



Rational numbers in primary schools in Portugal (1930-1974)

Os números racionais no ensino primário em Portugal (1930-1974)

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Abstract

The work focuses on the approach to non-negative rational numbers explored in official curriculum documents and textbooks for the initial training courses of primary school teachers. The analysis centred in these courses and primary education curricula and four manuals by reference authors in teacher training in Portugal, at the time. A document analysis was carried out, with a historical perspective. The first approach of rational numbers is done through the measurement of quantities, but, implicitly, the manuals also present different meanings for fractions in context. In the manuals the importance given to the different representations is highlighted, for the presentation of rational number and in two of these manuals the importance given to the different contexts for the fractions is highlighted. The analysed manuals follow the instructions of the official documents, in the proposed approach to rational numbers. However, it is also possible to observe some autonomy of the authors of the manuals in relation to the official curricular documents.

Keywords: Teacher training; Primary school; Non-negative rational numbers; History of mathematics education.

Resumo

O trabalho analisa a abordagem aos números racionais não negativos presentes nos documentos curriculares oficiais e nos manuais do curso de formação inicial dos professores do ensino primário. Analisaram-se os programas destes cursos, os programas do ensino primário e quatro manuais de autores de referência na formação de professores em Portugal, na época. Fez-se uma análise documental, com uma perspetiva histórica. A primeira abordagem e definição dos números racionais é feita através da medição de grandezas, mas, os manuais também apresentam diferentes significados para as frações nos exemplos com contexto. Nos manuais destaca-se a importância atribuída às diferentes representações, para a apresentação da definição de número racional e em dois destes manuais releva-se os diferentes contextos para as frações. Os manuais analisados seguem as indicações dos programas na abordagem proposta aos números racionais, no entanto, alguns autores evidenciam aspetos diferentes na abordagem que fazem a estes números.

Palavras-chave: Formação de professores; Ensino primário; Números racionais não negativos; História da educação matemática.

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Introduction

In this text we analyze the way in which the textbooks⁴ for the initial training of primary education teachers interpret the guidelines of the official curricular documents regarding the work to be developed in the approach to non-negative rational numbers, in the period from 1930 to 1974. The start date of the period under analysis is marked by a change in the format of the training courses for teachers at this level of education and its end marks the change in the initial training of these teachers that took place after a revolution that ended the dictatorship that prevailed in Portugal, since 1933. The study focuses on the teaching of rational numbers, focusing on the way in which the concept of rational number is worked on textbooks used in schools for training teachers of primary education, published within the time interval defined in the study. The teacher's professional knowledge to teach mathematics and to teach rational numbers, has been the subject of several investigations (see, for example, Behr, Harel, Post & Lesh, 1992; Monteiro & Pinto, 2005; Ni & Zhou, 2005; Nunes, Bryant & Watson; Pinto, 2011). However, Matos (2018) emphasizes the need to reflect on the relationship between research in the history of mathematics education and mathematics education, in which the history of education can contribute to the presentation of another point of view on the contents of mathematics studied.

The textbooks used in schools in charge of primary teacher's education are important sources of research to reconstruct the history of a particular school subject (Chervel, 1990). Valente (2007) highlights the role of textbooks and their connection with the development of school mathematics, stressing that mathematics is one of the subjects in which the relationship between textbooks and teaching is most evident. This close relationship between the subject and textbooks gives them a special status as a source for the history of school knowledge. Valente (2019) emphasizes that pedagogical textbooks, textbooks and teaching programs are reference documents, for a given time, for teaching work, where established knowledge is fixed, that is, objectified knowledge.

Non-negative rational numbers in school context

Klein, in his book "Elementary Mathematics from a Higher Point of View"⁵ considers fractions a second extension of the concept of number, after the introduction of negative numbers. For Klein (2009) fractions, as they are firstly approached in schools, have a concrete meaning but with a difference in relation to natural numbers. While in natural numbers we are dealing with numerable things, in fractions we move on to measurable things, that is, for measurement. As examples of measurable quantities, Klein (2009) presents the monetary systems, the weight system and mainly the length system, which he considers a more complete example.

⁴ In this article, the textbooks refer to the manuals used in the training of future teachers.

⁵ Portuguese translation of 2009 based on the English version of the first volume of Felix Klein's work, "Elementarmathematik vom hoheren Standpunkte aus", on Arithmetic, Algebra and Analysis, published in 1908 by Teubner, def Leipzig.

In the teaching of rational numbers in elementary levels, it is important to point out that different meanings of fractions in context are often highlighted. It is somewhat consensual that the works of Kieren (1976) were one of the first to consider that numbers consist of different constructs, and that for a global understanding of the concept of rational number, it would be necessary to understand these constructs and their confluence. Kieren (1976) presents seven different meanings that fractions can have. A few years later, Kieren (1988) reformulated his classification and presented five constructs that he considered fundamental in working with fractions: part-whole relationship, in which something is divided into equal parts; quotient, which depends on equitable sharing; the quotient between two whole numbers; measure, which compares two quantities; reason, as a relation between two parts of a whole; and an operator, which transforms the value of one number into another.

Monteiro and Pinto (2005) summarize the research carried out in this field and propose six meanings that fractions can assume in school contexts, at the elementary level, emphasizing that it is in the synthesis of this diversity and in the relationships that students are able to establish between situations, which the rational number sense can be established. Monteiro and Pinto (2005) define the different meanings as follows:

- the relationship between the part of a continuous or discrete whole, in which the fraction arises from the comparison between the part and the whole, the whole being the unit. The fraction refers to a fractional part of a single unit, such as a fifth of a sheet is painted or a fifth of a collection of 10 pencils are blue, the whole being the sheet of paper and the collection of pencils respectively. In this case the denominator indicates the number of parts the unit is divided into, and the numerator designates the number of selected parts;

- the quotient between two integers represented by the fraction a/b . This meaning is associated with equitative sharing, where the numerator represents the number of things to be shared and the denominator the number of recipients of this sharing. Monteiro and Pinto (2005) point out that this is a relationship between two quantities, but that it can also take on the meaning of a quantity, if one thinks that it is the quantity with which each of the receptors was left;

- the “part-part” ratio, which is understood as the ratio between two quantities of the same whole (e.g., the ratio between the number of boys and girls in a class is $3/2$ – reads “is 3 to 2” and corresponds to an intensive quantity;

- the ratio between two different quantities, which give rise to a new quantity, such as, for example, the ratio between the distance and the time required to cover it, the speed, which corresponds to an intensive quantity;

- the partitive operator and the multiplicative partitive operator, where the fraction transforms the cardinal of a discrete set, where the denominator indicates a division and the numerator a multiplication;

- the measure, in which one quantity is compared with another, taken as a unit. In this situation, the unit must be fractionated in such a way that an integer number of times is contained in the quantity to be measured.

Methodology

The study is part of historical research work, with data analysis having a descriptive and interpretive character, highlighting the common elements and the main differences between the different approaches proposed in the textbooks, and their relationship with the official curriculum documents that frame them.

The documentary corpus consists of textbooks or manuals published between 1930 and 1974 and official curriculum documents from the same period. We started by identifying the curricular documents published in the period under study. From this identification, and from the analysis of the literature about teacher education (Pintassilgo, 2006; Correia & Silva, 2002; Silva 2001), textbooks published between 1930 and 1974 were selected as well as sources in the archives of Higher Schools of Education, in the National Library of Portugal and in the General Secretariat of the Ministry of Education.

The selection of textbooks had the following criteria: textbooks in the area of mathematics or its teaching, which were explicitly aimed at training courses for primary education teachers, published in the chronological scope of the study by Portuguese authors, and which were fully available. In a second phase of source selection, the initially selected textbooks were discussed along with the main curricular changes that marked the initial training of primary school teachers from 1930 to 1974, to select a representative textbook for each period. The four textbooks we have selected are: Pimentel (1934), Gaspar and Ferreira (1944), Pinheiro (1961) and Gonçalves (1972, 1974).

The textbook authors

Changes made to the initial training course for primary school teachers in 1930 led to the elimination of most of the subjects from the science component of the specialty and the course was largely focused on the pedagogical component. Therefore, the four textbooks analyzed correspond to didactics subjects, focusing on aspects of teaching rational numbers. The analyzed textbooks are works of Alberto Pimentel Filho (1934), José Maria Gaspar and Orbelino Geraldês Ferreira (1944), José Moreirinhas Pinheiro (1961) and Gabriel Gonçalves (1972, 1974). The knowledge we have about the authors of the textbooks and their training is important to understand how they approach the teaching of rational numbers.

The authors of the textbooks analyzed in the period are mainly linked to pedagogy and didactics. In this group of authors, Alberto Pimentel Filho stands out, for being the only one with a training in science, being a doctor in medicine. He excelled in teaching at the Escola Normal de Lisboa and was the author of several textbooks in pedagogy and the history of education. José Maria Gaspar and Orbelino Geraldês Ferreira, co-authors of one of the

works, completed their primary school teacher training course at teacher training schools for primary teachers, in Coimbra. Later they obtained additional training, namely in pedagogical sciences. José Moreirinhas Pinheiro had an identical training and professional trajectory. He graduated to teach in primary education at the Escola do Magistério Primário de Coimbra, having obtained additional training in pedagogical sciences at the University of Coimbra. He taught in primary education before becoming a teacher at the Escola do Magistério Primário de Lisboa. He was author of several works in didactics and the history of education. Gabriel Gonçalves was a teacher at the Escola do Magistério do Porto, having later served as supervisor inspector at the Ministry of Education. He edited works in the field of didactics, namely arithmetic and Portuguese.

Results

This section presents the results of the official curricular documents analysis, both the programs of initial training courses for primary school teachers, and the primary school programs, as well as the analysis of the textbooks for the initial training of primary school teachers, regarding the work on rational numbers.

The official curriculum documents: initial primary school teachers' education programs and primary education programs

Despite the establishment of schools for the initial training of primary school teachers in 1930 (Decree No. 18.646, 07/19/1930), and the entire restructuring of the initial training course for primary school teachers, namely the study plan, the programs of the courses at this level of education were only changed in 1935 (Decree No. 25.311, 05/10/1935). In 1943 (Decree No. 32.269, 01/16/1943) there was a new alteration of the programs, which remained unchanged until 1974. The analysis of these programs, show that the subjects where the mathematics contents were explored were essentially from the pedagogical component of the course, such as Didactics, Special Didactics or Special Didactics of group B. The programs published in 1935 do not allow a breakdown of the contents to be taught in each subject. The programs published in 1943 refer, in the subject denominate Special Didactics, the content methodology for teaching fractions and decimals, but without specifying what this methodology should be.

Regarding the primary education programs implemented in the period of this study, the official curriculum documents were published in 1929 (Decree No. 16,730, of 13/04/1929), in 1937 (Decree No. 27.603, 03/29/1937), in 1960 (Decree-Law No. 43.369, of 12/02/1960) and in 1968 (Ordinance No. 23.485, of 07/16/1968).

In the 1929 programs, the work with rational numbers began in the 2nd grade, within the topic of Arithmetic, with the “concrete notion of an ordinary fraction, whose terms do not exceed ten”. In the 3rd grade, the work with the rational numbers followed with the decimal fractions and the four operations with decimal fractions. In the 4th grade, the ordinary fractions, the conversion of ordinary fractions into decimal and vice-versa, the simplification of fractions and the four operations with ordinary fractions were studied. The programs

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include Instructions, and in the 2nd grade program was indicated that ordinary fractions, with terms up to 10, should be worked in a concrete way, using objects considered appropriate or line segments. The 3rd grade program Instructions focused on decimal fractions and their decimal representation, as well as operations with decimals and procedures for correctly placing the comma in the results of operations with decimals. In the 4th grade, the Instructions focus on the conversion of ordinary fractions into decimals, the equivalence of fractions and their simplification, operations with fractions and work with mixed numbers. It was also recommended to solve numerous problems involving fractions.

In 1937, elementary primary education encompassed the first three years of schooling. In the Arithmetic programs, working with rational numbers began in the 2nd grade, with the study of the “proper fraction with digit numbers”. In Grade 3, this study involved the four operations with whole numbers and decimals. In the Observations included in the 1937 programs there are no specific indications for working with rational numbers.

In 1960, elementary primary education corresponded to the first four years of schooling. In the new programs there is a change in the initial approach to rational numbers, so the first work with this numbers is done through the decimal representation. The rational numbers study began in the 3rd grade, in the Arithmetic subject, with “measurements with already known linear units; writing and reading representative numbers of these measurements; comma use. Notions of tenth, hundredth and thousandth of any unit”. The 3rd grade program also included the writing and reading of decimal numbers and the four operations with decimal numbers. Other topics addressed in the 4th grade program were “Idea of the ordinary fraction. Conversion of ordinary fraction to decimal number (only in cases of finite decimal). Idea of fraction of a number and percentage”.

In the Observations included in the 1960 programs it was considered that the biggest obstacle to overcome in the 3rd grade was the decimal numbers. According to these Observations, decimal numbers “must be taught grounding on the meter and its submultiples”. Using the knowledge that students should previously have about the meter, it was recommended that it was partitioned into 10, 100 and 1000 equal parts. The programs provided guidance, noting that:

Having knowledge on these new units, they will measure lengths in which the meter enters an exact number of times and verify that these measurements result in whole numbers. They will then measure lengths in which the meter and the decimeter enter one or more times. The teacher will then show the students how these measurements are expressed as mixed decimal numbers, where the main unit is followed by a comma. Subsequently, and using the same practical method, centimeters and millimeters will be introduced. They will thus see that the rules learned in the formation of whole numbers are the same rules that regulate decimal numbers. The digits still have an absolute value and a position value. By removing the units, you will move from mixed decimal numbers to simple decimal numbers. Once familiarized with these concrete units, children will be able to accept the generalization, dividing any unit into tenths, hundredths, and thousandths. (Decree No. 42.994, 125, 05/28/1960, p. 1277)

In the 3rd grade the operations with decimal numbers, should be taught in comparison

with the operations with natural numbers already studied. It was considered that, by selecting suitable problems, it would not be difficult to make the student understand how to place the comma in the results obtained.

The Arithmetic program for the 4th grade had fractions as one of the essential subjects. The study of fractions should be restricted, starting from intuitive work and simple problems to explore the part-all relationship. In determining the fraction of numbers, the study of percentages was considered of special interest, due to the frequent use that was made of this representation.

In 1968, the elementary primary education corresponded to the first four years of schooling. The contents and the Instructions in the programs published in 1968 were very similar to those included in the 1960 programs, maintaining the first approach to rational numbers through decimal representation in the 3rd grade. On the 4th grade the work was focused on the study of fractions. It should be noted, however, that percentages were not studied in this program.

The textbooks

In the period of our study (1930-1974), we observed that in the textbook of Pimentel Filho (1934), in a topic called *Ordinary Fractions*, the author begins by underlining that the notion of fraction is a fundamental notion in the teaching of arithmetic, so it deserves special attention. The author discusses the interest that the notion of fraction can provoke in children, stressing that all the principles related to this content must be “exclusively induced from concrete, real cases, carried out directly by the students. More than in any other case, the passage from concrete notions to abstraction must here be slow and gradual.” (p. 147). It is an initial discussion that focuses on knowing the student's relationship with that specific content and with the need to concretize the different aspects to be worked.

In the third topic of the chapter, Pimentel Filho (1934) presents the fractions using concrete materials, he points out that the presentation must follow three phases, the presentation of the concrete unit, the presentation of the concrete fraction and the measurement of the fraction. After the work with concrete materials, the materials are represented pictorially and a relationship is later established with the verbal representation, with nomenclature exercises. The non-unit fractions are introduced in the same way, using the concrete resources and later the images. The illustrations are used to perform reading exercises for the fractions represented. In these early examples, Pimentel Filho (1934) privileges the introduction of the fraction as a relationship between the part and a whole of a continuous unit.

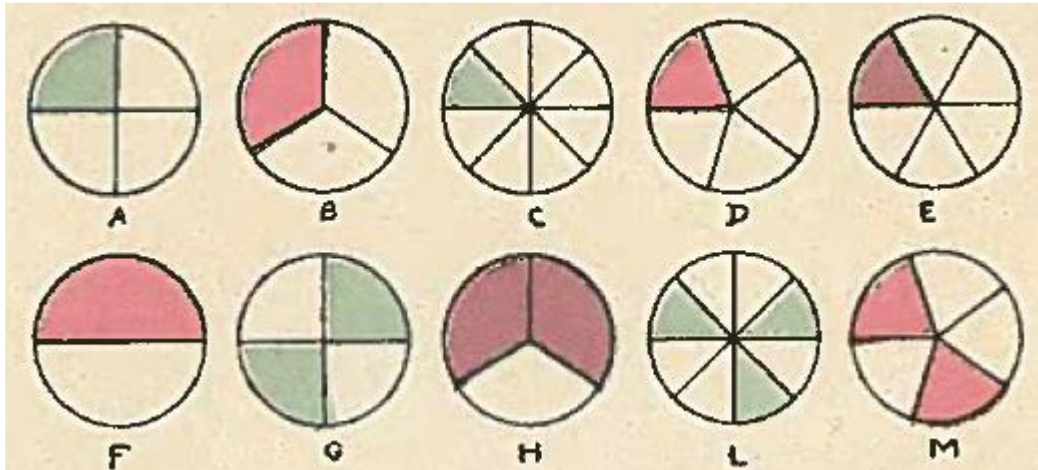


Figure 1 - Disks sectioned into different parts, representing different fractions of the unit, where the representation of non-unit fractions emerges

Source: Pimentel Filho, 1934, p. 150 (digitization, 100% do original)

After the concrete and pictorial presentation of fractions up to the ninth, and after establishing a relationship with verbal representation, Pimentel Filho (1934) suggests the presentation of the numerical representation of the fraction in three phases, numerical representation of the fraction, representation of fractional expressions, designation used by the author for the improper fraction, and representation of fractional numbers, designation used to identify the mixed numeral. In the first phase, he insists on the importance of the meaning of the numerator and denominator. He suggests that at first fractions should be written as words, and that this writing should only be abandoned as the reading of fractions is consolidated. In the second phase, he presents improper fractions that he calls fraction expressions, continuing to use the pictorial representation, which he relates to verbal and symbolic representation. In this phase, the presentation of the fraction as a relation between a part and a whole of a continuous unit continues to be emphasized.

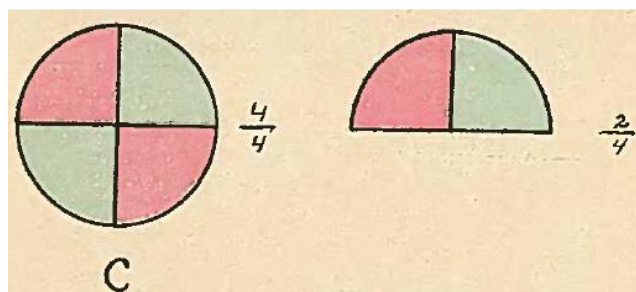


Figure 2 - Representation of a proposed approach to the improper fraction

Source: Pimentel Filho, 1934, p. 152

In the third phase, Pimentel Filho (1934) lay emphasis on mixed numerals, which he calls fractional numbers, defining them as those that are "formed by a whole number plus a fraction, such as $2 + \frac{2}{3}$, $5 + \frac{3}{4}$, etc." (Pimentel Filho, 1934, p. 154). In this phase he does not use the more usual symbolic notation of a mixed numeral, representing these numbers as an

addition of an integer with a fraction. Symbolic notation is only used when the operations of addition and subtraction are explored. It is noted that the verbal designation of mixed numeral is not used, nor later in the context of operations. The author exemplifies the turning of a fractional number into a fractional expression, meaning, turning a mixed numeral into an improper fraction, for example $2 + \frac{2}{3}$ would be turned into $\frac{8}{3}$ as two units are the same as $\frac{6}{3}$ “and joined to the loose $\frac{2}{3}$, is $\frac{8}{3}$ ” (Pimentel Filho, 1934, p. 154)

After addressing the phases that constitute a first step in the approach to fractions, Pimentel Filho (1934) presents nineteen exercises for the consolidation of the contents worked until then. Some of these exercises, which are essentially based on two authors, Bourlet and Groscurin, are as follow:

- 1.º Turn into halves, thirds, fourths, fifths ... ninths, 2, 3, 5, etc., whole numbers.
- 2.º John has 12 lead soldiers. ¿ If he gives half, how many will he be left with?
- 4.º How many pencils will be the $\frac{2}{5}$ of 25 pencils?
- 6.º ¿ If I want to divide a cheese by 8 persons, what portion of the cheese I will give to each one of them? What if I divide it by six persons? And by 5?
- 9.º I was given $\frac{17}{5}$ of oranges? Joining these $\frac{17}{5}$, how many oranges can I reconstitute?
¿ Are there any fifths left? How many?
- 18.º After having lost the $\frac{3}{5}$ of his marbles Paulo still have 12. ¿ How many marbles did he have? – (Groscurin) (Pimentel Filho, 1934, pp. 155-156)

We stress that, in the previous quotation the author presents some exercises (2nd, 4th, and 18th) that refer to situations in which the unit is a discrete set, and in which the fraction is understood as a multiplicative partitive operator, which had not been previously addressed till then in this work. Also noteworthy is the 6th exercise whose context refers to a situation of equitable sharing, with the fraction understood as a quotient.

In their textbook, Gaspar and Ferreira (1944) also begin by considering the teaching of fractions and decimals as essential in arithmetic. These authors note that teaching these numbers involves very abstract notions for children and, therefore, should be particularly intuitive, practical, and active. In the realization of the first notions, Gaspar and Ferreira (1944) suggest the use of the articulated meter or the use of capacity measures. These authors suggest a sequence for presenting the fractions to the students, which starts by working the half and then the fourth and eighth parts, because they can be obtained from the half and the fourth part of the half. The third, sixth and ninth parts would only be worked on after working on the tenth. The examples presented always refer to the fraction as part of a whole of a continuous unit. Gaspar and Ferreira (1944) advocate simultaneous teaching of fractions and decimal representation, considering that the decimal representation facilitates understanding because it gives a utilitarian and logical dimension to numbers, due to its relationship with the metric system.

In Pinheiro's (1961) textbook, the first approach to rational numbers is made from the decimal representation, decimal numbers as they are designated by Pinheiro (1961),

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following the instructions of the programs of the time⁶. In the program instructions, the work that must be done to start the decimal numbers is highlighted. According to these instructions, initiation to decimal numbers should be done from the study of the meter and its submultiples. Students should start by taking measurements where the meter entered a whole number of times. They would then measure using the meter and the decimeter representing in the form designated as mixed decimal, using the comma after the main unit⁷. They should repeat the process for the centimeter and the millimeter, observing the positions of the corresponding digits and establishing a relationship with the rules learned in the formation of whole numbers⁸. The students would then move on to the use of simple decimals, that is, those in which the main unit is zero. After working with the concrete units, students could generalize, dividing any unit into tenths, hundredths and thousandths. Operations with decimal numbers should be taught, establishing a parallelism with operations with whole numbers. The reference of the program instructions, which Pinheiro (1961) presents, suggests that he advocates an approach to non-negative rational numbers from their decimal representation, with the study of measures of length, as advocated by the program itself. After working with this unit, students should generalize any unit in tenths, hundredths and thousandths. It is this approach that Pinheiro (1961) makes in his work of didactics, which leads to a work centered on the whole relationship of a continuous unit.

In the work with fractions, the author emphasizes the pictorial representation, the verbal representation and later the relationship with the symbolic representation. The fraction is essentially presented as the part of a whole of a continuous unit. The initiation was done through unit fractions in a sequence identical to that proposed by Gaspar and Ferreira (1944). An example is also presented in which the fraction appears as a partitive multiplicative operator of a discrete unit. Pinheiro (1961) does not make explicit any indication to differentiate these two types of situations.

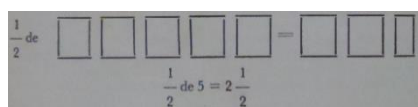


Figure 3 – Fraction as a partitive multiplicative operator of a discrete unit

Source: Pinheiro, 1961, p. 80

⁶ Primary education programs approved by Decree-Law No. 42:994 of May 28, 1960.

⁷ The mixed decimal designation is used in program instructions, as well as by Pinheiro (1961), to refer to a number that represents more than one unit, in its decimal representation, in which a comma separates the integer part from the non-integer part of the number. Program instructions also refer to single decimal as a number in its decimal representation, with a value less than unity.

⁸ In the programme instructions the integers are referred to, but what is part of primary school programs are only non-negative integers.

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Gonçalves (1974) proposal for the introduction to rational numbers focuses on work with decimal representation. Gonçalves (1974) option is justified by the guidelines of the primary education program in effect at the time.⁹

In fraction definition, Gonçalves (1974) states that fractions constitute “a new world, with its own types of units, quantities, numbers: new numeration, new notations and a different general operation.” (p. 142). The author considers that arithmetic presents the fraction as a case of a new reality of a new numbering and, therefore, its study should not be parallel to the study of whole numbers.

Gonçalves (1974, quoting Augustine, undated) highlights that the concept of a fractional number is more complex than the concept of a natural number and, therefore, requires greater maturity and mathematical knowledge from the child. Gonçalves (1974) distinguishes four different meanings of fractions in context, presenting examples that differentiate these meanings. The first example refers to what can be framed in the fraction as a part of a whole of a continuous unit “1) In the division of a continuous set it means "one or more of the equal parts into which that set has been divided».” (Gonçalves, 1974, p. 143, quotes in the original) the following figure was presented.

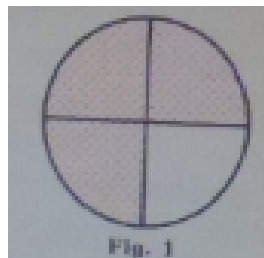


Figure 4 - Illustration of the fraction as part of a whole of a continuous unit
Source: Gonçalves, 1974, p. 143 (scanning, 100% of the original)

In the second example, this author presents the fraction, which can be framed as part of a whole of a discrete set, or multiplicative partitive operator “2) In the sharing of a discontinuous set, it means "one or more of the equal parts of that set» (of things, people, etc.).” (Gonçalves, 1974, p. 143, quotes in the original). To illustrate the fraction in this sense, the author presents the following figure.

⁹ At the time, the programs approved in Ordinance No. 23.485, Government Gazette, 167, 7/16/1968, 1019-36, were in effect.

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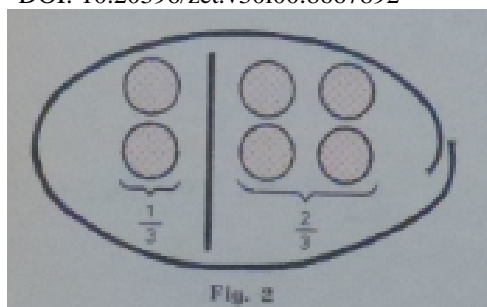


Figure 5 - Illustration of the fraction as part of a whole of a discrete unit
Source: Gonçalves, 1974, p. 143 (scanning, 100% of the original)

In the third concept that Gonçalves (1974) distinguishes in fractions, he presents an example that refers to what can be called a fraction as the quotient between two whole numbers, in a situation of equipartition.

3) It can mean the «quotient of two natural numbers (divisor \neq zero)». If I want to divide three bars of soap by 4 washerwomen, I can divide each bar into 4 parts, giving each washerwoman three quarters as there are three bars. See fig. 3 (Gonçalves, 1974, p. 143, quotes in the original).

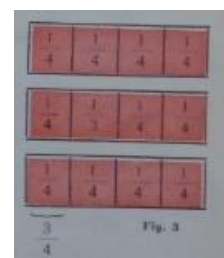


Figura 6 – Illustration of the cited figure
Fonte: Gonçalves, 1974, p. 143

This author also presents a fourth meaning that the concept of fraction can contain, referring to the fraction as a ratio “the ratio of the numerical properties of two sets”. The example presented for this case is as follows:

4) It can also mean "the ratio of the numerical properties of two sets". If, in a fruit bowl, there are 5 bananas and I eat two, the ratio between the bananas I ate and those in the fruit bowl is 2 to 5 $2/5$." (Gonçalves, 1974, p. 143, quotes marks in the original)

For Gonçalves (1974) this last meaning of the fraction is at the base of the study of percentage.

Gonçalves (1974) recommends that the development of the intuitive concept of fraction should be done through the equipartition of sets that he calls continuous, followed by the formation of subsets of a given set. It also defines the function of the numerator and denominator in the fraction, clarifying as follows what it means by a fractional number:

fractional number is an *idea*, and its symbolic representation is called a *fraction* (fractional number numeral), which can have the form a/b , and that a and b designate natural numbers, and can also refer to a as a dividend and b as a divisor, where $\mathbf{b} \neq \mathbf{0}$. (Gonçalves, 1974, p. 144, italics and bold in the original)

Still in the definition of fraction, Gonçalves (1974) distinguishes what he calls a fractional unit, when the whole unit is divided into equal parts and only one of these parts is taken, from the fractional quantity that results from the junction of several fractional units. It also emphasizes that the fraction can represent a quantity that is not a whole number but can also represent integer units. It is only after working with the notion of fraction that Gonçalves

(1974) introduced the nomenclature normally used in fractions as a fraction dash, which the author designates as fraction trace, numerator, denominator and terms of the fraction. Gonçalves (1974) also presents improper fractions, mentioning that this designation is because they refer to fractions that are worth more than the unit. In the previous example, this author also presents the improper fraction represented in the form of a mixed numeral, without verbally explaining the meaning of the integer part and the fractional part, presenting only the relationship between the figure and the symbolic representation. Verbally refers only to improper fractions, noting that children should note that the numerator is equal to or greater than the denominator¹⁰. And, in a footnote, he also emphasizes that the natural numbers must be considered as a subset of the fractional numbers, and that children must acquire this notion of fractional numbers.

Concluding remarks

With regard to the definition of a rational number, it is important to start by pointing out that, neither in textbooks nor in official curricular documents, is the designation of rational number used, having been identified designations as a fraction to refer to a number that has equal parts of the unit, but which is less than unity, or fractional number to refer to a number that has equal parts of unity, but which is greater than unity. Some designations used to designate proper fraction, improper fraction or mixed numeral are also different from those commonly found in textbooks in our days. Some of the designations used by authors such as Pimentel Filho (1934) seem to be related to the use of translations of textbooks in Castilian. It should be noted that the use of the designation rational number is very recent in the context of what is the 1st cycle of basic education in Portugal, corresponding to primary education, appearing for the first time associated with the Mathematics Program for Basic Education published in 2007 (Ponte et al., 2007).

The four textbooks analyzed in this work focus on the pedagogical component, responding to the changes that were made in the initial training courses for primary school teachers, in 1930, with the publication of programs in 1935 and 1943. The analysis of the programs of these courses showed that they are very generic, not detailing the contents to be taught in each subject.

In the work of Pimentel Filho (1934) the introduction of the fraction as a relationship between the part and a whole of a continuous unit is privileged. In the introduction to fractions, examples are also presented that refer to situations in which the fraction appears with the meaning of multiplicative partitive operator of a discrete set (Monteiro & Pinto, 2005). In this approach, the importance, and difficulties that this content offers students in initiation are highlighted, although specific difficulties of the content are not mentioned. It is an approach that privileges the relationship between the different representations, active, pictorial, verbal and symbolic, making ample use of images with color. In his work, the expression rational number is never used, being used instead designations such as fraction,

¹⁰ Gonçalves (1974) considers as improper fraction, fractions that represent numbers greater than or equal to the unit.

fractional expression to designate improper fraction, fractional number to designate mixed numeral, and decimal numbers. The proposal is centered on the presentation of a teaching sequence that favors initiation from fractions and only after decimals but does not present very formal definitions. Pimentel Filho (1934) discusses unit types at the beginning of the chapter on fractions. Pimentel Filho's textbook (1934) was edited when the 1929 primary education programs were in effect. In this work, it is possible to verify that the first approach to rational numbers is made from the representation in the form of a fraction, and this representation is also the first approach recommended in the programs in effect at the time. However, it should be noted that the textbook presents contents that go beyond what is established in the primary education program.

In the work of Gaspar and Ferreira (1944) it is worth highlighting the importance given to the indication of a teaching sequence with an initial focus on the fraction, but where it is argued that this representation should be worked in parallel with the decimal representation. In the initial work, which focuses on the unit fraction, always presented as part of a whole of a continuous unit, the authors value the relationship between the different representations, pictorial, verbal and only later the symbolic. The emphasis is given to the relationship between verbal and symbolic representation, not making use of the image as happened, for instance, in Pimentel Filho (1934). At the time of the edition of the textbook by Gaspar and Ferreira (1944), the primary education programs of 1937 were in effect, which, although they continued to present a first approach to rational numbers centered on representation in the form of a fraction, began to highlight and center the work on decimal representation. This is an approach that is also reflected in the textbook, which highlights the parallel work between the two forms of representation.

Pinheiro's work (1961) differs from previous works by proposing an initial approach to rational numbers based on their decimal representation. The author does not present a discussion about this option, and about the advantages and disadvantages of doing the initiation through fractions or decimals, justifying himself with the instructions of the primary education program of the time, primary education programs of 1960, which indicated this sequence and focused their work on the use of the meter and its submultiples and on the relationship that could be established between the organization of decimals and the decimal organization of whole numbers. Only after working with the meter unit did the students generalize the division of the unit into tenths, hundredths and thousandths to any unit. It is only after working with decimal representation that Pinheiro (1961) considers the initiation to fractions, highlighting in the proposed teaching sequence the initial work with unit fractions and the relationship between the different representations, pictorial, verbal and symbolic, in an identical order to the proposal by Gaspar and Ferreira (1944). The fraction is essentially characterized as the part of a whole of a continuous unit, although the author presents some examples in which the fraction appears as a partitive multiplicative operator applied to discrete units. In the work of Pinheiro (1961) the designation rational number is never used, using the fraction designation.

Gonçalves (1972, 1974) also opted for initiation centered on decimal representation, justifying this option with the guidelines of the primary education program of the time, primary education programs of 1968. However, it presents an explicit discussion about the possible teaching sequences, starting with the decimal representation or the representation in the form of a fraction. Gonçalves (1974) highlights some difficulties that this content can cause in children in its representation in fractions, because it represents a new numerical set, with different types of units and with a different notation and operation, as mentioned by Monteiro and Pinto (2005). In the work of Gonçalves (1974) the designation rational number is not used, being used the expression fractional number to identify the numbers that can be represented by a fraction. He also makes a distinction between the expression fractional unit, which he uses to designate unit fractions, and the fractional quantity that results from the addition of fractional units. This author refers to the complexity of the fractional number concept, distinguishing four different concepts when the fraction is presented in context: sharing a continuous set, sharing a discontinuous set, quotient of two integers and ratio. These four concepts can fit into the different meanings of fractions in context, as presented by Monteiro and Pinto (2005). Following the teaching of fractions, Gonçalves (1974) emphasizes that initiation must be done through what can be called the equitable sharing of continuous sets (Monteiro & Pinto, 2005). Following the teaching of fractions, Gonçalves (1974) emphasizes that initiation must be done through what can be called the equitable sharing of continuous sets (Monteiro & Pinto, 2005). In this initiation, he uses the relationship between pictorial representation and symbolic representation. In the work of Gonçalves (1974) it is still possible to verify that there is a need to distinguish the idea from its representation, which happens, for example, when the author distinguishes a fractional number from its representation in the form of a fraction and decimal number from its representation in decimal numeral.

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