematics, Wasserman et al. (2019) note that the instructional model of Wasserman et al. (2017)

> [...] considers the relationships between: (i) advanced mathematics; (ii) secondary mathematics; and (iii) teaching secondary mathematics. The frst two are about content; the third is about teaching. Content knowledge for teaching is developed as ideas from the frst and second are applied to the third. Our instructional model has three phases and “book-ends” the study of advanced mathematics by beginning and ending with a discussion of teaching. The pedagogical situations both motivate the

DOI: 10.20396/zet.v31i00.8667533

> study of the advanced mathematics so that students see the relevance of the content and provide an application for how the advanced mathematics might inform their teaching. (Wasserman et al., 2019, p. 7).

Starting the debate within the History of Mathematics, according to Roque (2015), during the 17th century irrational numbers were freely manipulated by mathematicians and their mathematical nature was not investigated. For example, "Pascal and Barrow claimed that irrational numbers should be understood only as symbols, not having an existence independent of continuous geometric quantities. A number like $\sqrt{3}$, for example, should be understood as a geometric quantity" (p. 434). At the end of the 18th century and the beginning of the 19th century, the numbers we now call irrational appeared in the studies of mathematicians such as Charles Sanders Peirce (1839-1914), George Cantor (1845-1918) and Richard Dedekind (1831-1916), including in the process of solving problems, but they were not yet defined, they were not even accepted as numbers. "All the names used to designate these numbers express the difficulty of admitting their existence or, rather, their mathematical citizenship: 'deaf' or 'inexpressible' numbers" (Roque, 2015, p. 409, emphasis added).

In the midst of discussions involving the concept of continuity, irrational numbers were grouped with rational numbers to form the set of real numbers. According to Bacha & Saito (2014, p. 15)

> It can be said that as he progressed in his study of continuity, Peirce was led to reject Cantor's view that the continuum was some geometric form made up of infinitely many points. While Cantor and Dedekind considered the irrational numbers to be the complement of the rationals, giving them scope over the real numbers, Peirce saw the relationship between rationals and irrationals differently. He concluded that there was a kind of proximity in the reals which, in fact, constituted a violation of continuity.

According to Bacha and Saito (2014), the concept of continuity was of paramount importance to Peirce, as it was necessary "to explain space, time, and motion, but also evolution, psychological development, science itself, in short, it would be a way to philosophical truth, but also to scientific truth in all fields" (p. 20). On the other hand, according to Bacha and Saito (2014), while Cantor explored the continuity of real numbers with his study of trigonometric series, Dedekind sought to characterize continuity or the idea of continuum to construct the definition of real numbers through the mathematical process known as Dedekind cuts. In his quest to create teaching processes involving basic topics in differential calculus, especially the study of limits, "Dedekind realized that geometric intuition, though a guide, was not strictly satisfactory, so he turned to a purely arithmetic study of continuity and irrational numbers" (p. 20).

Broetto and Santos-Wagner (2019, p. 740) point out that irrational numbers support the composition of the number system and that it is essential for students in basic education to understand them. "The insufficiency of rational numbers to measure any segment with a pre-established unit of measurement, such as the fact that the diagonal of a square is incommensurable in relation to its side, justifies the expansion of the numerical field."

In the case of the supposed situation reported, in addition to these facts and figures from the history of mathematics, it is important that the teacher know the results proved by George Cantor, such as that "the set R of real numbers is not enumerable" and that "the set Q of rational numbers is enumerable" because knowing this and also knowing that the union of

DOI: 10.20396/zet.v31i00.8667533

two enumerable sets is enumerable and that the set of real numbers is the union of the set of rational numbers and the set of irrational numbers, he would conclude that the set of irrational numbers is not enumerable and that there are therefore “many more” irrational numbers than rational ones, in the sense that a non-enumerable set (in this case, the irrational ones) has a higher cardinality than an enumerable set (in this case, the rational ones).

It's true that such arguments are completely beyond the level and objectives of secondary education, as is the rigorous presentation of the theory of real numbers as carried out in real analysis courses. However, as Klein (1932) and Broetto and Santos-Wagner (2019) point out, if teachers can relate the content they need to teach to what they've studied in undergraduate studies, especially in Real Analysis, they may be more confident in dialoguing with their students, clarifying doubts such as the ones we've exemplified.

According to Elias (2017, p. 207),

> When we are concerned with HCK involving these numbers, we necessarily need to understand how the rationals connect with the irrationals and, consequently, with the reals. If, on the one hand, the formal construction of rational numbers by equivalence classes is based on the operations already known for integers, on the other hand, the nature of the construction of real numbers is quite different, since it requires notions from Real Analysis, such as the limit.

According to Ball and Bass (2009), HCK "is not the kind of knowledge that teachers need to have in order to explain to their students; similarly, knowledge of the horizon does not create an imperative to act in a particular mathematical direction" (p. 10). Even so, having mastery of the content to be discussed with students, knowing the concepts involved in their entirety and, depending on the mathematical object discussed, investigating the historical context by analyzing the facts and characters that shaped and constituted the construction of the mathematical knowledge involved, can guide the teacher in a direction of greater commitment to teaching, avoiding distortions that can cause conceptual misunderstandings in the learning of mathematics and, as Ávila (2011) has already pointed out, in the process of teaching and learning irrational numbers.

Situation 2

According to Lima, Carvalho, Wagner and Morgado (2006, p. 61), "Equality $1 = 0.999...$ often perplexes the less experienced", and a teacher who, in their training process, has not had the opportunity to think about it, may feel uncomfortable and confused by issues related to it. To illustrate these issues and contribute to the debate about the importance of Real Analysis in mathematics teacher training and HCK, in Chart 4 we describe the situation involving the equality mentioned.

Chart 4 - Description of situation 2

After a math teacher has defined, in one of his high school classes, what a simple periodic tithe is, concluding that every simple periodic tithe represents a rational number, which is called a generating fraction, and going further, as in the Arithmetic compendiums, stating that "The generating fraction of a simple periodic tithe is a fraction whose numerator is the period and whose denominator is the number formed by as many nines as there are digits in the period" (Lima, 2006, p. 63), a student might ask their teacher the following question:

DOI: 10.20396/zet.v31i00.8667533

___ Teacher, if we find the generating fraction of the periodic tithe 0.999... the way we did in class, we get $\frac{9}{9}$ =1. Is it true that 0.999... is equal to 1 and that we can write 0.999... =1?

In order to contribute to the discussions in class, three other students can take part in the debate by answering the following:

___ I think that 0.999... is almost 1.

___ I think so too, in which case we should use the approximate sign and not the equal sign. For me, equal is equal.

___ Calm down, isn't $\frac{1}{3}$ = 0.333...? So if we do $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}$ we get, on the one hand, $\frac{3}{3}$ =1 and, on the other, 0.333... + 0.333... + 0.333... = 0.999... . So 0.999... =1, right professor?

Source: The authors

As well as dealing with Situation 1 the teacher needs to know Real Analysis topics, Situation 2 requires an understanding of subjects related to this discipline, and was designed by us to justify the importance of the presence of Real Analysis in the curriculum matrix of undergraduate mathematics courses, as well as to illustrate the need for greater proximity between the mathematics taught in these courses and the mathematics of basic education teacher practice. Unlike Situation 1, it does not use contributions from the History of Mathematics, dialoguing with characters and facts. Here, we'll start the discussion with three resolutions, as shown in Chart 5, which are discussed by Elias (2017).

Chart 5 - Three resolutions involving equality 1 = 0.999...

(1)

If, $\frac{1}{9} = 0.111\ldots$ $\frac{2}{9} = 0.222\ldots$ $\frac{3}{9} = 0.333\ldots$ $\frac{4}{9} = 0.444\ldots$ $\frac{5}{9} = 0.555\ldots$

then, $\frac{9}{9} = 0.999\ldots$ and if, $\frac{9}{9} = 1$, then, $1 = 0.999\ldots$ Chavante (2015)

(2)

$\frac{1}{3} = 0.333\ldots$ $3.\frac{1}{3} = 3.0.333\ldots$ $1 = 0.999\ldots$ Niven (1984)

(3)

$x = 0.999\ldots$ $10x - x = 9 + 0.999\ldots - 0.999\ldots$

$10x = 9.999\ldots$ $9x = 9$

$10x = 9 + 0.999\ldots$ $x = 1$ Niven (1984)

(1)

If $\frac{1}{9}$ = 0.111... $\frac{2}{9}$ = 0.222... $\frac{3}{9}$ = 0.333... $\frac{4}{9}$ = 0.444... $\frac{5}{9}$ = 0.555...

then $\frac{9}{9}$ = 0.999... and if $\frac{9}{9}$ = 1, then 1 = 0.999... Chavante (2015)

(2)

$\frac{1}{3}$ = 0.333... 3.$\frac{1}{3}$ = 3.0.333... 1 = 0.999... Niven (1984)

(3)

x = 0.999... $10x - x$ = 9 + 0.999... − 0.999...

$10x$ = 9.999... $9x = 9$

$10x$ = 9 + 0.999... $x = 1$ Niven (1984)

Source: Prepared by the authors based on Elias (2017)

DOI: 10.20396/zet.v31i00.8667533

According to Elias (2017), resolution (1), from Chart 5, formulated by Chavante (2015), provides an opportunity for discussion at elementary school level, more precisely with seventh grade students. It is a practical way of presenting this equality at this level of education, as it is based on a practical way of obtaining generating fractions of periodic diads, as presented in most seventh grade textbooks. However, as the author himself points out, "It is, as the name implies, a practical way of finding the generating fraction, but it does not justify anything" (Elias, 2017, p. 204).

Regarding resolution (2) of Chart 5 formulated by Niven (1984), and which is in line with the idea proposed by one of the students, i.e., $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=0.333\ldots+0.333\ldots+0.333 \rightarrow 1=0.999\ldots$. Elias (2017), based on Penteado (2004), points out that it is not always possible to carry out mathematical operations with infinite representations. Therefore, this situation is not mathematically consistent. As for resolution (3) of Chart 5, which is also presented by Niven (1984), according to Elias (2017, p. 205, emphasis added),

> We understand that this is a plausible justification for what is intended, but it also brings with it strangeness. When we multiply 0.9999... by 10, we end up with 9.9999... and we "believe" that the number of nines to the right of the comma is the same as the number of nines in 0.9999... because when we subtract 0.9999... from 9.9999... and zero out all the decimal places, we conclude that 9.9999...− 0.9999...= 9.

For Elias (2017), based on the discussions generated by the equality 0.999... = 1, other problematizations can be carried out, both with basic education students and in the process of training mathematics teachers, in order to encourage debates involving the set of irrational numbers. "For example, if 0.999... and 1 are different, then there must be a real number between them. In this case, what would be the arithmetic mean between these two numbers (0.999... and 1)?" (p. 208). Below, the reader can see which concepts and elements of Real Analysis are involved in Situation 2 and which could help a teacher experiencing a situation like this in the classroom.

Given the decimal expression α = 0.999... represented by the real number $\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\cdots$, it is in fact possible to say that α=1. In this case, we have a non-decreasing sequence of rational numbers

$$\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \cdots \leq \alpha_n \leq \cdots$$

where $\alpha_1=\frac{9}{10}$, $\alpha_2=\frac{99}{100}$, $\alpha_3=\frac{999}{1000}$, $\alpha_n=\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\cdots\frac{9}{10^n}$, which are values that get closer and closer to the real number $\alpha=1$ since $1-\alpha_1=0.1$, $1-\alpha_2=0.01$, $1-\alpha_3=0.001$ and, more generally, $1-\alpha_n=10^{-n}$, so that if n is large enough, the difference 1-α_n can become as small as desired. In this case, we say that the real number α=1 is the limit of the sequence of real numbers that we have presented. " The concept of limit – a linchpin in real analysis – is useful for reconciling tensions about infinite decimal expansions via convergent sequences or series (e.g., $0.\overline{9}=1$)" (Wasserman et al., 2017, p. 2).

DOI: 10.20396/zet.v31i00.8667533

Thus, according to Lima et al. (2006, p. 61), the only way to resolve the apparent paradox 1= 0.999... is to clarify that "the symbol 0.999... actually means the number whose approximate values are 0.9 0.99 0.999, etc.". And as we saw earlier, that number is 1.

We believe that mathematics teachers working in secondary schools need to know the mathematical objects described in order to feel confident when faced with questions from students and not run the risk of being influenced by their points of view and making mistakes in their answers. In addition, this knowledge can enable the teacher to promote productive interaction in the classroom. In this sense, we agree with Elias, Gereti and Savioli (2015) that for interaction to be productive, it must be within the reach of the students. For example, according to the authors, "when talking about the representations of a rational number, for example, teacher and student should be talking in the same direction, sharing knowledge" (p. 12). However, as Wasserman et al. (2017) and Wasserman et al. (2019) point out, in order for this to happen, it is essential that teachers be aware of and supported by the content they have studied in their undergraduate courses, and that they can make appropriate connections between what they have learned and what they are about to teach.

Some final considerations

The discussion about the presence of Real Analysis in undergraduate mathematics courses still raises many controversies, even though the research presented in this article shows that almost all public higher education institutions in Brazil maintain it in their curriculum, respecting the guidelines of the National Education Council. As we saw in Gomes (2013), the mathematics undergraduates and primary school teachers who took part in the research showed a certain aversion to Real Analysis because they considered it to be very difficult and formal, as well as assessing that there is a mismatch between what is learned in the subject and what is taught in elementary school. Despite this, they felt that Real Analysis plays an important role in the training of future math teachers. What we can see is that there is unanimity regarding the importance of learning this subject, but the reasons to explain this position remain, according to most people, somewhat unclear.

According to Wasserman et al. (2017, p. 2),

> Indeed, real analysis courses designed for teachers at times draw on these connections to make the ideas in real analysis more concretely related to secondary mathematics. However, when asked how courses such as real analysis affect their instruction, teachers rarely cite these topics or any other explicit connections between real analysis and secondary mathematics. Even when teachers are shown explanations from real analysis that resolve the tensions described above, such as a proof that $0.\overline{9} = 1$, many say that these explanations are not relevant for their teaching. To avoid misinterpretation, we do not claim that these ideas in analysis actually are irrelevant to the teaching of secondary mathematics, only that some teachers perceive them as such and most cannot connect these ideas to concrete pedagogical actions.

It is clear that this topic still needs to be widely discussed in order to establish more clearly the role of real analysis in the training of mathematics teachers, so that the difficulties

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that clearly exist in its teaching can be more consciously addressed. Jakobsen et al. (2012) state that the topics of "advanced mathematics for teachers need to be demonstrably related to the work of teaching in schools" (p. 4636). Following this direction, this article aims to bring into the debate the idea of content knowledge on the horizon, a theory still little discussed in Brazil because we understand that it can help bridge the gap between academic mathematics and school mathematics.

Situations 1 and 2 show the relationships that, although subtle, clearly exist between Real Analysis and the contents studied in Basic Education. The apparent lack of relationship between these contents is understandable, since the enumerability of sets, as in situation 1, and series, as in situation 2, are not topics that students necessarily have to learn, except in higher education courses. However, without their knowledge, the teacher may end up making less correct decisions when dealing with doubts such as those presented. Even if the students do not intend to build a career in mathematics, the teacher must be able to resolve their doubts in the best way possible, always respecting mathematical integrity. The situations mentioned were conceived as part of an attempt to continue a discussion that is known to be taking place, but which could and should take a more practical form. In other words, we intended to present situations that might occur in a mathematics classroom and to bring them here in the form of dialogues, with the aim of illustrating them in a somewhat more comprehensible way. We believe that the constant practice of imagining situations and trying to predict students' possible reactions, however difficult it may be, is important because it allows teachers to reflect on them and prepare themselves to deal with them in the best possible way.

According to Ball et al. (2008), HCK refers to the panorama of mathematical knowledge before, during and after one or more processes of teaching and learning mathematics. The construction of mathematical knowledge that takes place on the teacher's horizon goes through the past, present, and future in relation to the content present in the school curriculum and beyond. For Fernández and Figueiras (2014), HCK includes not only curricular content, but also the relationships between mathematical objects and different domains of knowledge, as well as other domains necessary for the development of the teacher's practice, which even go beyond the domain of the school context.

In this work, we believe that the study of HCK makes it possible to understand how mathematical objects relate to each other, how they intertwine and articulate. In other words, when dialoguing with students about a particular mathematical content, other mathematical subjects may come to light that the students already know about (supposedly learned in the past), as well as the establishment of links with other mathematical subjects that emerge soon after or that are already interconnected, as a result of the mathematical object that is being dialogued (present) and with a view to preparing the ground for the construction of mathematical knowledge of what is yet to come (future). We understand that the constitution of these links is validated mathematically in the process of initial training for mathematics teachers, within the scope of the subject Real Analysis.

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Research using the HCK involves the development of complex, articulated and dynamic skills, generally of a mathematical nature, but which can be linked to other areas of knowledge, and the pedagogical skills of the math teacher, with the aim of enabling the transition of mathematical knowledge from what the student should already know (past that becomes present), with what they are trying to learn (present with an eye on the future) and with what they still need to know (future that is made from the present in line with what was learned in the past). Studying the ideas that make up the core of the HCK is fundamental for the constitution of planning, the development of what has been planned, as well as the reflections that emerge from the math teacher's praxis.

Speaking of educational praxis, we agree with Wasserman et al. (2017) when they point out that believing that simply completing the subject of Real Analysis in a mathematics degree course will provide basic education teachers with good performance in their school mathematics praxis, i.e. discussing the contents of Real Analysis and understanding them alone will guarantee that these teachers will apply what they have learned in their school practices, is a transition that is difficult to achieve in educational contexts. " There is an implicit hope that as a byproduct of learning advanced mathematical content, teachers will respond differently to instructional situations in the future [...]" (p. 3).

In this sense, from the perspective of the SCK to be taught, we understand that it is necessary to focus on the teaching and learning process to be carried out by the teacher trainer responsible for the subject of Real Analysis in the undergraduate degree in Mathematics, in order to carry out an educational approach that starts from the praxis of Basic Education teachers, in order to promote educational discussions with their students in order to carry out dialogues or interlocutions between the contents of Academic Mathematics (present in Real Analysis) and the mathematical knowledge to be taught (School Mathematics) in the context of Basic Education. Thus, as Wasserman et al. (2017) and Wasserman et al. (2019) point out, Real Analysis educational actions should begin with the process of teaching and learning School Mathematics content, go through Academic Mathematics in Real Analysis, producing links between the mathematical content (of School Mathematics) from which the students' SCK begins and develops, and then return to the initial educational praxis with a teaching and learning process that helps to problematize and resignify mathematical content present in School Mathematics, with which Real Analysis has a strong connection.

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References

Almouloud, S., Silva, M. J. F., Miguel, M. I. R., & Fusco, C. A. S. (2008). Formação de professores de Matemática e apreensão significativa de problemas envolvendo provas e demonstrações. *EMP*, 10(2), 217-246. https://revistas.pucsp.br/index.php/emp/article/view/1744/1135

Ávila, G. S. (2011). *Análise Matemática para licenciatura.* São Paulo: Edgard Blücher.

Bacha, M. L., & Saito, F. (2014). Peirce e Cantor: um estudo preliminar sobre continuidade e infinitesimais. *RBHM*, 14(28), 1-23. https://doi.org/10.47976/RBHM2014v14n2801-23

Ball, D. L. (1993). With an Eye on the Mathematical Horizon: Dilemas of Teaching Elementary School Mathematics. *The Elementary School Journal***,** 93(4), 373-397. https://www.journals.uchicago.edu/doi/pdf/10.1086/461730

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching: What Makes It Special? *Journal of Teacher Education*, 59(5), 389-407. https://doi.org/10.1177%2F0022487108324554

Ball, D. L., & Bass, H. (2009). With an Eye on the Mathematical Horizon: Knowing Mathematics for Teaching to Learners´ Mathematical Futures. *Anais* do Jahrestagung der Gesellschaft für Didaktik der Mathematik, n.43, (pp. 1-12). Oldenburg. https://static1.squarespace.com/static/577fc4e2440243084a67dc49/t/579a39cebe65945c23e8b8cf/1469725134888/EyeOnMathHorizon.pdf

Broetto, G. C., & Santos-Wagner, V. M. P. (2019). O ensino de números irracionais na Educação Básica e na Licenciatura em Matemática: um círculo vicioso está em curso? *Bolema*, 33(64), 728-747. https://doi.org/10.1590/1980-4415v33n64a14

Chavante, E. R. (2015). *Convergências*: Matemática, 7º ano: anos finais: Ensino Fundamental. (Manual do Professor). São Paulo: Edições SM.

D'Amore, B. (2011). *Matemática, estupefação e poesia.* São Paulo: Livraria da Física.

Dysman, A. M., & Dysman, F.C. (2021). Análise Real na Licenciatura: do Pensamento Abissal à Ecologia de Saberes. *International Journal of Research in Mathematics Education – RIPEM*, 11(2), 336-359. https://doi.org/10.37001/ripem.v11i2.2558

Elias, H. R. (2017). *Fundamentos teórico-metodológicos para o ensino do Corpo dos Números Racionais na formação de professores de Matemática.* Tese de Doutorado em Ensino de Ciências e Educação Matemática. Londrina: Universidade Estadual de Londrina. http://www.bibliotecadigital.uel.br/document/?code=vtls000213550

Elias, H. R., Gereti, L. C. V., & Savioli, A. M. P. D. (2015). "Que horror! Uma coisinha tão Simples": Um Estudo sobre a Produção de Significados para Questões Matemáticas. *Anais do VI Seminário Internacional de Pesquisa em Educação Matemática*, (pp. 1-14). https://www.researchgate.net/publication/284442925.

Fernández, S., & Figueiras, L. (2014). Horizon Content Knowledge: Shaping MKT for a Continuous Mathematical Education. *REDIMAT*, 3(1), 7-29. https://hipatiapress.com/hpjournals/index.php/redimat/article/view/640/pdf

Gomes, D. O. (2013). *A Disciplina de Análise Segundo Licenciandos e Professores de Matemática da Educação Básica.* Dissertação de Mestrado em Educação Matemática.

DOI: 10.20396/zet.v31i00.8667533
Rio Claro: Universidade Estadual Paulista. https://repositorio.unesp.br/handle/11449/91056

Gomes, D. O., Otero-Garcia, S. C., Silva, L. D., & Baroni, R. L. S. (2015). Quatro ou Mais Pontos de Vista sobre o Ensino de Análise Matemática. *Bolema*, 29(53), 1242-1267. https://doi.org/10.1590/1980-4415v29n53a22

Gonçalves, K. V., & Fiorentini, D. (2023). Origens e apropriação cultural do Lesson Study: contribuições à aprendizagem do professor que ensina Matemática. *Revista Paranaense de Educação Matemática*, 12(29), 226-249. https://doi.org/10.33871/22385800.2023.12.29.226-249

Jakobsen, A., Thames, M. H., Ribeiro, C. M., & Delaney, S. (2012). Using Practice to Define and Distinguish Horizon Content Knowledge. *Anais do international Congress on Mathematical Education*, (pp. 4635-4644). Seul. https://www.researchgate.net/publication/258960150.

Klein, F**.** (1932). *Elementary mathematics from an advanced standpoint*. Londres: Macmillian and Co. Ltd..

Lima, E. L. (2007). *Matemática e Ensino.* Rio de Janeiro: Editora da SBM.

Lima, E. L., Carvalho, P. C. P., Wagner, E., & Morgado, A. C. (2006). *A Matemática do Ensino Médio.* Volume 1. Rio de Janeiro: Editora da SBM.

Marcelo-García, C. (2009). A identidade docente: constantes e desafios. *Revista Brasileira de Pesquisa sobre Formação Docente*, 1(1), 109-131. https://www.revformacaodocente.com.br/index.php/rbpfp/article/view/8

Martines, P. T. (2012). *O papel da disciplina de Análise segundo professores e coordenadores***.** Dissertação de Mestrado em Educação Matemática. Rio Claro: Universidade Estadual Paulista. https://repositorio.unesp.br/handle/11449/91033

Morais Filho, D. C. (2016). *Um convite à Matemática com técnicas de demonstração e notas históricas.* Rio de Janeiro: Editora da SBM.

Moreira, P. C., & David, M. M. M. S. (2005). *A Formação Matemática do Professor:* Licenciatura e Prática Docente. Belo Horizonte: Autêntica.

Moreira, P.C., Cury, H. N., & Vianna, C. R. (2005). Por que análise real na licenciatura? *Zetetiké*, 13(23), 11-42. https://doi.org/10.20396/zet.v13i23.8646978

Moreira, P.C., & Vianna, C. R. (2016). Por Que Análise Real na Licenciatura? Um Paralelo entre as Visões de Educadores Matemáticos e de Matemáticos. *Bolema*, 30(55), 515-534. https://doi.org/10.1590/1980-4415v30n55a11

Niven, I. (1984). *Números*: racionais e irracionais. Coleção Fundamentos da Matemática Elementar. Rio de Janeiro: SBM.

Penteado, C. B. (2004). *Concepções do professor do Ensino Médio relativas à densidade do conjunto dos números reais e suas reações frente a procedimento para a abordagem dessa propriedade*. Dissertação de Mestrado em Educação Matemática. São Paulo: Pontifícia Universidade Católica. https://tede2.pucsp.br/handle/handle/11180

Roque, T. (2015). *História da Matemática:* uma visão crítica, desfazendo mitos e lendas. Rio de Janeiro: Zahar.

DOI: 10.20396/zet.v31i00.8667533

Schulman, L. (1986). Those Who Understand: Knowledge Growth in Teaching. *Educational Researcher*, 15(2), 4-14. https://www.jstor.org/stable/1175860

Severino, A. J. (2007). *Metodologia do Trabalho Científico.* São Paulo, Cortez.

Silva, E. S., Andrade, F. C., & Santos, J. A. (2018). Explorando Uma Lista De Transmissão Para Refletir Sobre O Conhecimento Matemático Para o Ensino de Análise Combinatória. *REVEMAT.* 13(2), 210-227. http:// doi.org/105007/1981-1322.2018v13n2p210

Souza, A. C. C., Perez, G., Bicudo, I., Bicudo, M. A. V., Silva, M. G. P., Baldino, R. R., & Cabral, T. C. B. (1991). Diretrizes para a Licenciatura em Matemática. *Bolema*, 6(7), 90-99. https://www.periodicos.rc.biblioteca.unesp.br/index.php/bolema/article/view/10712

Wasserman, N. H., Fukawa-Connelly, T., Villanueva, M., Mejia-Ramos, J. P., & Weber, K. (2017). Making real analysis relevant to secondary teachers: Building up from and stepping down to practice. *PRIMUS*, 559–578. https://doi.org/10.1080/10511970.2016.1225874

Wasserman, N.H., Weber, K., Fukawa-Connelly T., & Mcguffey, W. (2019). Designing advanced mathematics courses to influence secondary teaching: fostering mathematics teachers' "attention to scope". *Journal of Mathematics Teacher Education*. 1-21. https://doi.org/10.1007/s10857-019-09431-6